ON THE WEAK CONVERGENCE RATE IN THE DISCRETIZATION OF ROUGH VOLATILITY MODELS

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ABSTRACT. We study the weak convergence rate in the discretization of rough volatility models. After showing a lower bound 2H under a general model, where H is the Hurst index of the volatility process, we give a sharper bound H + 1/2 under a linear model.

1. Introduction and the main results

The financial derivative pricing practice amounts to computing an expectation of a functional of an asset price process. When the asset price process is modelled by a diffusion, the computation is either by solving an associated PDE or by the Monte-Carlo method with a discretized process. The error in the computation of the expectation due to the discretization is called the weak error, in contrast to the strong error quantifying pathwise deviation. It is now well-known that for regular diffusion processes the weak error in the Euler-Maruyama discretization is $O(n^{-1})$, where *n* is the number of the discretization steps. See [9, 1, 4] among others.

Recently a class of stochastic volatility models called the rough volatility models has attracted attention. The volatility process under those models is not a diffusion but has a rougher path. This is the only class of continuous price models that are consistent to a power-law type term structure of implied volatility skew typically observed in equity option markets. The volatility processes of the major stock indices are also statistically estimated to be rough. See [5, 7, 6] for the details.

For example, under the rough Bergomi model introduced by [2], the asset price *S* follows

$$S_T = S_0 \exp\left\{\int_0^T \sqrt{V_t} \left\{\rho dW_t + \sqrt{1 - \rho^2} dW_t^{\perp}\right\} - \frac{1}{2} \int_0^T V_t dt\right\},\$$

$$V_t = V_0(t) \exp\left\{\int_0^t k(t, s) dW_s - \int_0^t k(t, s)^2 ds\right\},\$$

where (W, W^{\perp}) is a 2 dimensional standard Brownian motion, $V_0(t) = \mathbb{E}[V_t]$ is a deterministic function called the forward variance curve, $k(t, s) = \eta \sqrt{2H(t-s)^{H-1/2}}$, and $H \in (0, 1/2)$, $\rho \in (-1, 1)$ and $\eta > 0$ are parameters. The European option price with payoff f is expressed as

$$\mathsf{E}[f(S_T)] = \mathsf{E}\left[F\left(\int_0^T \sqrt{V_t} \mathrm{d}W_t, \int_0^T V_t \mathrm{d}t\right)\right],$$

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where

$$F(x,y) = \int_{\mathbb{R}} f\left(S_0 \exp\left\{\rho x + \sqrt{y(1-\rho^2)}z - \frac{1}{2}y\right\}\right) \phi(z) \mathrm{d}z,$$

where ϕ is the standard normal density. For these non-diffusion models the finite dimensional PDE pricing is not possible anymore. The Monte-Carlo pricing with a discretized stochastic integral

$$\int_{0}^{1} \sqrt{V_{\frac{[nt]}{n}}} dW_{t} = \sum_{i=0}^{n-1} \sqrt{V_{\frac{i}{n}}} (W_{\frac{i+1}{n}} - W_{\frac{i}{n}})$$

is proposed in [2], while the convergence rate of the the associated weak error is still an open problem.

Motivated by the rough volatility modelling, we are interested in the weak error rate in the discretization of a stochastic integral

$$X_1 := \int_0^1 \sigma(Y_t, t) dW_t, \ Y_t := \int_0^t (t-s)^{H-1/2} dW_s$$

for a sufficiently regular function σ . Under the rough Bergomi model, $\sqrt{V_t} = \sigma(Y_t, t)$ with

$$\sigma(y,t) = \sqrt{V_0(t)} \exp\left\{\frac{\eta\sqrt{2H}}{2}y - \frac{\eta^2}{4}t^H\right\}.$$

