STABILITY OF DEEP NEURAL NETWORKS VIA DISCRETE ROUGH PATHS

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ABSTRACT. Using rough path techniques, we provide a priori estimates for the output of Deep Residual Neural Networks in terms of both the input data and the (trained) network weights. As trained network weights are typically very rough when seen as functions of the layer, we propose to derive stability bounds in terms of the total *p*-variation of trained weights for any $p \in [1,3]$. Unlike the C^1 -theory underlying the neural ODE literature, our estimates remain bounded even in the limiting case of weights behaving like Brownian motions, as suggested in [Cohen, Cont, Rossier, Xu: "Scaling properties of deep residual networks", arXiv, 2021]. Mathematically, we interpret residual neural network as solutions to (rough) difference equations, and analyze them based on recent results of discrete time signatures and rough path theory.

1. INTRODUCTION

Since their introduction in 2016 [13], Residual Neural Networks (ResNets) have gained a vast amount of popularity as a preferred network architecture for Machine Learning applications. The general principle is that this architecture allows for deeper networks since it models only the residual change of the features at the output of each layer. This is achieved by introducing "skip connections" which – at some steps – adjust the output of a layer by adding an earlier layer's output (see Figure 1). These "blocks", formed by a sequence of layers connected by an identity mapping, are then stacked on top of each other in order to build the network.

The authors of the previously cited paper argue that this helps precondition the optimization solvers so that increasing the network depth does not result in severe numerical instabilities and performance degradation, as is observed in plain Neural Networks. In particular, this approach allows them to successfully train a Deep Neural Network with hundreds and even thousands [14] of layers.



FIGURE 1. Single block of the ResNet architecture

In a plain Neural Network, the input vector \mathbf{y}_{i+1} of the (i+1)-th hidden layer is given by an application of the weights and the activation function to the input of the previous hidden layer. In symbols

$$\mathbf{y}_{i+1} = \sigma(\theta_i \mathbf{y}_i)$$

where $\sigma \colon \mathbb{R}^{d_{i+1}} \to \mathbb{R}^{d_{i+1}}$ and θ_i is a $d_{i+1} \times d_i$ matrix. In the ResNet approach, this is modified so that the output to the next hidden layer is given as the sum of the *input* to the previous layer, plus the previous operations; that is,

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \sigma(\theta_i \mathbf{y}_i). \tag{1}$$

Here, it is assumed that the width of all layers is constant, but the approach can easily be adapted to the more familiar setting of varying widths by applying an appropriate projection to right-hand side of the last equation.

Remark 1.1. We simplify notation by leaving out the bias term in the update rule (1). The usual update rule

$$\mathbf{y}_{i+1} = \sigma(\theta_i \mathbf{y}_i + b_i)$$

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can be reproduced in the form (1) above by adding a column of consisting of ones to \mathbf{y}_i and an appropriate restriction on θ_i to map that column to another column of ones – in the appropriate dimension.

Remark 1.2. In this work, we assume that the architecture follows the update (1) at each layer. In the engineering practice, usually a few layers are skipped over. i.e. the true update may look as follows:

$$\widetilde{\mathbf{y}}_i = \sigma(\theta_i \mathbf{y}_i), \quad \mathbf{y}_{i+1} = \mathbf{y}_i + \sigma(\theta_i \widetilde{\mathbf{y}}_i),$$

skipping over one layer in the process.

It has been argued by several authors [6, 10, 11] that the update in eq. (1) can be seen as a step of the Euler scheme for a *controlled ODE* of the form

$$\dot{\mathbf{y}}(t) = \sigma(\theta(t)\mathbf{y}(t)), \quad \mathbf{y}(0) = \mathbf{y}_0.$$
⁽²⁾

Then, knowledge of stability and convergence of numerical schemes for such systems can be used to derive corresponding results for ResNets, especially since one expects that the behavior of the output layer of the network under consideration will follow closely that of the continuous-time solution of eq. (2) (that is, in the limit of infinite depth) for very deep architectures.

In this work, we go back one step and consider the situation of ResNets with many, but finitely many layers. Specifically, we consider finite difference equations of the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sum_{\mu=1}^d f_\mu(\mathbf{x}_k) (\mathbf{w}_{k+1}^\mu - \mathbf{w}_k^\mu), \quad \mathbf{x}_0 = \xi \in \mathbb{R}^m.$$
(3)

Here, in the simplest case of constant dimension d, \mathbf{x}_k denotes the vector of nodes at layer k (corresponding to \mathbf{y}_k above), and the increment matrix $(\mathbf{w}_{k+1}^{\mu} - \mathbf{w}_k^{\mu})_{\mu=1}^d$ corresponds to the matrix $\theta_k \in \mathbb{R}^{d \times d}$. Finally, the vector fields $f_{\mu} : \mathbb{R}^d \to \mathbb{R}^d$ take care of the matrix-vector-multiplication as well as of the non-linear activation function σ . We assume that the number of nodes d is constant over all the layers. For a more general and detailed view of the setting, we refer to Appendix A.

Already from this very cursory look, the reader may notice an apparent difference between (1) and (3): in the former formulation the nonlinearity is applied after the matrix multiplication, whereas the order of operations is reversed in our finite difference equation. However, when we consider a deep network, this difference essentially only effects the very first layer of the network, which is hit by the nonlinearity in (3) before any affine transform is applied. All other layers are treated exactly the same way by both architectures – assuming that the non-linearity is not applied to the output layer, as is customarily the case.

Hence, it does not come as a surprise that both formulations are essentially equivalent, as also pointed out in [18]. We refer to Appendix A for a detailed analysis in our setting.

As seen in (3), instead of using continuous-time techniques, our approach consists of analyzing the evolution of the sequence $(\mathbf{x}_0, \ldots, \mathbf{x}_N)$, where N is the depth of the network, obtained by iteration of eq. (1) directly at the discrete level. Seeing (3) as discretization of an ODE amounts to assuming that the weight sequence comes from a C^1 -path of finite variation. There are conceptual and numerical reasons, discussed below, that suggest a less restrictive view, formulated in the so-called *p*-variation scale. Recall that the *p*-variation seminorm of a sequence $(\mathbf{w}_0, \ldots, \mathbf{w}_N)$ is given by

$$\|\mathbf{w}\|_{p;[0,N]} := \left(\max_{s \in \mathcal{S}_{0,N}} \sum_{j=0}^{\#s} |\mathbf{w}_{s_{j+1}} - \mathbf{w}_{s_j}|^p\right)^{1/p}$$

where the maximum is taken over the set $S_{0,N}$ of all increasing subsequences

$$s = (s_0 = 0, s_1, \dots, s_m, s_{m+1} = N)$$

of $\{0, \ldots, N\}$ and we have set #s = m for such a sequence. We use analytic techniques borrowed from rough paths theory and the algebraic framework developed in [5] to contributes to our understanding of stability properties of deep neural networks. We have

Theorem 1.3. Suppose $\mathbf{x}, \tilde{\mathbf{x}}$ are two solutions to eq. (9) with initial conditions $\xi, \tilde{\xi}$ and driven by $\mathbf{w}, \tilde{\mathbf{w}}$ respectively.

• Let $1 \leq p < 2$ and $f_1, \dots, f_d \in \mathcal{C}_b^2$. Then $\sup_{k=0,\dots,N} |\mathbf{x}_k - \tilde{\mathbf{x}}_k| \leq 2c_{p,N}^{1/p} e^{c_{p,N} \|f\|_{\mathcal{C}_b^2}^p \|\tilde{\mathbf{w}}\|_{p;[0,N]}^p} (|\xi - \tilde{\xi}| + \|f\|_{\mathcal{C}_b^2} \|\mathbf{w} - \tilde{\mathbf{w}}\|_{p;[0,N]})$

holds, where $c_{p,N}$ is explicitly given in Theorem 5.4 below.

• Let $2 \leq p < 3$ and $f_1, \ldots, f_d \in C^3_b$. Then

$$\sup_{k=0,\dots,N} |\mathbf{x}_{k} - \tilde{\mathbf{x}}_{k}| \leq 2(c'_{p,N})^{1/p} e^{c_{p,N} \|f\|_{\mathcal{C}^{3}_{\mathrm{b}}}^{p} \left(\|\mathbf{W}\|_{p;[0,N]}^{p} + \|\tilde{\mathbf{W}}\|_{p;[0,N]}^{p} \right)} (|\xi - \tilde{\xi}| + \|f\|_{\mathcal{C}^{3}_{\mathrm{b}}} \rho_{p}(\mathbf{W}, \tilde{\mathbf{W}}))$$

holds, where $c_{p,N}$ is again explicitly given in Theorem 5.10 below.

The symbol \mathbf{W} denotes the discrete signature lift of the weight sequence \mathbf{w} appearing in eq. (3) (see Section 4) and $\|\|\cdot\|_p$ is an appropriately defined norm on the spaces of lifts (Theorem 5.10). This inequality holds *uniformly* over input data. In practice, the weight matrices are randomly initialized with random i.i.d. values so typically the trained weights are also random. Our estimates hold pathwise, in the sense that the depend only on a single initialization of the weight matrices. Typically the size of the constants $c_{p,N}, c'_{p,N}$ appearing in Theorem 1.3 can be very large, but they remain uniformly bounded as $N \to \infty$ for all fixed $p \in [1,3)$. The fact that these constant can take on large values is also a consequence of the pathwise nature of our estimates, in the sense that they control the worst-case behavior of the network. We expect that under some assumptions on the distribution on the weights, some tighter control can be obtained for the average-case behavior. In the continuous-time setting, the corresponding analysis has been performed by e.g. Cass, Litterer and Lyons [1].

To see how our a priori estimate compares to what the C^1 theory would imply, we ran a simple numerical experiment¹, by first training a ResNet128 using the MNIST dataset and then computing the *p*-variation of the weights and their lift (Figure 2c). The jump observed at p = 2 is produced by switching from the standard *p*-variation norm $\|\cdot\|_p$ to the augmented *p*-variation norm $\|\cdot\|_p$. To put Figure 2c into context, note that the classical C^1 analysis estimate corresponds to the case p = 1 in our theory. (Figure 2a shows one entry of the matrices \mathbf{w}_k plotted against the time index k as well as the same entry of the differences $\mathbf{w}_{k+1} - \mathbf{w}_k$. Specifically, we plot the entry with indices (0,0). Similarly, Figure 2b shows the value of two entries of the vector of nodes \mathbf{x}_k plotted against the layer k. In this case, we chose the entries with indices 0 and 32, respectively. The choices of particular entries are arbitrary.

The roughness of the driving weight matrices depicted in Figure 2a might seem surprising at first sight. But recall the usual (random) initialization practice of the weights before the start of training: weights are typically initialized to be independent across layers and nodes and, in the case of constant dimension *d*, also identically distributed. There are several popular choices for the distribution itself, including normal and uniform distributions. Hence, (possibly after a proper re-scaling reflecting a choice of "time", and possibly in some asymptotic sense) the path of initialized weight matrices correspond to a matrix-valued Brownian motion, sampled in discrete time. As indicated by Figure 2a, the training does not seem to fundamentally change the picture: While trained weights are certainly no longer i.i.d., they still seem to exhibit the roughness of sample paths of a Brownian motion. We refer to [2] for an in-depth study of scaling properties of deep residual neural networks.

Accepting that the weights of deep residual neural networks behave like Brownian motions even after training, and considering the case of many layers (e.g., 128 layers as used in Figure 2), Figure 2c becomes clear. Indeed, paths of Brownian motion have finite *p*-variation only for p > 2 in the continuous time limit, hence we expect explosion of the *p*-variation for p < 2 even in the discrete case when the number of steps becomes large. In particular, the C^1 analysis p = 1 is expected to yield very poor results in the case of deep residual neural networks, if no regularization techniques are used to enforce smoothness.

The article is organized as follows. In Section 2 we review classical stability results from ODE theory and their application to the design of stable residual architectures, and their counterparts in the discrete setting. In Section 3 we introduce the basic tools of discrete rough analysis needed in order to extend the previously mentioned results to the *p*-variation topology. Next, in Section 4 we review the algebraic theory of the so-called iterated-sums signature of a time series. Finally in Section 5 we prove stability bounds for residual architectures in the *p*-variation norms, for $p \in [1, 3)$.

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¹Code available on GitHub, at https://github.com/ntapiam/resnets.







(B) Evolution of selected features, rescaled to lie in the interval [-1, 1].



(C) *p*-variation norm of the weights for $p \in [1, 3]$.

FIGURE 2. ResNet128 trained to MNIST data.

2. STABILITY IN FINITE-VARIATION NORM

Classical analytical tools can be exploited to understand the behavior of deep ResNets by comparing their behavior to a limiting ODE system of the form eq. (2) [6, 10, 11]. The main tool for this kind of analysis is Grönwall's inequality, which we now recall.

Theorem 2.1. Let $u, \alpha, \beta \colon [0, T] \to \mathbb{R}$ be continuous functions, with α non-decreasing and $\min(\alpha, 0) \in L^1$, such that

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s) \,\mathrm{d}s$$

for all $t \in [0, T]$. Then

$$u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s) \,\mathrm{d}s\right)$$

for all $t \in [0, T]$.

It is a standard result that, together with a priori bounds for solutions to eq. (2), this result implies the following stability bound [8, Theorem 3.15].

