Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Computational finance – Lecture 2

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Outline

Example (Black–Scholes model)

$S_T = S_0 \exp\left(\sigma W_T + \left(r - \frac{1}{2}\sigma^2\right) T\right) \sim \mathrm{LN}(?, ?)$, $W_T \sim \mathcal{N}(0, T)$. Simulate?

Simulation of non-uniform distributions

Transformation of independent uniform random variables $U_1, U_2, \ldots$.

Theorem (Inversion method)

Let F be a c.d.f. and let $F^{-1}(u) := \inf\{\, x \mid F(x) \ge u \,\}$. For $U \sim \mathcal{U}([0, 1[)$, $X := F^{-1}(U) \sim F$.

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- Exponential distribution: $\mathrm{Exp}(\lambda)$ has c.d.f. $F(x) := 1 - e^{-\lambda x}$, $F^{-1}(u) = -\frac{1}{\lambda}\log(1-u)$. Hence, $X := -\frac{1}{\lambda}\log(U) \sim \mathrm{Exp}(\lambda)$.
- Sometimes no explicit expression of the quantile function exists, but efficient numerical approximations are available. E.g., $\Phi^{-1}(u) = \sqrt{2}\,\mathrm{erf}^{-1}(2u-1)$.
- Transfer of structural properties from U to X.

Exercise

Generate samples from $X^* := \max(X_1, \dots, X_L)$, given that F^{-1} is available.

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Setting:

- Target distribution: X has density $f : \mathbb{R}^d \to \mathbb{R}_+$
- Auxiliary distribution: Y has density $g : \mathbb{R}^d \to \mathbb{R}_+$ and can be sampled in some way.
- There is $c \geq 1$ s.t. $f(x) \leq cg(x)$, $x \in \mathbb{R}^d$.

Algorithm (Acceptance–Rejection method)

1. Generate (independent) instances $U \sim \mathcal{U}([0,1])$ and Y.
2. If $U \leq \frac{f(Y)}{cg(Y)}$, return Y; else go back to 1.

Idea: The $(x_1, \ldots, x_d)$-component of a uniform distribution on the hypograph $\subset \mathbb{R}^d \times \mathbb{R}_+$ of f has distriution of X.

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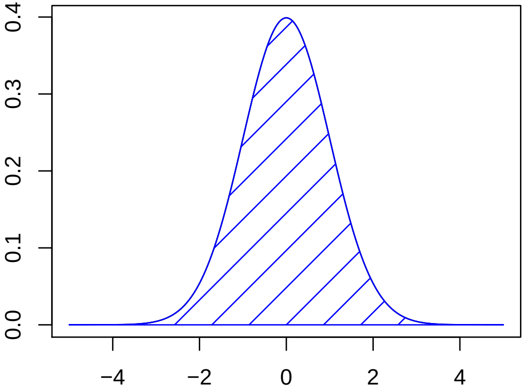

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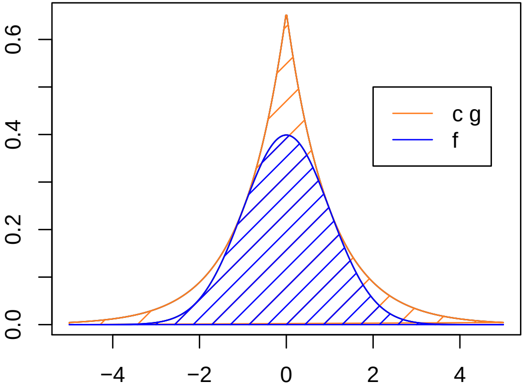

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Proof

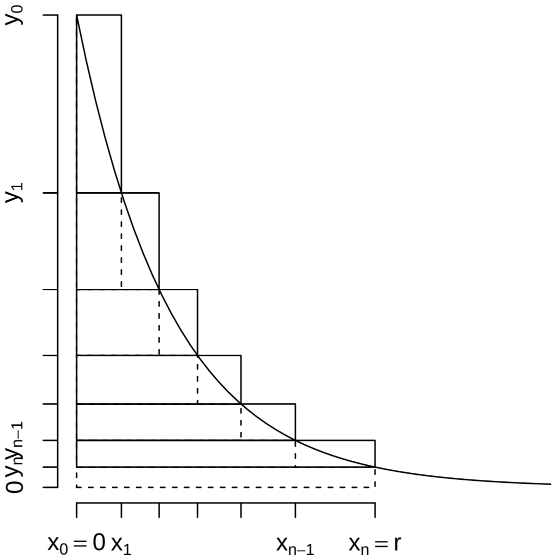
- Very flexible algorithm, requiring knowledge of the density.
- The number of uniform r.v.s required to generate one sample from the target distribution is random. In fact, the number N of steps required satisfies $N \sim \mathrm{Geo}(1/c)$, with $E[N] = c$.