Noting that *Y* is Gaussian and so the random sequence $(W_{\frac{i}{n}}, Y_{\frac{i}{n}}), i = 0, 1, ...$ can be efficiently sampled using the Cholesky decomposition of the covariance matrix, a numerically feasible discretized integral is given by

$$X_1^n := \int_0^1 \sigma\left(Y_{\frac{[nt]}{n}}, \frac{[nt]}{n}\right) \mathrm{d}W_t.$$

The associated weak error with respect to a test function *f* is given by $\varepsilon_n(f, 1)$, where

$$\varepsilon_n(f,a) = \mathsf{E}[f(X_1)] - \mathsf{E}[f((1-a)X_1 + aX_1^n)]$$

for $a \in [0, 1]$.

Recently Bayer et al. [3] has considered the case $\sigma(y, t) = y$ and showed that the weak error rate is at least H + 1/2, that is, $\varepsilon_n(f, 1) = O(n^{-H-1/2})$ for $f \in C_b^k$ with $k \ge 1/H$. The approach taken by [3] is based on an infinite dimensional Markov representation of Y. With a finite dimensional Markov approximation to Y, the existing techniques for diffusions are then applicable. In this article we take a different approach for the same problem. We apply the duality approach introduced by [4] that is based on the Malliavin calculus and known to be effective to deal with non-Markov models.

Our first result is for a general time homogeneous $\sigma(y, t) = \sigma(y)$ but not sharp.

Theorem 1.1. If $\sigma \in C_p^2$ and $f \in C_p^3$, then $\varepsilon_n(f, a) = O(n^{-2H})$ uniformly in $a \in [0, 1]$.

The second result is sharper but only for σ being linear as in [3].

Theorem 1.2. If $\sigma(y) = y$ and $f \in C_p^{2m+1}$, then $\varepsilon_n(f, 1) = O(n^{-((2mH) \land (H+1/2))})$.

Corollary 1.1. If $\sigma(y) = y$ and $f \in C_p^k$ with $k \ge 2 + 1/(2H)$, then $\varepsilon_n(f, 1) = O(n^{-(H+1/2)})$.

In comparison with [3], we allow test functions of polynomial growth, and for H < 1/4 we require less regularity on test functions to ensure the same rate. The proof is shorter as we see below.

It is still an open problem whether the rate H + 1/2 is sharp even for the linear model. See [3] for some numerical experiments that suggest it is the case. It would be straightforward to extend Theorem 1.1 to include the case that the volatility function σ is time-dependent. It seems however difficult to extend our proof of Theorem 1.2 to include the case that σ is nonlinear; the linearity results in a recursive structure of the weak error which we exploit as a key.

2. The duality approach

We give the proofs of Theorem 1.1 and 1.2 in this section. Let

$$k_n(s,t) = (t-s)^{H-1/2} - \left(\frac{[nt]}{n} - s\right)_+^{H-1/2}$$

Here and in the sequel, we mean by $(x)_{+}^{\alpha}$

$$(x)^{\alpha}_{+} = x^{\alpha} \mathbb{1}_{(0,\infty)}(x)$$

for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}$. We start with a lemma.

Lemma 2.1. For any $\alpha \in (-1/2 - H, 1/2 - H)$,

$$\int_0^1 \int_0^t (t-s)^{\alpha} |k_n(s,t)| \mathrm{d}s \mathrm{d}t = O(n^{-\alpha - H - 1/2})$$

and

$$\int_0^1 \int_0^t \left(\frac{[nt]}{n} - s \right)_+^{\alpha} |k_n(s, t)| ds dt = O(n^{-\alpha - H - 1/2}).$$

Proof. Changing variable as t - s = (t - [nt]/n)u,

$$\int_0^1 \int_0^t (t-s)^{\alpha} |k_n(s,t)| ds dt$$

= $\int_0^1 \left(t - \frac{[nt]}{n}\right)^{\alpha + H + 1/2} \int_0^{t/(t-[nt]/n)} u^{\alpha} \left|u^{H-1/2} - (u-1)^{H-1/2}\right| du dt.$