Theorem 2.2. Let $\mathbf{x}, \tilde{\mathbf{x}}$ be solutions to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t)$$

started respectively from $\xi, \tilde{\xi} \in \mathbb{R}^n$ and driven by $\mathbf{w}, \tilde{\mathbf{w}} \in C^1([0,T], \mathbb{R}^d)$. If $f \in \operatorname{Lip}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, the bound

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,T]} \leq e^{2\|f\|_{\mathrm{Lip}}\|\tilde{\mathbf{w}}\|_{1;[0,T]}} (|\xi - \tilde{\xi}| + \|f\|_{\mathrm{Lip}}\|\mathbf{w} - \tilde{\mathbf{w}}\|_{1;[0,T]})$$

The usefulness of these results in the previously mentioned references comes from the fact that for smooth enough driving signals, the behavior of the discrete control system defined in eq. (3) will be well approximated by the continuous-time limiting system. However, this relies on the assumption that the driving path \mathbf{w} is indeed smooth, and that we are considering enough time steps, i.e., the network is deep enough. It turns out that in practice neither of these assumptions might be satisfied (see Figure 2a). The main goal of this paper is to show that both these assumptions can be removed while retaining the stability results.

In this section we show how to obtain such a bound in the finite time-horizon regime, i.e., working directly at the discrete level. In the current literature the smoothness assumption is sometimes circumvented by penalizing the L^1 norm (or C^1 in continuous-time models) of the weights during training in order to enforce the necessary smoothness. As before, the main tool is a discrete version of Grönwall's inequality (see e.g. [16, Lemma A.3]).

Theorem 2.3. Let $c \ge 0$ and φ_i and v_i be non-negative sequences. If

$$\varphi_j \leqslant c + \sum_{i=1}^{j-1} v_i \varphi_i$$

for all $j \ge 1$, then

$$\varphi_j \leqslant c \prod_{i=1}^{j-1} (1+v_i) \leqslant c \exp\left(\sum_{i=1}^{j-1} v_i\right)$$

for all $j \ge 1$.

Let us consider solutions $\mathbf{x}, \tilde{\mathbf{x}}$ to eq. (3), driven resp. by $\mathbf{w}, \tilde{\mathbf{w}}$ and started resp. from two different initial conditions $\xi, \tilde{\xi} \in \mathbb{R}^m$. Suppose furthermore that the vector fields f_{μ} are Lipschitz and bounded. We denote by L(f) the Lipschitz constant of $f \colon \mathbb{R}^m \to \mathbb{R}^m$.

Considering the difference $\mathbf{z}_k \coloneqq \mathbf{x}_k - \tilde{\mathbf{x}}_k$ and letting $\Delta_k = \mathbf{w}_k - \tilde{\mathbf{w}}_k$, we can immediately observe that

$$\mathbf{z}_{k+1} - \mathbf{z}_k = \mathbf{x}_{k+1} - \mathbf{x}_k - (\tilde{\mathbf{x}}_{k+1} - \tilde{\mathbf{x}}_k)$$
$$= \sum_{\mu=1}^d f_\mu(\mathbf{x}_k)(\mathbf{w}_{k+1}^\mu - \mathbf{w}_k^\mu) - \sum_{\mu=1}^d f_\mu(\tilde{\mathbf{x}}_k)(\tilde{\mathbf{w}}_{k+1}^\mu - \tilde{\mathbf{w}}_k^\mu)$$

Therefore

$$|\mathbf{z}_{k+1} - \mathbf{z}_{k}| \leq \sum_{\mu=1}^{d} |f_{\mu}(\mathbf{x}_{k})| |\Delta_{k+1}^{\mu} - \Delta_{k}^{\mu}| + \sum_{\mu=1}^{d} |f_{\mu}(\mathbf{x}_{k}) - f_{\mu}(\tilde{\mathbf{x}}_{k})| |\tilde{\mathbf{w}}_{k+1}^{\mu} - \tilde{\mathbf{w}}_{k}^{\mu}|.$$

Hence, we see that

$$|\mathbf{z}_{k+1} - \mathbf{z}_k| \leq ||f||_{\infty} |\Delta_{k+1} - \Delta_k| + L(f)|\mathbf{z}_k| |\tilde{\mathbf{w}}_{k+1} - \tilde{\mathbf{w}}_k|.$$

Performing a telescopic sum we obtain that

$$|\mathbf{z}_{k}| \leq L(f) \sum_{j=0}^{k-1} |\mathbf{z}_{j}| |\tilde{\mathbf{w}}_{j+1} - \tilde{\mathbf{w}}_{j}| + |\mathbf{z}_{0}| + ||f||_{\infty} \sum_{j=0}^{k-1} |\Delta_{j+1} - \Delta_{j}|$$

The second term in the right-hand side is bounded by

$$|\mathbf{z}_0| + ||f||_{\infty} ||\Delta||_{1;[0,N]}$$

Therefore, we obtain from Theorem 2.3 that

$$|\mathbf{z}_{k}| \leq (|\mathbf{x}_{0} - \tilde{\mathbf{x}}_{0}| + ||f||_{\infty} ||\Delta||_{1;[0,N]}) \prod_{j=0}^{k-1} (1 + L(f)|\tilde{\mathbf{w}}_{j+1} - \tilde{\mathbf{w}}_{j}|).$$

Using the elementary estimate $1 + x \leq e^x$ we may finally obtain

$$\sup_{k=0,\dots,N} |\mathbf{x}_{k} - \tilde{\mathbf{x}}_{k}| \leq e^{L(f) \|\tilde{\mathbf{w}}\|_{1;[0,N]}} (|\mathbf{x}_{0} - \tilde{\mathbf{x}}_{0}| + \|f\|_{\infty} \|\mathbf{w} - \tilde{\mathbf{w}}\|_{1;[0,N]}).$$
(4)

Therefore, it is possible to obtain Lipschitz stability results at the discrete-time level. In the continuoustime case, it is possible to prove such theorems with respect to the whole range of p-variation topologies, for any $p \in [1, \infty)$. In the rest of the article we introduce the analogous techniques for treating the discrete-time case and we show how to obtain the desired bounds for $p \in [1, 3)$. The main difficulty in this case is that Theorem 2.3 is not well adapted to the weaker topologies, so a new generalization is needed (see Theorem 3.13). Indeed, directly applying Theorem 2.3 in the p-variation norm would lead to a bound like eq. (4) constant depending on N, which is unbounded as $N \to \infty$.

3. Elements of rough analysis

We begin with a brief overview of classical results present in the rough analysis literature. We remark that many of these results are usually stated in terms of continuous-time variables which introduces certain additional difficulties. In our case, no such difficulties arise so the statements and proofs of analogous results become simpler.

3.1. **Discrete controls.** We recall that in the setting of [17] a control function (or simply a control) is a function $\omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ which is super-additive, in the sense that $\omega(s, u) + \omega(u, t) \leq \omega(s, t)$ for all s < u < t. In the continuous-time setting, the main motivation for introducing control functions is to measure the size of the increments of a function in a more flexible way than what the natural control $\omega(s, t) = |t - s|$ allows.

Definition 3.1 ([3]). A (discrete) control is a triangular array of non-negative numbers ($\omega_{k,l} : k < l$) such that $\omega_{k,k} = 0$ and

$$\omega_{k,l} + \omega_{l,m} \leqslant \omega_{k,m}$$

for all k < l < m

Remark 3.2. Observe that for a control ω the maps $l \mapsto \omega_{k,l}$ and $k \mapsto \omega_{k,l}$ are non-decreasing and non-increasing, respectively. Indeed, if $0 \leq k < l < m \leq N$ then

$$\omega_{k,l} \leqslant \omega_{k,l} + \omega_{l,m} \leqslant \omega_{k,m}$$

and

$$\omega_{k,m} \ge \omega_{k,l} + \omega_{l,m} \ge \omega_{l,m}$$

Now we collect some results on how to produce new controls out of any given control.

Lemma 3.3. Let ω be a control and $\varphi \colon [0, \infty) \to [0, \infty)$ an increasing convex function such that $\varphi(0) = 0$. Then $\tilde{\omega}_{k,l} \coloneqq \varphi(\omega_{k,l})$ is also a control.

Proof. Since φ is convex and $\varphi(0) = 0$ we have that

$$\varphi(\lambda(x+y)) \leq \lambda \varphi(x+y)$$

for any $\lambda \in [0, 1]$. Choosing $\lambda = \frac{x}{x+y}$ we obtain

$$\varphi(x) \leq \frac{x}{x+y}\varphi(x+y)$$

Similarly, $\varphi(y) \leq \frac{y}{x+y}\varphi(x+y)$ so that

$$\varphi(x) + \varphi(y) \le \varphi(x+y),$$

i.e. φ is super-additive.

Therefore, if $0 \leq k < l < m \leq N$,

$$\begin{split} \tilde{\omega}_{k,l} + \tilde{\omega}_{l,m} &= \varphi(\omega_{k,l}) + \varphi(\omega_{l,m}) \\ &\leq \varphi(\omega_{k,l} + \omega_{l,m}) \\ &\leq \varphi(\omega_{l,m}) = \tilde{\omega}_{l,m} \end{split}$$

where the last inequality follows from the monotonicity of φ .

Remark 3.4. In particular, this implies that if ω is a control, then ω^{α} is also a control, for any $\alpha > 1$.

Lemma 3.5. Let $\omega, \tilde{\omega}$ be two controls. If $\alpha, \beta > 0$ are such that $\alpha + \beta \ge 1$, then $\hat{\omega}_{k,l} \coloneqq \omega_{k,l}^{\alpha} \tilde{\omega}_{k,l}^{\beta}$ is also a control.

Proof. Let $\theta \coloneqq \alpha + \beta$. By Lemma 3.3, it is enough to show that

$$z_{k,l} \coloneqq \omega_{k,l}^{\frac{\alpha}{\theta}} \tilde{\omega}_{k,l}^{\frac{\beta}{\theta}}$$

is a control, since then $\hat{\omega}_{k,l} = z_{k,l}^{\theta}$ will also be a control. Since $\frac{\alpha}{\theta} + \frac{\beta}{\theta} = 1$, Hölder's inequality implies that

$$z_{k,l} + z_{l,m} \leq (\omega_{k,l} + \omega_{l,m})^{\frac{\alpha}{\theta}} (\tilde{\omega}_{k,l} + \tilde{\omega}_{l,m})^{\frac{\beta}{\theta}}$$
$$\leq \omega_{k,m}^{\frac{\alpha}{\theta}} \tilde{\omega}_{k,m}^{\frac{\beta}{\theta}}$$

and the proof is finished.

3.2. *p*-variation. In the following we will deal with *time series*, which are finite sequences of vectors $\mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_N) \in (\mathbb{R}^d)^N$. We will use the convention of indexing time steps with lower indices and lowercase Latin letters, and spatial components with upper indices and lowercase Greek letters, so for example $\mathbf{w}_k^{\mu} \in \mathbb{R}$ refers to the μ -th component of the *k*-th entry in the time series \mathbf{w} . The main reason for making this distinction is that the ranges for both sets of variables is different: indeed, note that Greek letter indices always belong to the set $\{1, \dots, d\}$, while Latin letter indices belong to the set $\{0, \dots, N\}$.

We will also need to deal with general *triangular arrays*, which are collections of vectors of the form $(\Xi_{k,l}: 0 \leq k < l \leq N)$. For any time series we define a triangular array $(\mathbf{w}_{k,l})$ by setting $\mathbf{w}_{k,l} \coloneqq \mathbf{w}_l - \mathbf{w}_k$.

Definition 3.6. Given p > 0, we define the p-variation with respect to a fixed choice of norm $|\cdot|$ on \mathbb{R}^d , by

$$\|\mathbf{w}\|_{p;[k,l]} \coloneqq \left(\max_{s \in \mathcal{S}_{k,l}} \sum_{j=0}^{\#s} |\mathbf{w}_{s_{j+1}} - \mathbf{w}_{s_j}|^p\right)^{1/p}$$

where the maximum is taken over the set $S_{k,l}$ of all increasing subsequences

 $s = (s_0 = k, s_1, \dots, s_m, s_{m+1} = l)$

of $\{k, k+1, \ldots, l-1, l\}$ and we have set #s = m for such a sequence. For a triangular array Ξ one can also define its p-variation as

$$\|\Xi\|_{p;[k,l]} := \left(\sup_{s \in \mathcal{S}_{k,l}} \sum_{j=0}^{\#s} |\Xi_{s_j,s_{j+1}}|^p\right)^{1/p}.$$

We observe that in the case where $\Xi_{k,l} = \mathbf{w}_l - \mathbf{w}_k$ both definitions coincide. Since the trivial sequence $(k, l) \in S_{k,l}$ we obtain immediately the bound

$$|\Xi_{k,l}| \le ||\Xi||_{p;[k,l]} \tag{5}$$

for any p > 0. In the particular case where $\Xi_{k,l} = \mathbf{w}_l - \mathbf{w}_k$ we also obtain

$$\|\mathbf{w}\|_{\infty}\coloneqq \sup_{k=0,...,N} |\mathbf{w}_k|\leqslant |\mathbf{w}_0|+\|\mathbf{w}\|_{p;[0,N]}.$$

Proposition 3.7. Let Ξ be a triangular array and $p \ge 0$. Then $\omega_{k,l} := \|\Xi\|_{p;[k,l]}^p$ is a control.