Example

The double exponential distribution has density $g(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. With the standard normal density $f = \varphi$, we have

$$f(x) \le cg(x), \quad x \in \mathbb{R}, \quad c := \sqrt{\frac{2e}{\pi}}, \text{ since } \frac{\varphi(x)}{g(x)} = \sqrt{\frac{2}{\pi}}e^{-x^2/2+|x|} \le c.$$

- The auxiliary distribution g does not need to be a standard distribution.
- Ziggurat algorithm proposes to use a tailor made distribution with step-function as density, closely following the target density f up to the tails.
- Rectangles and tail chosen to have same areas.
- Specialized tail sampling distribution assumed to exist.
- Choose large number of rectangles.



Algorithm (Box–Muller method)

1. *Generate ind.* $U_1, U_2 \sim \mathcal{U}([0,1[)$;
2. *Set* $\theta := 2\pi U_2$, $\rho := \sqrt{-2\log(U_1)}$;
3. *Return two independent standard normal r.v.s* $X_1 := \rho\cos(\theta)$, $X_2 := \rho\sin(\theta)$.

Algorithm (Polar method)

1. *Generate ind.* $U_1, U_2 \sim \mathcal{U}(]-1,1[)$ *and set* $S := U_1^2 + U_2^2$;
2. *If* $S < 1$, *set* $r := \sqrt{\frac{-2\ln(S)}{S}}$ *return the independent standard normals* $Y_1 := rU_1$, $Y_2 := rU_2$; *else, return to 1.*

Generation of d-dimensional r.v. $X \sim \mathcal{N}(\mu, \Sigma)$

1. Generate d independent r.v.s $Z_1, \ldots, Z_d \sim \mathcal{N}(0,1)$, $Z := (Z_1, \ldots, Z_d)$;
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1. Explain why c in the acceptance–rejection method can only be greater than or equal to 1. What does $c = 1$ imply?
2. Provide a method for generating double exponential random variables using only *one* uniform random number per output. Moreover, justify the bound c in the example on generating normal r.v.s by the acceptance–rejection method.
3. Show that (X_1, X_2) generated by the Box–Muller method have the two-dimensional standard normal distribution.
   Hint: Show that the density of the two-dimensional uniform variate (U_1, U_2) is transformed to the density of the two-dimensional standard normal distribution.
4. Show that (Y_1, Y_2) generated by the polar method have the two-dimensional standard normal distribution.
5. Implement the different methods for generating Gaussian random numbers and compare their efficiency.

Given a r.v. X taking values in some space $\mathcal{X}$, and a measurable function $f : \mathcal{X} \to \mathbb{R}$. Let $I[f;X] := E[f(X)]$, provided that $E[|f(X)|] < \infty$.

Definition (Monte Carlo simulation)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. copies of X. Define

$$I_M[f;X] := \frac{1}{M} \sum_{i=1}^{M} f(X_i).$$

Theorem

$$I_M[f;X] \xrightarrow[a.s.]{M\to\infty} I[f;X].$$

Theorem

Suppose that $\sigma^2(f;X) := \operatorname{var} f(X) < \infty$. *Letting* $\epsilon_M(f;X) := I[f;X] - I_M[f;X]$, *we have*

$$E\left[\epsilon_M(f;X)^2\right] = \frac{\sigma^2(f;X)}{M}, \qquad \lim_{M\to\infty} P\left(\frac{\sigma(f;X)a}{\sqrt{M}} \le \epsilon_M \le \frac{\sigma(f;X)b}{\sqrt{M}}\right) = \Phi(b) - \Phi(a)$$

for the standard normal c.d.f. Φ.

Proof.

□

- The Monte Carlo method has rate of convergence $\frac{1}{2}$ in the MSE sense.

- Suppose that $X \sim \mathcal{U}([0,1[^d)$, d large, f smooth (enough).
- Classical numerical integration: iterative application of 1D quadrature (e.g., trapezoidal rule) using Fubini's theorem. Assume rate of convergence k in Δx in dimension 1.
- Cost model: computation cost is assumed to be proportional to the number of evaluations of the integrand f.

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Monte Carlo

- Error $\simeq \frac{1}{\sqrt{M}}$, cost $\simeq M$. Hence, error $\simeq$ cost$^{-1/2}$
- Error is random.
- Explicit and sharp error control in MSE sense.

Classical numerical integration

- $\Delta x = \frac{1}{N}$ ...step size of 1D grid.
- Error $\simeq \frac{1}{N^k}$, cost $\simeq N^d$. Hence, error $\simeq$ cost$^{-k/d}$.
- Error is deterministic.
- No sharp and computable error control.