This is $O(n^{-\alpha-H-1/2})$ as claimed since

$$\int_0^2 u^{\alpha} \left| u^{H-1/2} - (u-1)_+^{H-1/2} \right| < \mathrm{d}u < \infty$$

for
$$\alpha > -1/2 - H$$
 and

$$\int_{2}^{\infty} u^{\alpha} \left| u^{H-1/2} - (u-1)^{H-1/2} \right| du = \frac{1}{|H-1/2|} \int_{2}^{\infty} u^{\alpha} \int_{u-1}^{u} v^{H-3/2} dv du$$

$$\leq \frac{2^{3/2-H}}{|H-1/2|} \int_{2}^{\infty} u^{\alpha+H-3/2} du$$

$$= \frac{2^{1+\alpha}}{(1/2-H)(1/2-\alpha-H)}$$

for $\alpha < 1/2 - H$, using that $v^{H-3/2} \le (u-1)^{H-3/2}$ for $u - 1 \le v \le u$ and

$$\sup_{u>2}\frac{(u-1)^{H-3/2}}{u^{H-3/2}}=2^{3/2-H}.$$

The latter is similarly proved by a variable-change [nt]/n - s = (t - [nt]/n)u. 2.1. **Proof of Theorem 1.1.** We take the duality approach introduced by [4]. We have

$$X_1 - X_1^n = \int_0^1 \sigma_n(t) (Y_t - Y_{[nt]/n}) dW_t = \int_0^1 \sigma_n(t) \int_0^t k_n(s, t) dW_s dW_t,$$

where

$$\sigma_n(t) = \int_0^1 \sigma'((1-\theta)Y_{[nt]/n} + \theta Y_t) \mathrm{d}\theta.$$

Let

$$F_n(a) = \int_0^a f'((1-b)X_1 + bX_1^n) \mathrm{d}b.$$

Then, using the duality formula (the Malliavin integration-by-parts formula; see [8] for the details), we have

$$\varepsilon_n(f,a) = \mathsf{E}[F_n(a)(X_1 - X_1^n)]$$

= $\mathsf{E}[\int_0^1 D_t F_n(a)\sigma_n(t) \int_0^t k_n(s,t) dW_s dt]$
= $\int_0^1 \int_0^t \mathsf{E}[D_s(D_t F_n(a)\sigma_n(t))]k_n(s,t) ds dt,$

where D denotes the Malliavin-Shigekawa derivative. Note that

$$\begin{aligned} D_t F_n(a) &= \int_0^a f^{(2)} ((1-b)X_1 + bX_1^n) ((1-b)D_t X_1 + bD_t X_1^n) db, \\ D_t X_1 &= \sigma(Y_t) + \int_t^1 \sigma'(Y_u)(u-t)^{H-1/2} dW_u, \\ D_t X_1^n &= \sigma(Y_{[nt]/n}) + \int_t^1 \sigma'(Y_{[nu]/n}) \left(\frac{[nu]}{n} - t\right)_+^{H-1/2} dW_u, \\ D_s \sigma_n(t) &= \int_0^1 \sigma^{(2)} ((1-\theta)Y_{[nt]/n} + \theta Y_t) \left((1-\theta) \left(\frac{[nt]}{n} - s\right)_+^{H-1/2} + \theta(t-s)_+^{H-1/2}\right) d\theta, \\ D_s D_t F_n(a) \\ (1) &= \int_0^a f^{(2)} ((1-b)X_1 + bX_1^n) ((1-b)D_s D_t X_1 + bD_s D_t X_1^n) db \\ &+ \int_0^a f^{(3)} ((1-b)X_1 + bX_1^n) ((1-b)D_s X_1 + bD_s X_1^n) ((1-b)D_t X_1 + bD_t X_1^n) db, \end{aligned}$$