Proof. Indeed, if $s' \in S_{k,l}$ and $s'' \in S_{l,m}$ then $s = (s', s'') \in S_{k,m}$ and so

$$\sum_{j=0}^{\#s'} |\Xi_{s_j,s_{j+1}}|^p + \sum_{j'=0}^{\#s''} |\Xi_{s'_{j'},s'_{j'+1}}|^p \le \|\Xi\|_{p;k,m}^p$$

and super-additivity follows from taking the supremum over $S_{k,l}$ and $S_{l,m}$.

Remark 3.8. Since the set $S_{k,l}$ is finite, the *p*-variation norm of Ξ is finite for any p > 0 and triangular array Ξ . This should be contrasted with the usual setting for rough paths, where one deals with paths in continuous time; in that setting, the *p*-variation norm can become infinite and this introduces a number of analytical problems which are not present in the present context.

Remark 3.9. The *p*-variation defines a quasi-norm for $0 (i.e. the triangle inequality fails), and a semi-norm for <math>p \ge 1$ on time series, since all constant sequences have vanishing *p*-variation. For $p \ge 1$, it becomes a norm on triangular arrays.

Lemma 3.10. Let $0 \le p < q < \infty$. Then $\|\Xi\|_{q;[k,l]} \le \|\Xi\|_{p;[k,l]}$

Proof. Observe that, since $\frac{q}{p} > 1$, the inequality

$$\sum_{j=0}^{\#s} |\Xi_{s_j,s_{j+1}}|^q \leqslant \left(\sum_{j=0}^{\#s} |\Xi_{s_j,s_{j+1}}|^p\right)^{q/p}$$

holds for any $s \in \mathcal{S}_{k,l}$.

Given a triangular array Ξ , we define another collection $(\delta \Xi_{k,l,m} : 0 \leq k < l < m)$ by

$$\delta \Xi_{k,l,m} \coloneqq \Xi_{k,m} - \Xi_{k,l} - \Xi_{l,m}$$

In the special case where $\Xi_{k,l} = \mathbf{w}_l - \mathbf{w}_k$ we see that $\delta \Xi_{k,l,m} = 0$. The operator δ satisfies the following product rule: if w is a time series and Ξ is a triangular array, consider the triangular array $\mathbf{Z}_{k,l} \coloneqq \mathbf{w}_k \Xi_{k,l}$. Then

$$\delta \mathbf{Z}_{k,l,m} = \mathbf{w}_k \delta \Xi_{k,l,m} - \mathbf{w}_{k,l} \Xi_{l,m}.$$
(6)

Finally we collect here some standard results for further reference.

Lemma 3.11. Let Ξ be a triangular array and $p \ge 0$. Suppose there is a control w such that

$$|\Xi_{k,l}| \leqslant C\omega_{k,l}^{1/p}$$

for all $0 \leq k < l \leq N$ and some constant C > 0. Then,

$$|\Xi||_{p;[k,l]} \leqslant C\omega_{k,l}^{1/p}$$

for all $0 \leq k < l \leq N$.

Proof. By hypothesis the inequality

$$|\Xi_{k,l}|^p \leqslant C^p \omega_k$$

holds for all $0 \leq k < l \leq N$. By superadditivity of w, if $s \in S_{k,l}$ then also

$$\sum_{j=0}^{\#s} |\Xi_{s_j,s_{j+1}}|^p \leqslant C^p \omega_{k,l}$$

The desired bound follows upon taking the maximum over $s \in S_{k,l}$.

Lemma 3.12. Assume that $p \ge 1$ and

$$|\mathbf{x}_{k,l}| \leq C \omega_{k,l}^{1/2}$$

 $|\mathbf{x}_{k,l}| \leq C \omega_{k,l}^{1/p}$ for all $0 \leq k < l$ such that $\omega_{k,l} \leq 1$ or if l = k + 1. Then

$$\|\mathbf{x}\|_{p;[k,l]} \leq 2C(\omega_{k,l}^{1/p} \vee \omega_{k,l})$$

for all $0 \leq k < l$.

Proof. We show that the inequality $|\mathbf{x}_{k,l}| \leq C \tilde{\omega}_{k,l}^{1/p}$ holds for all $0 \leq k < l \leq N$, where $\tilde{\omega}_{k,l} \coloneqq \omega_{k,l}^p \lor \omega_{k,l}$ which is a control by Lemma 3.3. The conclusion then follows from Lemma 3.11.

If k < l are such that $\omega_{k,l} \leq 1$ then there is nothing to show, since in this case $\tilde{\omega}_{k,l}^{1/p} = \omega_{k,l}^{1/p}$. Suppose now that k < l are such that $\omega_{k,l} > 1$. Inductively define $j_0 = k < j_1 < \cdots < j_M < j_{M+1} = l$ by setting

$$j_{u+1} \coloneqq \max\{j > j_u : \omega_{j_u,j} \le 1\} \land (j_u+1)$$

By super-additivity of ω we immediately get that $M+1 \leq 2\omega_{k,l}$. Also, by definition $|\mathbf{x}_{j_u,j_{u+1}}| \leq C\omega_{j_u,j_{u+1}}^{1/p}$ for u = 0, 1, ..., r. Thus, by the triangle inequality we obtain that

$$\begin{aligned} |\mathbf{x}_{k,l}| &\leq C \sum_{u=0}^{M} \omega_{j_u,j_{u+1}}^{1/p} \\ &\leq C(M+1) \\ &\leq 2C\omega_{k,l} \\ &= 2C\tilde{\omega}_{k,l}^{1/p}. \end{aligned}$$

Finally, we show the following result, known as the rough Grownall Lemma. It is a slight variation of [4, Lemma 2.12], adapted to our particular setting.

Theorem 3.13. Let \mathbf{z} be a time series and suppose there exist controls $\omega, \tilde{\omega}$ such that

$$|\mathbf{z}_{k,l}| \leq C \left(\max_{j=0,\dots,l} |\mathbf{z}_j|\right) \omega_{k,l}^{1/\kappa} + \tilde{\omega}_{k,l}^{1/\rho}$$

whenever $\omega_{k,l} \leq L$ or l = k + 1, for some constants C > 0 and $\kappa, \rho \geq 1$. Then,

$$\max_{j=0,\dots,N} |\mathbf{z}_j| \leq 2 \exp\left(\frac{\omega_{0,N}}{\alpha L}\right) \left\{ |\mathbf{z}_0| + \max_{j=0,\dots,N} \left(\tilde{\omega}_{0,j}^{1/\rho} \left(1 + 2\frac{\omega_{0,j}}{\alpha L}\right)^{1-1/\rho} \exp\left(-\frac{\omega_{0,j}}{\alpha L}\right) \right) \right\}$$

where $\alpha \coloneqq \min(1, \frac{1}{L(2Ce^2)^{\kappa}})$.

Proof. Define the sequences

$$G_k \coloneqq \max_{j=0,\dots,k} |\mathbf{z}_j|, \ H_k \coloneqq G_k \exp\left(-\frac{\omega_{0,k}}{\alpha L}\right), \ H_k^* \coloneqq \max_{j=0,\dots,k} H_j$$

Subdivide the interval $\{0, \ldots, N\}$ into $j_0 = 0 < j_1 < \cdots < j_K < j_{K+1} = N$ where j_u is the largest integer in $\{j_{u-1} + 1, \ldots, N\}$ such that $\omega_{j_{u-1}, j_u} \leq \alpha L$ or $j_u = j_{u-1} + 1$ if such an integer does not exist. We note that by subadditivity we necessarily have, for each $u = 1, \ldots, K$, that

$$u \leqslant 1 + 2\frac{\omega_{0,j_u}}{\alpha L}.$$

Indeed, by definition we have that for each r, $\omega_{j_{r-1},j_r+1} > \alpha L$, hence if $j \in \{j_{u-1} + 1, \ldots, j_u\}$ we have

$$0 \leq \omega_{j_{u-1}+1,j} \leq 2\omega_{0,j} - \sum_{r=0}^{u-2} \omega_{j_r,j_{r+1}+1} \leq 2\omega_{0,j} - \alpha L(u-1),$$

that is,

$$u \leqslant 1 + 2\frac{\omega_{0,j}}{\alpha L}.$$

Now, for $j_{u-1} < j \leq j_u$ we have

$$\begin{aligned} |\mathbf{z}_{0,j}| &\leqslant \sum_{r=0}^{u-2} |\mathbf{z}_{j_r,j_{r+1}}| + |\mathbf{z}_{j_{u-1},j}| \\ &\leqslant \sum_{r=0}^{u-2} \left(CG_{t_{r+1}} \omega_{j_r,j_{r+1}}^{1/\kappa} + \tilde{\omega}_{j_r,j_{r+1}}^{1/\rho} \right) + CG_j \omega_{j_{u-1},j}^{1/\kappa} + \tilde{\omega}_{j_{u-1},j}^{1/\rho} \\ &\leqslant C(\alpha L)^{1/\kappa} \sum_{r=0}^{u-1} G_{j_{r+1}} + u^{1-1/\rho} \tilde{\omega}_{0,j}^{1/\rho}. \end{aligned}$$

We bound the first term on the right-hand side by

$$\sum_{r=0}^{u-1} G_{j_{r+1}} = \sum_{r=0}^{u-1} H_{j_{r+1}} \exp\left(\frac{\omega_{0,j_{r+1}}}{\alpha L}\right)$$
$$\leqslant H_N^* \sum_{r=1}^u e^r$$
$$\leqslant H_N^* e^{u+1}.$$

Combining this with the previous bound we obtain

$$G_j \leq |\mathbf{z}_0| + C(\alpha L)^{1/\kappa} e^{u+1} H_N^* + u^{1-1/\rho} \tilde{\omega}_{0,j}^{1/\rho}$$

and so

$$H_j \leqslant \left(|\mathbf{z}_0| + \tilde{\omega}_{0,j}^{1/\rho} \left(1 + 2\frac{\omega_{0,j}}{\alpha L} \right)^{1-1/\rho} \right) \exp\left(-\frac{\omega_{0,j}}{\alpha L}\right) + C(\alpha L)^{1/\kappa} \mathrm{e}^2 H_N^*.$$

This implies that

$$H_N^* \leq |\mathbf{z}_0| + \max_{j=0,\dots,N} \left\{ \tilde{\omega}_{0,j}^{1/\rho} \left(1 + 2\frac{\omega_{0,j}}{\alpha L} \right)^{1-1/\rho} \exp\left(-\frac{\omega_{0,j}}{\alpha L}\right) \right\} + C(\alpha L)^{1/\kappa} \mathrm{e}^2 H_N^*$$

and so, by our choice of α we obtain

$$\max_{j=0,\dots,N} |\mathbf{z}_j| = G_N \leqslant H_N^* \exp\left(\frac{\omega_{0,N}}{\alpha L}\right)$$
$$\leqslant 2 \exp\left(\frac{\omega_{0,N}}{\alpha L}\right) \left\{ |\mathbf{z}_0| + \max_{j=0,\dots,N} \left(\tilde{\omega}_{0,j}^{1/\rho} \left(1 + 2\frac{\omega_{0,j}}{\alpha L}\right)^{1-1/\rho} \exp\left(-\frac{\omega_{0,j}}{\alpha L}\right) \right) \right\}$$
e are done.

and we

3.3. The Sewing Lemma. At the core of the theory of rough integration lies the Sewing Lemma [7,9]. Therefore, it is tightly connected with the solution theory of differential equations driven by rough signals. Since our main aim is to perform a precise analysis of the behavior of discrete equations driven by irregular time-series, it is no doubt that its discrete analogue will play a prominent rôle here as well.

We begin by showing some preliminary results.

Lemma 3.14. Suppose $s \in S_{k,l}$ of length #s = m. For any given control ω , there exists an integer j^* with $1 \leq j^* \leq m$ such that

$$\omega_{s_{j^{*}-1},s_{j^{*}+1}} \leqslant \frac{2}{m} \omega_{k,l}.$$

Proof. Suppose, on the contrary, that for any $1 \leq j \leq m$ we have that

$$\omega_{s_{j-1},s_{j+1}} > \frac{2}{m}\omega_{k,l}.$$

Then this would imply that

$$2\omega_{k,l} < \sum_{j=1}^m \omega_{s_{j-1},s_{j+1}} \leqslant 2\omega_{k,l}$$

by super-additivity, which is a contradiction.

Proposition 3.15 (Discrete sewing). Let $(\Xi_{k,l}: 0 \leq k \leq l \leq N)$ be a triangular array, and suppose that there exist two controls ω and $\tilde{\omega}$ such that

$$|\delta \Xi_{k,l,m}| \leqslant \omega_{k,l}^{\alpha} \tilde{\omega}_{l,m}^{\beta}$$

for some $\alpha, \beta > 0$ with $\alpha + \beta > 1$, and for all $0 \le k < l \le N$. Then

$$\left|\sum_{j=k}^{l-1} \Xi_{j,j+1} - \Xi_{k,l}\right| \leq 2^{(\alpha+\beta)} \zeta_N(\alpha+\beta) \omega_{k,l}^{\alpha} \tilde{\omega}_{k,l}^{\beta}$$

for all $0 \leq k < l \leq N$, where ζ_N denotes the partial sum of Riemann's zeta function

$$\zeta_N(s) \coloneqq \sum_{n=1}^N n^{-s}.$$

Proof. By Remark 3.2 we deduce that $|\delta \Xi_{k,l,m}| \leq \omega_{k,m}^{\alpha} \tilde{\omega}_{k,m}^{\beta}$, and Lemma 3.5 implies that $\hat{\omega} := \omega^{\frac{\alpha}{\theta}} \tilde{\omega}^{\frac{\beta}{\theta}}$ is a control.

Now we apply a Young-style argument to estimate the above difference. First we observe that if l - k = 1then the bound is trivial since the left-hand side vanishes. Therefore we assume that $l - k \ge 2$. By Lemma 3.14 we can find an index $k < j^* < l$ such that

$$\hat{\omega}_{j*-1,j*+1} \leq \frac{2}{(l-k-1)}\hat{\omega}_{k,l}.$$

Hence, if we denote by $s := (k, k + 1, ..., j^* - 1, j^* + 1, ..., l)$ we have

$$\left|\sum_{j=k}^{l-1} \Xi_{j,j+1} - \sum_{s} \Xi_{s_j,s_{j+1}}\right| = |\delta \Xi_{j*-1,j*,j*+1}| \le \left(\frac{2}{l-k-1}\right)^{\theta} \hat{\omega}_{k,l}^{\theta}.$$

Then we can apply Lemma 3.14 again to the sequence s to obtain a "coarser" sequence s', containing one less point, and such that

$$\left|\sum_{s} \Xi_{s_j,s_{j+1}} - \sum_{s'} \Xi_{s'_j,s'_{j+1}}\right| \leqslant \left(\frac{2}{l-k-2}\right)^{\theta} \hat{\omega}_{k,l}^{\theta}.$$

Continuing in this way we obtain a sequence of coarsenings of the full sequence until we get to $s^* = (k, l)$, and by using the triangular inequality we then deduce the estimate

$$\left|\sum_{j=k}^{l-1} \Xi_{j,j+1} - \Xi_{k,l}\right| \leqslant 2^{\theta} \sum_{r=1}^{l-k-1} \frac{1}{r^{\theta}} \hat{\omega}_{k,l}^{\theta}$$

from where the conclusion follows.

We will also need the following generalization of the Sewing Lemma, whose proof is straightforward.

Proposition 3.16 (Generalized discrete sewing). Suppose that Ξ is a triangular array as before. Suppose that there are controls ω_r and $\tilde{\omega}_r$, and exponents α_r , $\beta_r > 0$ such that $\alpha_r + \beta_r > 1$ for all r = 1, ..., n. If

$$|\delta \Xi_{k,l,m}| \leqslant \sum_{r=1}^{n} \omega_{r;k,l}^{\alpha_r} \tilde{\omega}_{r;l,m}^{\beta_r}$$

then

$$\left|\sum_{j=k}^{l-1} \Xi_{j,j+1} - \Xi_{k,l}\right| \leq 2^{\hat{\theta}} \zeta_N(\hat{\theta}) \sum_{r=1}^n \omega_{r;k,l}^{\alpha_r} \tilde{\omega}_{r;k,l}^{\beta_r}$$

where $\hat{\theta} \coloneqq \min_{r=1,\dots,n} \{ \alpha_r + \beta_r \}.$

4. LIFTING TIME SERIES

Inspired by the theory of rough paths, we introduce an augmentation or lift of a given time series \mathbf{w} . Recall that the convention of using lowercase Latin letters as sub-indices to index time, and lowercase Greek letters to index spatial components is in place.

Definition 4.1. Given a time series \mathbf{w} , we call its lift the triangular array of d-by-d matrices \mathbb{W} defined by

$$\mathbb{W}_{k,l}^{\mu\nu} \coloneqq \sum_{j=k}^{l-1} (\mathbf{w}_{j}^{\mu} - \mathbf{w}_{k}^{\mu}) (\mathbf{w}_{j+1}^{\nu} - \mathbf{w}_{j+1}^{\nu}).$$

We write $\mathbf{W} \coloneqq (\mathbf{w}, \mathbb{W})$.

The main purpose of this lift is to provide "second order information" about the time series. It is, first of all, a discrete analogue of an iterated integral as in the rough path setting, but it may be interpreted as a generalized quadratic covariation of the components of \mathbf{w} . The lift \mathbb{W} is part of a much larger structure, known as the *iterated-sums signature* of \mathbf{w} [5].

We now record a basic property of \mathbb{W} for later use:

Theorem 4.2. The time series lift \mathbb{W} of a time series **w** satisfies Chen's identity: for all indices $0 \le k < l < m \le N$ and $\mu, \nu \in \{1, \ldots, d\}$ we have

$$\delta \mathbb{W}_{k,l,m}^{\mu\nu} = \mathbf{w}_{k,l}^{\mu} \mathbf{w}_{l,m}^{\nu}.$$

Given $p \in [2,3)$, a pair $\mathbf{W} = (\mathbf{w}, \mathbb{W})$ consisting of a time series and its lift, and indices $0 \le k \le l \le N$, we define a semi-norm

$$\|\mathbf{W}\|_{p;[k,l]} \coloneqq \|\mathbf{w}\|_{p;[k,l]} + \|\mathbb{W}\|_{p/2;[k,l]}^{1/2}, \tag{7}$$

and a pseudometric

$$\rho_p(\mathbf{W}, \tilde{\mathbf{W}}) \coloneqq \|\mathbf{w} - \tilde{\mathbf{w}}\|_{p;[0,N]} + \|\mathbb{W} - \tilde{\mathbb{W}}\|_{p/2;[0,N]}.$$
(8)

We note that both can be turned into a proper norm (resp. metric) if we add the absolute value of the initial value.

5. Controlled difference equations

In this section we consider equations of the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sum_{\mu=1}^d f_\mu(\mathbf{x}_k)(\mathbf{w}_{k+1}^\mu - \mathbf{w}_k^\mu), \quad \mathbf{x}_0 = \xi \in \mathbb{R}^m$$
(9)

for some vector fields f_1, \ldots, f_d on \mathbb{R}^m , and where k ranges between 0 and some fixed time horizon $n \in \mathbb{N}$. Our main aim is to obtain some control over the size of the end-point value \mathbf{x}_n of the solution.

In view of the previous sections, and in particular of the bound in eq. (5), we will try to obtain good estimates for the *p*-variation norm $\|\mathbf{x}\|_{p;[0,n]}$. Of course, such estimates will require some assumptions on the vector fields. It turns out that we will not only be able to control the "large scale" behavior of \mathbf{x} , but we will also obtain a cascade of estimates of some remainder terms, reminiscent of a Taylor expansion.

The techniques needed to obtain those bounds will depend crucially on $p \in [1, \infty)$. At first, we distinguish two basic regimes: $p \in [1, 2)$ and $p \in [2, \infty)$. By analogy with the rough paths literature, we call the former the *young regime*, and the latter the *rough regime* – even though there is strictly no notion of roughness in our setting. The rough regime can be further subdivided into the cases where $p \in [n, n + 1)$, which we call the *level n rough regime*. The terminology will make itself clear later down the road.

A central tool for constructing solutions to ODEs driven by rough paths are the so-called *controlled* paths, introduced by Gubinelli [9]. See also [12]. In a nutshell, the notion of "controlledness" contains all the necessary analytical estimates needed for the definition of a rough integral which then is used to give sense to solutions of Rough Differential Equations. In the present setting no such definition is needed since there are no divergences appearing from considering eq. (9). Nonetheless, we can still derive similar bounds. Note however that in our case the estimates are proven rather than assumed.

Given a vector field $f : \mathbb{R}^m \to \mathbb{R}^m$ of class $\mathcal{C}^n_{\mathbf{b}}$, i.e. it and all its derivatives up to order n are bounded, we define

$$\|f\|_{\mathcal{C}^n_{\mathbf{b}}} \coloneqq \max_{k=1,\dots,n} \|D^k f\|_{\infty}.$$

If $f = (f_1, \ldots, f_d)$ is a collection of vector fields on \mathbb{R}^n of class $\mathcal{C}^n_{\mathrm{b}}$ (or, equivalently, a map in $\mathcal{C}^n_{\mathrm{b}}(\mathbb{R}^n, \mathbb{R}^{dn})$), we define

$$\|f\|_{\mathcal{C}^n_{\mathrm{b}}} \coloneqq \max_{\mu=1,\dots,d} \|f_{\mu}\|_{\mathcal{C}^n_{\mathrm{b}}}.$$

Lemma 5.1. Suppose $f \in C_b^2$ and let $\mathbf{x}, \tilde{\mathbf{x}}$ be two time series. Then

$$\|f(\mathbf{x}) - f(\tilde{\mathbf{x}})\|_{p;[k,l]} \leq 2^{(p-1)/p} \|f\|_{\mathcal{C}^{2}_{\mathrm{b}}} \Big(\|\mathbf{x} - \tilde{\mathbf{x}}\|_{p;[k,l]}^{p} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,l]}^{p} \Big)^{1/p}$$

Furthermore, if $f \in C^3_b$ and we let

$$T_{k,l} \coloneqq f(\mathbf{x}_l) - f(\mathbf{x}_k) - Df(\mathbf{x}_k)\delta\mathbf{x}_{k,l}$$

and similarly for $\tilde{\mathbf{x}}$, then

$$\|T - \tilde{T}\|_{p/2;[k,l]} \leq 2^{(p-2)/p} \|f\|_{\mathcal{C}^{3}_{b}} \Big[\|\mathbf{x} - \tilde{\mathbf{x}}\|_{p;[k,l]}^{p} (\|\mathbf{x}\|_{p;[k,l]}^{p} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p})^{1/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,l]}^{p/2} \Big]^{2/p} \Big]^{2/p} \Big]$$

Proof. Suppose first that $f \in \mathcal{C}^1_{\mathrm{b}}$. By the Fundamental Theorem of Calculus we may write

$$f(\mathbf{x}_l) - f(\mathbf{x}_k) = \int_0^1 Df(\mathbf{x}_k + \tau \delta \mathbf{x}_{k,l}) \delta \mathbf{x}_{k,l} \, \mathrm{d}\tau.$$

Therefore, by adding and subtracting cross terms, we see that

$$\begin{aligned} |f(\mathbf{x}_{l}) - f(\mathbf{x}_{k}) - (f(\tilde{\mathbf{x}}_{l}) - f(\tilde{\mathbf{x}}_{k}))| &\leq \int_{0}^{1} |Df(\mathbf{x}_{k} + \tau \delta \mathbf{x}_{k,l}) (\delta \mathbf{x}_{k,l} - \delta \tilde{\mathbf{x}}_{k,l})| \, \mathrm{d}\tau \\ &+ \int_{0}^{1} |(Df(\mathbf{x}_{k} + \tau \delta \mathbf{x}_{k,l}) - Df(\tilde{\mathbf{x}}_{k} + \tau \delta \tilde{\mathbf{x}}_{k,l})) \delta \tilde{\mathbf{x}}_{k,l}| \, \mathrm{d}\tau \end{aligned}$$

The right-hand side is bounded by

$$\|f\|_{\mathcal{C}^{2}_{\mathrm{b}}}(\|\mathbf{x}-\tilde{\mathbf{x}}\|_{p;[k,l]}+\|\tilde{\mathbf{x}}\|_{p;[k,l]}\|\mathbf{x}-\tilde{\mathbf{x}}\|_{\infty;[0,l]}) \leq 2^{1-1/p}\|f\|_{\mathcal{C}^{2}_{\mathrm{b}}}\Big(\|\mathbf{x}-\tilde{\mathbf{x}}\|_{p;[k,l]}^{p}+\|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p}\|\mathbf{x}-\tilde{\mathbf{x}}\|_{\infty;[0,l]}^{p}\Big)^{1/p},$$

and the result follows from Lemma 3.11.