$$D_{s}D_{t}X_{1} = \sigma'(Y_{t})(t-s)^{H-1/2} + \int_{t}^{1} \sigma^{(2)}(Y_{u})(u-s)^{H-1/2}(u-t)^{H-1/2} dW_{u},$$

$$D_{s}D_{t}X_{1}^{n} = \sigma'(Y_{[nt]/n}) \left(\frac{[nt]}{n} - s\right)_{+}^{H-1/2} + \int_{t}^{1} \sigma^{(2)}(Y_{[nu]/n}) \left(\frac{[nu]}{n} - s\right)_{+}^{H-1/2} \left(\frac{[nu]}{n} - t\right)_{+}^{H-1/2} dW_{u}.$$

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Therefore, denoting by *C* a generic constant which is independent of *a*, we have by Minkowski's integral inequality that for any *n* and 0 < s < t < 1, $|\mathsf{E}[D_s(D_tF_n(a)\sigma_n(t))]| \le ||D_sD_tF_n(a)||_2 ||\sigma_n(t)||_2 + ||D_tF_n(a)||_2 ||D_s\sigma_n(t)||_2$

$$\leq C \left(||D_s D_t F_n(a)||_2 + ||D_t F_n(a)||_2 \left(|t-s|^{H-1/2} + \left(\frac{[nt]}{n} - s\right)_+^{H-1/2} \right) \right)$$

$$\leq C \left(||D_s D_t X_1||_4 + ||D_s D_t X_1^n||_4 + (||D_s X_1||_6 + ||D_t X_1||_6 + ||D$$

The result then follows from Lemma 2.1 with $\alpha = H - 1/2$.

2.2. **Proof of Theorem 1.2.** When $\sigma(y) = y$, we have $\sigma_n(t) = 1$,

$$D_t X_1 = \int_0^1 |t - s|^{H - 1/2} \mathrm{d}W_s, \ D_s D_t X_1 = |t - s|^{H - 1/2}$$

and

and

$$D_{t}X_{1}^{n} = \int_{0}^{1} \left(\frac{[nt]}{n} - s\right)_{+}^{H-1/2} + \left(\frac{[ns]}{n} - t\right)_{+}^{H-1/2} dW_{s}$$
with $D_{s}D_{t}X_{1}^{n} = ([nt]/n - s)_{+}^{H-1/2} + ([ns]/n - t)_{+}^{H-1/2}$. By (1),
 $D_{s}D_{t}F_{n}(a)$

$$= f^{(2)}(X_{1}) \int_{0}^{a} ((1 - b)D_{s}D_{t}X_{1} + bD_{s}D_{t}X_{1}^{n})db$$

$$+ \int_{0}^{a} (f^{(2)}((1 - b)X_{1} + bX_{1}^{n}) - f^{(2)}(X_{1}))((1 - b)D_{s}D_{t}X_{1} + bD_{s}D_{t}X_{1}^{n})db$$

$$+ \int_{0}^{a} f^{(3)}((1 - b)X_{1} + bX_{1}^{n})((1 - b)D_{s}X_{1} + bD_{s}X_{1}^{n})((1 - b)D_{t}X_{1} + bD_{t}X_{1}^{n})db.$$
Therefore we have