Now, assume that $f \in C_{\rm b}^2$. By iterated application of the Fundamental Theorem of Calculus we may now write

$$T_{k,l} = \int_0^1 \int_0^\tau D^2 f(\mathbf{x}_k + \nu \delta \mathbf{x}_{k,l}) (\delta \mathbf{x}_{k,l}, \delta \mathbf{x}_{k,l}) \, \mathrm{d}\nu \mathrm{d}\tau$$

Inserting appropriate cross terms we obtain

$$\begin{aligned} |T_{k,l} - \tilde{T}_{k,l}| &\leq \int_0^1 \int_0^\tau \left| D^2 f(\mathbf{x}_k + \nu \delta \mathbf{x}_{k,l}) (\delta \mathbf{x}_{k,l}, \delta \mathbf{x}_{k,l}) - D^2 f(\mathbf{x}_k + \nu \delta \mathbf{x}_{k,l}) (\delta \tilde{\mathbf{x}}_{k,l}, \delta \tilde{\mathbf{x}}_{k,l}) \right| \mathrm{d}\nu \mathrm{d}\tau \\ &+ \int_0^1 \int_0^\tau \left| \left[D^2 f(\mathbf{x}_k + \nu \delta \mathbf{x}_{k,l}) - D^2 f(\tilde{\mathbf{x}}_k + \nu \delta \tilde{\mathbf{x}}_{k,l}) \right] (\delta \tilde{\mathbf{x}}_{k,l}, \delta \tilde{\mathbf{x}}_{k,l}) \right| \mathrm{d}\nu \mathrm{d}\tau. \end{aligned}$$

We now note that by symmetry of $D^2 f(\mathbf{x})$, it holds that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we have the identity

$$D^2 f(\mathbf{x})(\mathbf{a}, \mathbf{a}) - D^2 f(\mathbf{x})(\mathbf{b}, \mathbf{b}) = D^2 f(\mathbf{x})(\mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}).$$

Hence, the first term may be bounded by

$$\|f\|_{\mathcal{C}^{3}_{\mathrm{b}}} 2^{1-1/p} \Big\{ \|\mathbf{x} - \tilde{\mathbf{x}}\|_{p;[k,l]}^{p/2} (\|\mathbf{x}\|_{p;[k,l]}^{p} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p})^{1/2} \Big\}^{2/p}$$

The second term can be bounded, as before, by

$$\|f\|_{\mathcal{C}^3_{\mathrm{b}}} \Big(\|\tilde{\mathbf{x}}\|_{p;[k,l]}^p\|\mathbf{x}-\tilde{\mathbf{x}}\|_{\infty;[0,l]}^{p/2}\Big)^{2/p}$$

Putting both terms together and proceeding as before we obtain the bound

$$|T - \tilde{T}|_{p/2;[k,l]} \leq 2^{1-2/p} \|f\|_{\mathcal{C}^{3}_{\mathrm{b}}} \Big[\|\mathbf{x} - \tilde{\mathbf{x}}\|_{p;[k,l]}^{p/2} (\|\mathbf{x}\|_{p;[k,l]}^{p} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p})^{1/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,l]}^{p/2} \Big]^{2/p}. \quad \Box$$

5.1. The Young regime. In this regime, we can easily obtain good bounds with minimal assumptions on the f_i . These bounds have already been shown by Davie [3], but it will be an enlightening exercise to go through the proof in full details, since it will lay the foundations for our approach in the rough regime. Also, our methods are slightly different and already in this case they highlight the importance of the rôle played by the Sewing Lemma (Propositions 3.15 and 3.16) and the rough Grönwall lemma (Theorem 3.13).

Before beginning we define the remainder

$$R_{k,l} \coloneqq \mathbf{x}_{k,l} - \sum_{\mu=1}^{d} f_{\mu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu}$$
(10)

so that

$$\mathbf{x}_{k,l} = \sum_{\mu=1}^{a} f_{\mu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu} + R_{k,l}$$

Theorem 5.2. Let $1 \leq p < 2$, and suppose that $f = (f_1, \ldots, f_d)$ is a collection of vector fields in \mathbb{R}^n , of class $\mathcal{C}^1_{\mathbf{b}}$. The bound

$$\|\mathbf{x}\|_{p;[k,l]} \leq 2 \left(2^{p} C_{p,N}^{p-1} \|f\|_{\mathcal{C}_{\mathbf{b}}^{1}}^{p} \|\mathbf{w}\|_{p;[k,l]}^{p} \vee 2 \|f\|_{\mathcal{C}_{\mathbf{b}}^{1}} \|\mathbf{w}\|_{p;[k,l]} \right)$$

$$C_{p,N} \coloneqq 2^{2/p} \zeta_{N}(2/p).$$
(11)

holds, with

Proof. Consider the triangular array $\Xi_{k,l} \coloneqq \sum_{\mu} f_{\mu}(\mathbf{x}_k) \mathbf{w}_{k,l}^{\mu}$. By eq. (6) we immediately see that

$$\delta \Xi_{k,l,m} = -\sum_{\mu=1}^d (f_\mu(\mathbf{x}_l) - f_\mu(\mathbf{x}_k)) \mathbf{w}_{l,m}^\mu$$

so that the usual Lipschitz bound implies

$$|\delta \Xi_{k,l,m}| \leq ||f||_{\mathcal{C}^1_{\mathrm{b}}} ||\mathbf{x}||_{p;[k,l]} ||\mathbf{w}||_{p;[k,l]}$$

and the hypothesis of Proposition 3.15 is satisfied since 2/p > 1. Thus, we obtain

$$\left|\sum_{j=k}^{l-1} \Xi_{j,j+1} - \Xi_{k,l}\right| \leq C_{p,N} \|f\|_{\mathcal{C}_{\mathbf{b}}^{1}} \|\mathbf{x}\|_{p;[k,l]} \|\mathbf{w}\|_{p;[k,l]}$$

with $C_{p,N} \coloneqq 2^{2/p} \zeta_N(2/p)$. Now, we observe that by eq. (9),

$$\sum_{j=k}^{l-1} \Xi_{j,j+1} = \mathbf{x}_{k,l}$$

thus obtaining

$$|R_{k,l}| \leq C_{p,N} \|f\|_{\mathcal{C}^1_{\mathbf{h}}} \|\mathbf{x}\|_{p;[k,l]} \|\mathbf{w}\|_{p;[k,l]}.$$
(12)

By Lemma 3.11, the same bound holds if we replace $|R_{k,l}|$ on the left-hand side by $||R||_{p/2;[k,l]}$.

Using the relation between the remainder R and the increments of ${\bf x}$ we get

$$\|\mathbf{x}_{k,l}\| \leq C_{p,N} \|f\|_{\mathcal{C}_{\mathbf{b}}^{1}} \|\mathbf{x}\|_{p;[k,l]} \|\mathbf{w}\|_{p;[k,l]} + \|f\|_{\mathcal{C}_{\mathbf{b}}^{1}} \|\mathbf{w}\|_{p;[k,l]}$$

for all $0 \leq l < k \leq N$. We deduce that

$$\|\mathbf{x}\|_{p;[k,l]}^{p} \leq 2^{p-1} C_{p}^{p} \|f\|_{\mathcal{C}_{b}^{1}}^{p} \|\mathbf{x}\|_{p;[k,l]}^{p} \|\mathbf{w}\|_{p;[k,l]}^{p} + 2^{p-1} \|f\|_{\mathcal{C}_{b}^{1}}^{p} \|\mathbf{w}\|_{p;[k,l]}^{p}$$

If we now consider a pair k < l such that $\bar{\omega}_{k,l}^{1/p} \coloneqq 2C_{p,N} \|f\|_{\mathcal{C}^1_{\mathbf{b}}} \|\mathbf{w}\|_{p;[k,l]} \leq 1$, we obtain

$$\|\mathbf{x}\|_{p;[k,l]}^p \leq 2^p \|f\|_{\mathcal{C}^1}^p \|\mathbf{w}\|_{p;[k,l]}^p = C_{p,N}^{-p} \bar{\omega}_{k,l}$$

for all such (k, l). In particular

$$|\mathbf{x}_{k,l}| \leqslant C_{p,N}^{-1} \bar{\omega}_{k,l}^{1/p}$$

By eq. (9) the same inequality also holds when l = k + 1. From Lemma 3.12 we then get

$$\begin{aligned} \mathbf{x} \|_{p;[k,l]} &\leq 3C_{p,N}^{-1} \left(\bar{\omega}_{k,l} \vee \bar{\omega}_{k,l}^{1/p} \right) \\ &= 3C_p^{-1} \left(2^p \|f\|_{\mathcal{C}^1_{\mathrm{b}}}^p C_{p,N}^p \|\mathbf{w}\|_{p;[k,l]}^p \vee 2\|f\|_{\mathcal{C}^1_{\mathrm{b}}} C_{p,N} \|\mathbf{w}\|_{p;[k,l]} \right) \end{aligned}$$

from where the result follows.

Remark 5.3. The hypothesis on the vector fields f, namely $f \in C_{\rm b}^1$, can be relaxed to $f \in \operatorname{Lip}^{\gamma-1}$ for some $\gamma \in (p, 2]$, meaning that f need not be differentiable but we merely need the existence of positive constant L such that

$$|f(\mathbf{x}) - f(\tilde{\mathbf{x}})| \leq L|\mathbf{x} - \tilde{\mathbf{x}}|^{\gamma - 1}$$

for all $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbb{R}^n$.

Finally we show that

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Theorem 5.4. Let $1 \leq p < 2$ and suppose $\mathbf{x}, \tilde{\mathbf{x}}$ are two solutions to eq. (9) with initial conditions $\xi, \tilde{\xi}$ and driven by $\mathbf{w}, \tilde{\mathbf{w}}$ respectively. If furthermore $f_1, \ldots, f_d \in C_b^2$ are such that $\max_{\mu=1,\ldots,d} \|f_{\mu}\|_{C_b^2} \leq L$, then

$$\sup_{k=0,...,N} |\mathbf{x}_k - \tilde{\mathbf{x}}_k| \leq 2c_{p,N}^{1/p} e^{c_{p,N}L^p(\|\mathbf{w}\|_{p;[0,N]}^p + \|\tilde{\mathbf{w}}\|_{p;[0,N]}^p)} (|\xi - \tilde{\xi}| + L\|\mathbf{w} - \tilde{\mathbf{w}}\|_{p;[0,N]})$$

holds, where

$$c_{p,N} \coloneqq (4e^2)^p (4^{p-1}C_{p,N}^p + 1)$$

and $C_{p,N}$ is as in Theorem 5.2.

Proof. In order to make the notation more compact we also define the controls

$$\varepsilon_{k,l} \coloneqq \|\mathbf{w} - \tilde{\mathbf{w}}\|_{p;[k,l]}^p, \quad \omega_{k,l} \coloneqq \|\mathbf{w}\|_{p;[k,l]}^p + \|\tilde{\mathbf{w}}\|_{p;[k,l]}^p$$

Now, we define $\mathbf{z}_k \coloneqq \mathbf{x}_k - \tilde{\mathbf{x}}_k$ and notice that

$$|\mathbf{z}_{k,l}| \leq |R_{k,l} - \tilde{R}_{k,l}| + \sum_{\mu=1}^{d} |f_{\mu}(\mathbf{x}_{k})\mathbf{w}_{k,l} - f_{\mu}(\tilde{\mathbf{x}}_{k})\tilde{\mathbf{w}}_{k,l}|.$$

For the second term we have the bound

$$\sum_{\mu=1}^{a} |f_{\mu}(\mathbf{x}_{k})\mathbf{w}_{k,l}^{\mu} - f_{\mu}(\tilde{\mathbf{x}}_{k})\tilde{\mathbf{w}}_{k,l}^{\mu}| \leq ||f||_{\infty}\varepsilon_{k,l}^{1/p} + ||Df||_{\infty}|\mathbf{z}_{k}|||\tilde{\mathbf{w}}||_{p;[k,l]}$$
$$\leq L\Big(\varepsilon_{k,l}^{1/p} + |\mathbf{z}_{k}|||\tilde{\mathbf{w}}||_{p;[k,l]}\Big).$$

To bound the first term, we use the Sewing Lemma with the germ $\Xi_{k,l} \coloneqq \sum_{\mu} f_{\mu}(\mathbf{x}_k) \mathbf{w}_{k,l}^{\mu} - \sum_{\mu} f_{\mu}(\tilde{\mathbf{x}}_k) \tilde{\mathbf{w}}_{k,l}^{\mu}$. First we compute

$$\delta \Xi_{k,l,m} = -\sum_{\mu=1}^{d} (f_{\mu}(\mathbf{x}_{l}) - f_{\mu}(\mathbf{x}_{k})) \mathbf{w}_{l,m}^{\mu} - \sum_{\mu=1}^{d} (f_{\mu}(\tilde{\mathbf{x}}_{l}) - f_{\mu}(\tilde{\mathbf{x}}_{k})) \tilde{\mathbf{w}}_{l,m}^{\mu}$$

so that

$$\begin{aligned} |\delta \Xi_{k,l,m}| &\leq \|f\|_{\mathcal{C}^{1}_{\mathrm{b}}} \|\mathbf{x}\|_{p;[k,l]} \varepsilon_{l,m}^{1/p} + \sum_{i=1}^{d} \|f_{\mu}(\mathbf{x}) - f_{\mu}(\tilde{\mathbf{x}})\|_{p;[k,l]} \|\tilde{\mathbf{w}}^{\mu}\|_{p;[l,m]} \\ &\leq L \|\mathbf{x}\|_{p;[k,l]} \varepsilon_{l,m}^{1/p} + \max_{\mu=1,\dots,d} \|f_{\mu}(\mathbf{x}) - f_{\mu}(\tilde{\mathbf{x}})\|_{p;[k,l]} \|\tilde{\mathbf{w}}\|_{p;[l,m]} \end{aligned}$$

Hence by Lemma 5.1

$$\begin{aligned} \left| \sum_{j=k}^{l-1} \Xi_{j,j+1} - \Xi_{k,l} \right| &= |R_{k,l} - \tilde{R}_{k,l}| \\ &\leq 2^{1-1/p} L C_{p,N} \bigg(\|\mathbf{x}\|_{p;[k,l]} \varepsilon_{k,l}^{1/p} + \Big(\|\mathbf{z}\|_{\infty;[k,l]}^p \|\tilde{\mathbf{x}}\|_{p;[k,l]}^p + \|\mathbf{z}\|_{p;[k,l]}^p \Big)^{1/p} \|\tilde{\mathbf{w}}\|_{p;[k,l]} \Big) \end{aligned}$$

Now, on the one hand, we see that

$$\begin{aligned} |\mathbf{z}_{k,l}| &\leq |R_{k,l} - \tilde{R}_{k,l}| + \sum_{\mu=1}^{d} |f_{\mu}(\mathbf{x}_{k})\mathbf{w}_{k,l}^{\mu} - f_{\mu}(\tilde{\mathbf{x}}_{k})\tilde{\mathbf{w}}_{k,l}^{\mu} \\ &\leq |R_{k,l} - \tilde{R}_{k,l}| + L\big(\|\mathbf{z}\|_{\infty;[k,l]}\|\tilde{\mathbf{w}}\|_{p;[k,l]} + \varepsilon_{k,l}\big), \end{aligned}$$

so the bound

$$\|\mathbf{z}\|_{p;[k,l]} \leq 8^{1-1/p} L C_{p,N} \Big\{ \|\mathbf{x}\|_{p;[k,l]}^p \varepsilon_{k,l} + \Big(\|\mathbf{z}\|_{\infty;[k,l]}^p \|\tilde{\mathbf{x}}\|_{p;[k,l]}^p + \|\mathbf{z}\|_{p;[k,l]}^p \Big) \|\tilde{\mathbf{w}}\|_{p;[k,l]}^p + \|\mathbf{z}\|_{\infty;[0,l]}^p \|\tilde{\mathbf{w}}\|_{p;[k,l]}^p + \varepsilon_{k,l} \Big\}^{1/p}$$
holds. Therefore, for any pair of indices $k < l$ such that $8^{p-1} L^p C_{p,N}^p \omega_{k,l} \leq \frac{1}{2}$, we have that

$$\|\mathbf{z}\|_{p;[k,l]}^{p} \leq 2^{3p-2} L^{p} C_{p,N}^{p} \Big(1 + \|\mathbf{x}\|_{p;[k,l]}^{p}\Big) \varepsilon_{k,l} + \|\mathbf{z}\|_{\infty;[k,l]}^{p} \Big(2^{3p-2} L^{p} C_{p,N}^{p} \|\tilde{\mathbf{w}}\|_{p;[k,l]}^{p} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p}\Big)$$

By the a priori estimate in Theorem 5.2, we see that

$$\|\mathbf{x}\|_{p;[k,l]} \leq 2L \|\mathbf{w}\|_{p;[k,l]}, \quad \|\tilde{\mathbf{x}}\|_{p;[k,l]} \leq 2L \|\tilde{\mathbf{w}}\|_{p;[k,l]}$$

so that

$$\|\mathbf{z}\|_{p;[k,l]}^p \leq A_p \varepsilon_{k,l} + A_p \|\mathbf{z}\|_{\infty;[0,l]}^p \omega_{k,l}.$$

where $A_p \coloneqq 2^p L^p (4^{p-1} C_{p,N}^p + 1)$. On the other hand, when l = k + 1 we have

$$\begin{aligned} \mathbf{z}_{k+1} - \mathbf{z}_k &| \leq \sum_{\mu=1}^d \left| f_{\mu}(\mathbf{x}_k) \mathbf{w}_{k,k+1}^{\mu} - f_{\mu}(\tilde{\mathbf{x}}_k) \tilde{\mathbf{w}}_{k,k+1}^{\mu} \right| \\ &\leq A_p^{1/p} \varepsilon_{k,l}^{1/p} + A_p^{1/p} \|\mathbf{z}\|_{\infty;[0,k+1]} \|\tilde{\mathbf{w}}\|_{p;[k,l]} \end{aligned}$$

Finally, by using Theorem 3.13 we obtain

$$|\mathbf{x}_N - \tilde{\mathbf{x}}_N| \leq 2c_{p,N} e^{c_{p,N} \|\tilde{\mathbf{w}}\|_{p;[0,N]}^p} \left(|\xi - \tilde{\xi}| + \|\mathbf{w} - \tilde{\mathbf{w}}\|_{p;[0,N]} \right)$$

where we have used that $x \mapsto (1+cx)^{\alpha} e^{-x}$ is decreasing over $[0,\infty)$ as long as $c\alpha \leq 1$, and

$$c_{p,N} \coloneqq 2^p e^{2p} A_p. \qquad \Box$$

Remark 5.5. As before, the hypothesis on the vector fields can be relaxed to requiring that $f \in \operatorname{Lip}^{\gamma}$ for some $\gamma \in (p, 2]$. In this case this means that $f \in \mathcal{C}_{\mathrm{b}}^{1}$ and there is a constant L > 0 such that

$$\|Df(\mathbf{x}) - Df(\tilde{\mathbf{x}})\| \leq L|\mathbf{x} - \tilde{\mathbf{x}}|^{\gamma - 1}$$

for all $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbb{R}^n$.

5.2. The case of $2 \le p < 3$. We show analogues of the results in the previous section for the case where now we take $p \in [2,3)$.

We keep the previous notations, i.e., we consider eq. (9) and but redefine R in eq. (10) as

$$R_{k,l} \coloneqq \mathbf{x}_{k,l} - \sum_{\mu=1}^{d} f_{\mu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu} - \sum_{\mu,\nu=1}^{d} D f_{\nu}(\mathbf{x}_{k}) f_{\mu}(\mathbf{x}_{k}) \mathbb{W}_{k,l}^{\mu\nu},$$
(13)

and we furthermore consider

$$I_{k,l} \coloneqq \mathbf{x}_{k,l} - \sum_{\mu=1}^{d} f_{\mu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu}$$
(14)

$$J_{k,l}^{\mu} \coloneqq f_{\mu}(\mathbf{x}_{l}) - f_{\mu}(\mathbf{x}_{k}) - \sum_{\nu=1}^{d} Df_{\mu}(\mathbf{x}_{k}) f_{\nu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\nu}$$
(15)

$$= f_{\mu}(\mathbf{x}_{l}) - f_{\mu}(\mathbf{x}_{k}) - Df_{\mu}(\mathbf{x}_{k})\delta\mathbf{x}_{k,l} + Df_{\mu}(\mathbf{x}_{k})I_{k,l}.$$
 (16)

where in eqs. (13) and (15), \mathbb{W} denotes the iterated-sums lift of \mathbf{w} .

Definition 5.6. For $\mu, \nu \in \{1, \ldots, d\}$ we define the vector field $F_{\mu\nu} \colon \mathbb{R}^n \to \mathbb{R}^n$

$$F_{\mu\nu}(\mathbf{x}) \coloneqq Df_{\nu}(\mathbf{x})f_{\mu}(\mathbf{x})$$

Observe that by successive application of the chain rule one can show that if

$$\|F_{\mu\nu}\|_{C_{\mathbf{b}}^{k}} \leq (2^{k+1} - 1) \|f\|_{C_{\mathbf{b}}^{k+1}}^{2}$$

for all $k \ge 0$ as long as the norm on the right-hand side is finite.

Lemma 5.7. Let $p \in [2,3)$ and $f \in C_b^2$. The bound

$$\max_{\mu=1,\dots,d} \|J^{\mu}\|_{p/2;[k,l]} \leq 2^{1-2/p} \|f\|_{\mathcal{C}^{2}_{\mathrm{b}}} \left(\|I\|_{p/2;[k,l]}^{p/2} + \frac{1}{2} \|\mathbf{x}\|_{p;[k,l]}^{p} \right)^{2/p}.$$

holds.

Proof. Performing a first-order Taylor expansion on f_i we see that

$$J_{k,l}^{\mu} = Df_{\mu}(\mathbf{x}_k) \left(\mathbf{x}_{k,l} - \sum_{\nu=1}^d f_{\nu}(\mathbf{x}_k) \mathbf{w}_{k,l}^{\nu} \right) + \frac{1}{2} D^2 f_{\mu}(\mathbf{x}_k + \theta \mathbf{x}_{k,l}) (\mathbf{x}_{k,l}, \mathbf{x}_{k,l})$$

for some $\theta \in (0, 1)$. Thus

$$\begin{aligned} |J_{k,l}^{\mu}| &\leq \|f\|_{\mathcal{C}^{2}_{\mathrm{b}}} \left(|I_{k,l}| + \frac{1}{2} |\mathbf{x}_{k,l}|^{2} \right) \\ &\leq 2^{1-p/2} \|f\|_{\mathcal{C}^{2}_{\mathrm{b}}} \left(\|I\|_{p/2;[k,l]}^{p/2} + \frac{1}{2} \|\mathbf{x}\|_{p;[k,l]}^{p} \right)^{2/p}. \end{aligned}$$

The proof is concluded by applying Lemma 3.11.

Theorem 5.8. Let $p \in [2,3)$, and suppose that **x** solves eq. (9) with $f \in C_b^2$. The bounds

$$\|\mathbf{x}\|_{p;[k,l]} \leq K_p(\|\mathbf{W}\|_{p;[k,l]}^p \vee \|\mathbf{W}\|_{p;[k,l]})$$
$$\|I\|_{p/2;[k,l]} \leq K'_p(\|\mathbf{W}\|_{p;[k,l]}^{2p} \vee \|\mathbf{W}\|_{p;[k,l]}^2)$$

hold, with

$$K_p \coloneqq 9 \times 2^{6(1-1/p)} \Big(1 \vee 6^{1-1/p} 8^{(1-1/p)(1-2/p)} C_{p,N}^{1-1/p} \Big), \quad K'_p \coloneqq 3 \times 2^{1-2/p} \Big(1 + K_p^2 \Big).$$

Proof. As before, by scaling we may assume that $||f||_{C_b^2} \leq 1$. Consider the triangular array

$$\Xi_{k,l} \coloneqq \sum_{\mu=1}^d f_\mu(\mathbf{x}_k) \mathbf{w}_{k,l}^\mu + \sum_{\mu,\nu=1}^d F_{\mu\nu}(\mathbf{x}_k) \mathbb{W}_{k,l}^{\mu\nu}.$$

We immediately see that

$$\delta \Xi_{k,l,m} = -\sum_{\mu=1}^{d} \left(f_{\mu}(\mathbf{x}_{l}) - f_{\mu}(\mathbf{x}_{k}) \right) \mathbf{w}_{l,m}^{\mu} + \sum_{\mu,\nu=1}^{d} \left\{ F_{\mu\nu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu} \mathbf{w}_{l,m}^{\nu} - \left(F_{\mu\nu}(\mathbf{x}_{l}) - F_{\mu\nu}(\mathbf{x}_{k}) \right) \mathbb{W}_{l,m}^{\mu\nu} \right\}$$
$$= -\sum_{\mu=1}^{d} J_{k,l}^{\mu} \mathbf{w}_{k,l}^{\mu} - \sum_{\mu,\nu=1}^{d} \left(F_{\mu\nu}(\mathbf{x}_{l}) - F_{\mu\nu}(\mathbf{x}_{k}) \right) \mathbb{W}_{l,m}^{\mu\nu}.$$

Since $f \in C_b^2$, the function $F_{\mu\nu}$ is in C_b^1 for all $\mu, \nu \in \{1, \ldots, d\}$ and $||F_{\mu\nu}||_{C_b^1} \leq 3$. Therefore, we have the *p*-variation estimate

$$||F_{\mu\nu}(\mathbf{x})||_{p;[k,l]} \le 3||\mathbf{x}||_{p;[k,l]}$$

Hence, we see that

$$|\delta \Xi_{k,l,m}| \leq \sum_{\mu=1}^{d} \|J^{\mu}\|_{p/2;[k,l]} \|\mathbf{w}^{\mu}\|_{p;[l,m]} + 3\|\mathbf{x}\|_{p;[k,l]} \sum_{\mu,\nu=1}^{d} \|\mathbb{W}^{\mu\nu}\|_{p/2;[l,m]}.$$

By Proposition 3.16 we see that

$$|R_{k,l}| \leq 3C_{p,N} \left(\sum_{\mu=1}^{d} \|J^{\mu}\|_{p/2;[k,l]} \|\mathbf{w}^{\mu}\|_{p;[k,l]} + \|\mathbf{x}\|_{p;[k,l]} \|\mathbb{W}\|_{p/2;[k,l]} \right).$$
(17)

Now we note that

$$|I_{k,l}| \leq |R_{k,l}| + \sum_{\mu,\nu=1}^{d} \left| F_{\mu\nu}(\mathbf{x}_k) \mathbb{W}_{k,l}^{\mu\nu} \right| \leq |R_{k,l}| + \|\mathbb{W}\|_{p/2;[k,l]}$$

so that, by eq. (17) and Lemma 5.7, we obtain

$$\begin{aligned} |I_{k,l}| &\leq 3 \times 2^{1-2/p} C_{p,N} \left\{ (\|I\|_{p/2;[k,l]}^{p/2} + \|\mathbf{x}\|_{p;[k,l]}^{p})^{2/p} \|\mathbf{w}\|_{p;[k,l]} + \|\mathbf{x}\|_{p;[k,l]} \|\mathbb{W}\|_{p/2;[k,l]} \right\} + \|\mathbb{W}\|_{p/2;[k,l]} \\ &\leq 3 \times 4^{1-2/p} C_{p,N} \left\{ (\|I\|_{p/2;[k,l]}^{p/2} + \|\mathbf{x}\|_{p;[k,l]}^{p}) \|\mathbf{w}\|_{p;[k,l]}^{p/2} + \|\mathbf{x}\|_{p;[k,l]}^{p/2} \|\mathbb{W}\|_{p/2;[k,l]}^{p/2} \right\}^{2/p} + \|\mathbb{W}\|_{p/2;[k,l]}. \end{aligned}$$