Therefore, we have

$$\varepsilon_n(f,a) = r_{n,1}(f,a) + r_{n,2}(f,a) + r_{n,3}(f,a),$$

where

$$\begin{split} r_{n,1}(f,a) &= \mathsf{E}[f^{(2)}(X_1)] \int_0^a g_n(b) \mathrm{d}b, \\ r_{n,2}(f,a) &= \int_0^a \mathsf{E}[(f^{(2)}((1-b)X_1 + bX_1^n) - f^{(2)}(X_1))]g_n(b) \mathrm{d}b, \\ g_n(b) &= \int_0^1 \int_0^t \left((1-b)(t-s)^{H-1/2} + b\left(\frac{[nt]}{n} - s\right)_+^{H-1/2} \right) k_n(s,t) \mathrm{d}s \mathrm{d}t, \\ r_{n,3}(f,a) &= \int_0^a \int_0^1 \int_0^t h_n(s,t,b) k_n(s,t) \mathrm{d}s \mathrm{d}t \mathrm{d}b, \\ h_n(s,t,b) &= \mathsf{E}[f^{(3)}((1-b)X_1 + bX_1^n)((1-b)D_sX_1 + bD_sX_1^n)((1-b)D_tX_1 + bD_tX_1^n)]. \end{split}$$

Notice that

$$r_{n,2}(f,a) = -\int_0^a \varepsilon_n(f^{(2)},b)g_n(b)\mathrm{d}b$$

and that $r_{n,3}(f, a) = O(n^{-H-1/2})$ uniformly in $a \in [0, 1]$ for $f \in C_p^3$ since

$$\int_0^1 \int_0^t |k_n(s,t)| \mathrm{d}s \mathrm{d}t = O(n^{-H-1/2})$$

by Lemma 2.1. Therefore,

$$\varepsilon_n(f,a) = E[f^{(2)}(X_1)] \int_0^a g_n(b) \mathrm{d}b - \int_0^a \varepsilon_n(f^{(2)},b) g_n(b) \mathrm{d}b + O(n^{-H-1/2}).$$

By induction, we obtain

(2)

$$\varepsilon_{n}(f,1) = \sum_{i=1}^{m-1} (-1)^{i-1} \mathsf{E}[f^{(2i)}(X_{1})] \int_{0}^{1} \cdots \int_{0}^{b_{i-1}} \prod_{j=1}^{i} g_{n}(b_{j}) db_{j} + (-1)^{m-1} \int_{0}^{1} \int_{0}^{b_{1}} \cdots \int_{0}^{b_{m-2}} \varepsilon_{n}(f^{(2(m-1))}, b_{m-1}) \prod_{i=1}^{m-1} g_{n}(b_{i}) db_{i} + O(n^{-H-1/2})$$

for $m \ge 2$ and $f \in C_p^{2m-1}$. We are going to show that the first sum is $O(n^{-1})$. Note that

$$\int_0^1 \cdots \int_0^{b_{i-1}} \prod_{j=1}^i g_n(b_j) db_j = \frac{1}{i!} \left(\int_0^1 g_n(b) db \right)^i$$

and that

$$\begin{split} \int_0^1 g_n(b) db &= \frac{1}{2} \int_0^1 \int_0^t ((t-s)^{H-1/2} + ([nt]/n - s)^{H-1/2}_+) k_n(s,t) ds dt \\ &= \frac{1}{2} \int_0^1 \int_0^t ((t-s)^{2H-1} - ([nt]/n - s)^{2H-1}_+) ds dt \\ &= \frac{1}{4H} \int_0^1 \left(t^{2H} - \left(\frac{[nt]}{n}\right)^{2H} \right) dt \\ &= \frac{1}{2} \int_0^1 \int_{[nt]/n}^t s^{2H-1} ds dt = O(n^{-1}). \end{split}$$

Therefore, from (2), we have

$$\varepsilon_n(f,1) = (-1)^{m-1} \int_0^1 \int_0^{b_1} \cdots \int_0^{b_{m-2}} \varepsilon_n(f^{(2(m-1))}, b_{m-1}) \prod_{i=1}^{m-1} g_n(b_i) db_i + O(n^{-H-1/2}).$$

By Lemma 2.1, we have $g_n(b) = O(n^{-2H})$ uniformly in $b \in [0, 1]$. Since $f \in C_p^{2m+1}$ by the assumption, we have $\varepsilon_n(f^{(2(m-1))}, b_{m-1}) = O(n^{-2H})$ uniformly in b_{m-1} by Theorem 1.1, and thus we conclude.

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