Taking $\frac{p}{2}$ -variation we obtain

$$\begin{split} \|I\|_{p/2;[k,l]} &\leqslant 3 \times 8^{1-2/p} C_{p,N} \Big\{ (\|I\|_{p/2;[k,l]}^{p/2} + \|\mathbf{x}\|_{p;[k,l]}^{p}) \|\mathbf{w}\|_{p;[k,l]}^{p/2} + \|\mathbf{x}\|_{p;[k,l]}^{p/2} \|\mathbb{W}\|_{p/2;[k,l]}^{p/2} \Big\}^{2/p} + 2^{1-2/p} \|\mathbb{W}\|_{p/2;[k,l]} \\ & \text{If } 0 \leqslant k < l \leqslant N \text{ are such that } 3 \times 8^{1-2/p} C_{p,N} \|\|\mathbf{W}\|_{p;[k,l]} \leqslant \frac{1}{2} \text{ then} \end{split}$$

$$\|I\|_{p/2;[k,l]} \leq 3 \times 2^{1-2/p} (\|\mathbf{x}\|_{p;[k,l]}^2 + \|\mathbb{W}\|_{p/2;[k,l]}).$$
(18)

Finally, noting that

$$|\mathbf{x}_{k,l}| \leq |I_{k,l}| + \sum_{\mu=1}^{d} |f_{\mu}(\mathbf{x}_{k})\mathbf{w}_{k,l}^{\mu}| \leq ||I||_{p/2;[k,l]} + ||\mathbf{w}||_{p;[k,l]}.$$

we obtain, by taking p-variation, that

$$\begin{aligned} \|\mathbf{x}\|_{p;[k,l]} &\leq 2^{1-1/p} (\|I\|_{p/2;[k,l]} + \|\mathbf{w}\|_{p;[k,l]}) \\ &\leq 3 \times 2^{2-3/p} \|\mathbf{x}\|_{p;[k,l]}^2 + 3 \times 2^{2-3/p} \|\mathbb{W}\|_{p/2;[k,l]} + 2^{1-1/p} \|\mathbf{w}\|_{p;[k,l]} \end{aligned}$$

Let $c_1 \coloneqq 3 \times 2^{2-3/p}$, $c_2 \coloneqq 2^{1-1/p}$. Multiplying both sides by c_1 and using our hypothesis on the interval [k, l] we obtain that

$$c_1 \|\mathbf{x}\|_{p;[k,l]} \le (c_1 \|\mathbf{x}\|_{p;[k,l]})^2 + c_1(c_1 + c_2) \|\mathbf{W}\|_{p;[k,l]}.$$

Set $c \coloneqq c_1(c_1 + c_2)$. Reducing further the size of the interval if necessary, we may assume that $c_1(c_1 + c_2) \| \mathbf{W} \|_{p;[k,l]} \leq \frac{1}{4}$, so that we must necessarily have that one of the following inequalities hold:

$$4\|\mathbf{x}\|_{p;[k,l]} \ge \frac{1+\sqrt{1-4c}\|\mathbf{W}\|_{p;[k,l]}}{2} \ge \frac{1}{2}, \quad 4\|\mathbf{x}\|_{p;[k,l]} \le \frac{1-\sqrt{1-4c}\|\mathbf{W}\|_{p;[k,l]}}{2} \le 2c\|\|\mathbf{W}\|_{p;[k,l]}.$$

In fact, the second inequality holds if $\|\mathbf{x}\|_{p;[k,l]} \leq \frac{1}{8}$. Applying Lemma 3.12, we obtain

$$\|\mathbf{x}\|_{p;[k,l]} \leq K_p \left(\|\mathbf{W}\|_{p;[k,l]} \vee \|\mathbf{W}\|_{p;[k,l]}^p \right)$$

with

$$K_p \coloneqq 9 \times 2^{6(1-1/p)} \left(1 \vee 6^{1-1/p} 8^{(1-1/p)(1-2/p)} C_{p,N}^{1-1/p} \right)$$

This shows the first estimate.

Now replace this bound in eq. (18) and use the fact that $\|\mathbb{W}\|_{p/2;[k,l]} \leq \|\mathbb{W}\|_{p;[k,l]}^2$ to obtain

$$\|I\|_{p/2;[k,l]} \leq 3 \times 2^{1-2/p} \left(1 + K_p^2\right) \|\mathbf{W}\|_{p;[k,l]}^2$$

Finally, we prove our main result, namely the stability bound for the evolution of features through the network. But first, we extend Lemma 5.7 to bound the difference of the remainders J and \hat{J} for solutions of difference equations driven by different noises.

Lemma 5.9. Let \mathbf{x} and $\tilde{\mathbf{x}}$ be solutions to eq. (9) driven by \mathbf{w} and $\tilde{\mathbf{w}}$, respectively. Then, for all $0 \leq k < \infty$ $l \leqslant N$ we have

$$\max_{\mu=1,\dots,d} \|J^{\mu} - \tilde{J}^{\mu}\|_{p/2;[k,l]} \leq 2^{2-4/p} \|f\|_{\mathcal{C}^{3}_{\mathbf{b}}} \Big\{ \|I - \tilde{I}\|_{p/2;[k,l]} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,l]} \Big(\|\tilde{I}\|_{p/2;[k,l]} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{2} \Big) \\ + \|\mathbf{x} - \tilde{\mathbf{x}}\|_{p;[k,l]} (\|\mathbf{x}\|_{p;[k,l]} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}) \Big\}$$

Proof. Using eq. (16) we see that

$$J_{k,l}^{\mu} - \tilde{J}_{k,l}^{\mu} = T_{k,l}^{\mu} - \tilde{T}_{k,l}^{\mu} + B_{k,l}$$

where,

$$B_{k,l} \coloneqq Df_{\mu}(\mathbf{x}_k)I_{k,l} - Df_{\mu}(\tilde{\mathbf{x}}_k)I_{k,l},$$

$$T_{k,l}^{\mu} \coloneqq f_{\mu}(\mathbf{x}_l) - f_{\mu}(\mathbf{x}_l) - Df_{\mu}(\mathbf{x}_k)\delta\mathbf{x}_{k,l},$$

and \tilde{T}^i is defined similarly.

Adding and subtracting cross terms we obtain the following bound for the B term:

$$|B_{k,l}| \leq ||f||_{\mathcal{C}^{3}_{\mathrm{b}}}(|I_{k,l} - I_{k,l}| + |\mathbf{x}_{k} - \tilde{\mathbf{x}}_{k}||I_{k,l}|),$$

so that

$$\|B\|_{p/2;[k,l]} \leq 2^{1-2/p} \|f\|_{\mathcal{C}^3_{\mathbf{b}}} \Big(\|I - \tilde{I}\|_{p/2;[k,l]}^{p/2} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,l]}^{p/2} \|\tilde{I}\|_{p/2;[k,l]}^{p/2} \Big)^{2/p} .$$

From Lemma 5.1 we obtain that

$$\|T - \tilde{T}\|_{p/2;[k,l]} \leq 2^{1-2/p} \|f\|_{\mathcal{C}^{3}_{\mathrm{b}}} \Big[\|\mathbf{x} - \tilde{\mathbf{x}}\|_{p;[k,l]} (\|\mathbf{x}\|_{p;[k,l]} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}) + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{2} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty;[0,l]} \Big]$$
proof is finished.

and the proof is finished.

Theorem 5.10. Let $2 \leq p < 3$ and suppose $\mathbf{x}, \tilde{\mathbf{x}}$ are two solutions to eq. (9) with initial conditions $\xi, \tilde{\xi}$ and driven by $\mathbf{w}, \tilde{\mathbf{w}}$ respectively. If furthermore $f_1, \ldots, f_d \in \mathcal{C}^3_b$, then

$$\sup_{k=0,\dots,N} |\mathbf{x}_{k} - \tilde{\mathbf{x}}_{k}| \leq 2c'_{p,N} e^{c_{p,N} \|f\|_{\mathcal{C}^{3}_{\mathrm{b}}}^{p} \left(\|\mathbf{W}\|_{p;[0,N]}^{p} + \|\tilde{\mathbf{W}}\|_{p;[0,N]}^{p} \right)} (|\xi - \tilde{\xi}| + \|f\|_{\mathcal{C}^{3}_{\mathrm{b}}} \rho_{p}(\mathbf{W}, \tilde{\mathbf{W}}))$$

holds, where

$$c_{p,N} \coloneqq 2^p e^{2p} (L_p + K_p^2 + K_p')^p, \quad c'_{p,N} = 2^{1-2/p} c_{p,N}^{1/p}$$

with

$$L_p \coloneqq 4^{3/2 - 2/p} \times 7^{2 - 3/p} \times C_{p,N}$$

the constant $C_{p,N}$ appears in Proposition 3.15 and K_p, K'_p are as in Theorem 5.8.

Proof. We divide the proof in several steps. Below we denote

$$\Delta \mathbf{x}_k \coloneqq \mathbf{x}_k - \tilde{\mathbf{x}}_k$$
$$\Delta J_{k,l}^{\mu} \coloneqq J_{k,l}^{\mu} - \tilde{J}_{k,l}^{\mu}$$
$$\Delta I_{k,l} \coloneqq I_{k,l} - \tilde{I}_{k,l}$$

and we consider the controls

$$\varepsilon_{k,l} \coloneqq \|\mathbf{w} - \tilde{\mathbf{w}}\|_{p;[k,l]}^p$$
$$\omega_{k,l} \coloneqq \|\mathbf{x}\|_{p;[k,l]}^p + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^p$$
$$E_{k,l} \coloneqq \|\mathbb{W} - \tilde{\mathbb{W}}\|_{p/2;[k,l]}^{p/2}.$$

We also assume, without loss of generality, that $\|f\|_{\mathcal{C}^3_{\mathrm{b}}} \leqslant 1.$

Step 1. We estimate the difference of the remainders R and \tilde{R} as defined in eq. (13) via the Sewing Lemma. To this end, consider the germ

$$\Xi_{k,l} \coloneqq \sum_{\mu=1}^{d} f_{\mu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu} + \sum_{\mu,\nu=1}^{d} F_{\mu\nu}(\mathbf{x}_{k}) \mathbb{W}_{k,l}^{\mu\nu} - \sum_{\mu=1}^{d} f_{\mu}(\tilde{\mathbf{x}}_{k}) \tilde{\mathbf{w}}_{k,l}^{\mu} - \sum_{\mu,\nu=1}^{d} F_{\mu\nu}(\mathbf{x}_{k}) \tilde{\mathbb{W}}_{k,l}^{\mu\nu}$$

A standard calculation, using Chen's identity Theorem 4.2 yields

$$\delta \Xi_{k,l,m} = -\sum_{\mu=1}^{d} (f_{\mu}(\mathbf{x}_{l}) - f_{\mu}(\mathbf{x}_{k})) \mathbf{w}_{l,m}^{\mu} + \sum_{\mu,\nu=1}^{d} \left(F_{\mu\nu}(\mathbf{x}_{k}) \mathbf{w}_{k,l}^{\mu} \mathbf{w}_{l,m}^{\nu} - (F_{\mu\nu}(\mathbf{x}_{l}) - F_{\mu\nu}(\mathbf{x}_{k})) \mathbb{W}_{l,m}^{\mu\nu} \right) \\ + \sum_{\mu=1}^{d} (f_{\mu}(\tilde{\mathbf{x}}_{l}) - f_{\mu}(\tilde{\mathbf{x}}_{k})) \tilde{\mathbf{w}}_{l,m}^{\mu} - \sum_{\mu,\nu=1}^{d} \left(F_{\mu\nu}(\tilde{\mathbf{x}}_{k}) \tilde{\mathbf{w}}_{k,l}^{\mu} \tilde{\mathbf{w}}_{l,m}^{\nu} - (F_{\mu\nu}(\tilde{\mathbf{x}}_{l}) - F_{\mu\nu}(\tilde{\mathbf{x}}_{k})) \mathbb{W}_{l,m}^{\mu\nu} \right) \\ = -\sum_{\mu=1}^{d} J_{k,l}^{\mu} \mathbf{w}_{l,m}^{\mu} - \sum_{\mu,\nu=1}^{d} (F_{\mu\nu}(\mathbf{x}_{l}) - F_{\mu\nu}(\mathbf{x}_{k})) \mathbb{W}_{l,m}^{\mu\nu} + \sum_{\mu=1}^{d} \tilde{J}_{k,l}^{\mu} \tilde{\mathbf{w}}_{l,m}^{\mu} + \sum_{\mu,\nu=1}^{d} (F_{\mu\nu}(\tilde{\mathbf{x}}_{l}) - F_{\mu\nu}(\tilde{\mathbf{x}}_{k})) \mathbb{W}_{l,m}^{\mu\nu}$$
Therefore

Therefore

$$\begin{split} |\delta\Xi_{k,l,m}| &\leq \sum_{\mu=1}^{d} \|\Delta J^{\mu}\|_{p/2;[k,l]} \|\mathbf{w}^{\mu}\|_{p;[l,m]} + \sum_{i=1}^{d} \|\tilde{J}^{\mu}\|_{p;[k,l]} \|\Delta \mathbf{w}^{\mu}\|_{p;[l,m]} \\ &+ \sum_{\mu,\nu=1}^{d} \|F_{\mu\nu}(\mathbf{x}) - F_{\mu\nu}(\tilde{\mathbf{x}})\|_{p;[k,l]} \|\mathbb{W}^{\mu\nu}\|_{p/2;[l,m]} + \sum_{\mu,\nu=1}^{d} \|F_{\mu\nu}(\tilde{\mathbf{x}})\|_{p;[k,l]} \|\mathbb{W}^{\mu\nu} - \tilde{\mathbb{W}}^{\mu\nu}\|_{p/2;[l,m]}. \end{split}$$

Hence, by the Sewing Lemma we obtain that

$$|R_{k,l} - \tilde{R}_{k,l}| \leq C_{p,N} \Big\{ \|\Delta J\|_{p/2;[k,l]} \|\mathbf{w}\|_{p;[k,l]} + \|\tilde{J}\|_{p/2;[k,l]} \varepsilon_{k,l}^{1/p} + \|\Delta F\|_{p;[k,l]} \|\mathbf{W}\|_{p/2;[k,l]} + \|F(\tilde{\mathbf{x}})\|_{p;[k,l]} E_{k,l}^{2/p} \Big\}.$$
(19)

Step 2. We now use the relation $I_{k,l} = R_{k,l} + \sum_{\mu,\nu=1}^{d} F_{\mu\nu}(\mathbf{x}_k) \mathbb{W}_{k,l}^{\mu\nu}$ to obtain

$$\begin{aligned} |I_{k,l} - \tilde{I}_{k,l}| &\leq |R_{k,l} - \tilde{R}_{k,l}| + \sum_{\mu,\nu}^{d} |F_{\mu\nu}(\mathbf{x}_{k}) - F_{\mu\nu}(\tilde{\mathbf{x}}_{k})| |\mathbb{W}_{k,l}^{\mu\nu}| + \sum_{\mu,\nu=1}^{d} |F_{\mu\nu}(\tilde{\mathbf{x}}_{k})| |\mathbb{W}_{k,l}^{\mu\nu} - \tilde{\mathbb{W}}_{k,l}^{\mu\nu}| \\ &\leq |R_{k,l} - \tilde{R}_{k,l}| + \|\Delta \mathbf{x}\|_{\infty;[0,l]} \|\mathbb{W}\|_{p/2;[k,l]} + E_{k,l}^{2/p}. \end{aligned}$$

Using Lemmas 5.7 and 5.9 and eq. (19) we see that the right-hand side is bounded by

$$C_{p,N}\left\{ \left(\|\Delta I\|_{p/2;[k,l]} + \|\Delta \mathbf{x}\|_{\infty;[0,l]} (\|\tilde{I}\|_{p/2;[k,l]} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^2) + \|\Delta \mathbf{x}\|_{p;[k,l]} \omega_{k,l}^{1/p} \right) \|\mathbf{w}\|_{p;[k,l]} + (\|\tilde{I}\|_{p/2;[k,l]} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^2) \varepsilon_{k,l}^{1/p} + (\|\Delta \mathbf{x}\|_{p;[k,l]} + \|\Delta \mathbf{x}\|_{\infty;[0,l]} \|\tilde{\mathbf{x}}\|_{p;[k,l]}) \|\mathbb{W}\|_{p/2;[k,l]} + \|\tilde{\mathbf{x}}\|_{p;[k,l]} E_{k,l}^{2/p} \right\} + \|\Delta \mathbf{x}\|_{\infty;[0,l]} \|\mathbb{W}\|_{p/2;[k,l]} + E_{k,l}^{2/p}$$

Defining the control

$$Q_{k,l} \coloneqq 24^{p/2-1} C_p^{p/2} \left\{ \left(\|\Delta I\|_{p/2;[k,l]}^{p/2} + \|\Delta \mathbf{x}\|_{\infty;[0,l]}^{p/2} (\|\tilde{I}\|_{p/2;[k,l]}^{p/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p}) + \|\Delta \mathbf{x}\|_{p;[k,l]}^{p/2} \omega_{k,l}^{1/2} \right) \|\mathbf{w}\|_{p;[k,l]}^{p/2} \\ + \left(\|\tilde{I}\|_{p/2;[k,l]}^{p/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p} \right) \varepsilon_{k,l}^{1/2} + \left(\|\Delta \mathbf{x}\|_{p;[k,l]}^{p} + \|\Delta \mathbf{x}\|_{\infty;[0,l]}^{p} \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p} \right)^{1/2} \|W\|_{p/2;[k,l]}^{p/2} \\ + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p/2} E_{k,l} \right\}$$

we obtain the bound

$$|\Delta I_{k,l}| \leq 3^{1-2/p} \Big(Q_{k,l} + \|\Delta \mathbf{x}\|_{\infty;[0,l]}^{p/2} \|\mathbf{W}\|_{p/2;[k,l]}^{p/2} + E_{k,l} \Big)^{2/p}.$$

By Lemma 3.11 the same bound holds for $\|\Delta I\|_{p/2;[k,l]}$. Now, taking k < l close enough such that

$$\left\| \mathbf{W} \right\|_{p;[k,l]} \leq \frac{1}{2 \times 72^{1-2/p} \times C_{p,N}}$$

we see that

$$\|\Delta I\|_{p/2;[k,l]} \leq 2 \times 72^{1-2/p} C_{p,N} \Big\{ \|\Delta \mathbf{x}\|_{\infty;[0,l]} \tilde{U}_{k,l}^{2/p} + \|\Delta \mathbf{x}\|_{p;[k,l]} \omega_{k,l}^{1/p} + \tilde{U}_{k,l}^{2/p} \varepsilon_{k,l}^{1/p} + V_{k,l}^{1/p} \|\mathbb{W}\|_{p/2;[k,l]}^{1/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]} E_{k,l}^{2/p} \Big\}$$

where now

$$\tilde{U}_{k,l} \coloneqq \|\tilde{I}\|_{p/2;[k,l]}^{p/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p} \\
V_{k,l} \coloneqq \|\Delta \mathbf{x}\|_{p;[k,l]}^{p} + \|\Delta \mathbf{x}\|_{\infty;[0,l]}^{p}\|\tilde{\mathbf{x}}\|_{p;[k,l]}^{p}$$

are new controls as well. **Step 3.** We now use the fact that

 $\mathbf{x}_{k,l} = I_{k,l} + \sum_{\mu=1}^d f_\mu(\mathbf{x}_k) \mathbf{w}_{k,l}^\mu$

to obtain that

$$\begin{aligned} |\Delta \mathbf{x}_{k,l}| &\leq |\Delta I_{k,l}| + |\Delta \mathbf{x}_k| \sum_{\mu=1}^d |\tilde{\mathbf{w}}_{k,l}^{\mu}| + \sum_{\mu=1}^d |\Delta \mathbf{w}_{k,l}^{\mu}| \\ &\leq \|\Delta I\|_{p/2;[k,l]} + \|\Delta \mathbf{x}\|_{\infty;[0,l]} \|\mathbf{w}\|_{p;[k,l]} + \varepsilon_{k,l}^{1/p}. \end{aligned}$$

From the previous bound on $\|\Delta I\|_{p/2;[k,l]}$ we get that

$$\begin{aligned} |\Delta \mathbf{x}_{k,l}| &\leq 2 \times 72^{1-2/p} C_{p,N} \Big\{ \|\Delta \mathbf{x}\|_{\infty;[0,l]} \tilde{U}_{k,l}^{2/p} + \|\Delta \mathbf{x}\|_{p;[k,l]} \omega_{k,l}^{1/p} + \tilde{U}_{k,l}^{2/p} \varepsilon_{k,l}^{1/p} \\ &+ V_{k,l}^{1/p} \|\mathbb{W}\|_{p/2;[k,l]}^{1/2} + \|\tilde{\mathbf{x}}\|_{p;[k,l]} E_{k,l}^{2/p} + \|\Delta \mathbf{x}\|_{\infty;[0,l]} \|\mathbf{w}\|_{p;[k,l]} + \varepsilon_{k,l}^{1/p} \Big\} \end{aligned}$$

Taking p-variation we see that

$$\begin{split} \|\Delta \mathbf{x}\|_{p;[k,l]} &\leq L_p \Big\{ \|\Delta \mathbf{x}\|_{\infty;[0,l]} \tilde{U}_{k,l}^{2/p} + \|\Delta \mathbf{x}\|_{p;[k,l]} \omega_{k,l}^{1/p} + \tilde{U}_{k,l}^{2/p} \varepsilon_{k,l}^{1/p} + V_{k,l}^{1/p} \|\mathbb{W}\|_{p/2;[k,l]}^{1/2} \\ &+ \|\tilde{\mathbf{x}}\|_{p;[k,l]} E_{k,l}^{2/p} + \|\Delta \mathbf{x}\|_{\infty;[0,l]} \|\mathbf{w}\|_{p;[k,l]} + \varepsilon_{k,l}^{1/p} \Big\} \end{split}$$

with $L_p \coloneqq 2 \times 7^{1-1/p} \times 24^{1-2/p} \times C_{p,N}$. Using again the fact that k < l are chosen so that $\|\mathbf{W}\|_{p;[k,l]} \leq L_p^{-1} < 1$, and the a priori estimate in Theorem 5.8 we obtain that

$$\|\Delta \mathbf{x}\|_{p;[k,l]} \leq L_p \|\Delta \mathbf{x}\|_{\infty;[0,l]} (\tilde{U}_{k,l}^{2/p} + \|\|\mathbf{W}\|_{p;[k,l]} + \|\|\tilde{\mathbf{W}}\|_{p;[k,l]}) + L'_p (\varepsilon_{k,l}^2 + E_{k,l})^{2/p}.$$

Now, we notice that

$$\tilde{U}_{k,l} = \|\mathbf{x}\|_{p;[k,l]}^p + \|\tilde{I}\|_{p/2;[k,l]}^{p/2} \leqslant (K_p^p + (K_p')^{p/2}) \|\|\tilde{\mathbf{W}}\|_{p;[k,l]}^p$$

so that

$$\tilde{U}_{k,l}^{2/p} \leq L_p^{-1} (K_p^2 + K_p') ||\!| \tilde{\mathbf{W}} ||\!|_{p;[k,l]}$$

Trivial estimates show that the same bound holds when l = k + 1, so by the rough Grönwall lemma and an argument similar to the Young case we obtain

$$\|\Delta x\|_{\infty;[0,N]} \leq c'_p e^{c_{p,N}(\|\mathbf{W}\|_{p;[0,N]}^p + \|\mathbf{\tilde{W}}\|_{p;[0,N]}^p)} (|\mathbf{x}_0 - \tilde{\mathbf{x}}_0| + \rho_p(\mathbf{W}, \mathbf{\tilde{W}})).$$

where

$$c_{p,N} \coloneqq 2^p e^{2p} (L_p + K_p^2 + K_p')^p, \quad c'_{p,N} = 2^{1-2/p} c_{p,N}.$$

APPENDIX A. COMPARISON OF ARCHITECTURES

In this section we show that the residual architectures mentioned in the introduction, namely

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \sigma(\mathbf{y}_k, \theta_k), \tag{20}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sum_{\mu=1}^d f_\mu(\mathbf{x}) \mathbf{w}_{k,k+1}^\mu$$
(21)

are related to each other, in the sense that any evolution represented by eq. (20) may be obtained as a projection of the evolution governed by (21). Note that we allow the dimensions of the noises, as well as the vector fields, to be different in each architecture. Consider the maps

$$\mathfrak{Y}\colon \mathbb{R}^m \times C(\mathbb{R}^m \times \mathcal{M}_{m \times m}, \mathbb{R}^m) \times \mathcal{M}_{m \times m}^N \to \mathbb{R}^m, \quad \mathfrak{X}\colon \mathbb{R}^n \times C(\mathbb{R}^n, \mathbb{R}^n)^d \times (\mathbb{R}^d)^N \to \mathbb{R}^n$$

defined by

$$\mathfrak{Y}(\mathbf{y},\sigma,\theta) \coloneqq \mathbf{y}_N, \quad \mathfrak{X}(\mathbf{x},f,\mathbf{w}) = \mathbf{x}_N$$

where \mathbf{y}_k and \mathbf{x}_k solve eqs. (20) and (21) respectively, with $\mathbf{y}_0 = \mathbf{y}$, $\mathbf{x}_0 = \mathbf{x}$. The following result draws on ideas by Kidger, Morrill, Foster and Lyons [15].

Proposition A.1. Fix $d = m^2 + 1$ and n = m + d, and set $\pi \colon \mathbb{R}^n \to \mathbb{R}^m$ be the projection onto the first *m* coordinates. Then the inclusion

$$\mathfrak{Y}(\mathbb{R}^m \times C(\mathbb{R}^m \times \mathcal{M}_{m \times m}, \mathbb{R}^m) \times \mathcal{M}_{m \times m}^N) \subset \pi \circ \mathfrak{X}(\mathbb{R}^n \times C(\mathbb{R}^n, \mathbb{R}^n)^d \times (\mathbb{R}^d)^N)$$

holds.

The content of this is result is that we may emulate the non-linear evolution of eq. (20) by a linear control system of greater dimension as in eq. (21). Since π is a Lipschitz map, this has no repercussion for our estimates. Therefore, it suffices to study systems linear in the control.

Proof. Let **w** be the $(m^2 + 1)$ -dimensional noise obtained by flattening of the θ matrices and adding a time component, i.e., for $\mu \in \{1, \ldots, m^2\}$ set

$$\mathbf{w}_k^{\mu} = \sum_{j=0}^k heta_j(\lfloor \mu/m
floor, \mu \mod m)$$

and $\mathbf{w}_k^d = k$. Let $\tilde{\pi} \colon \mathbb{R}^n \to \mathbb{R}^d$ denote the projection onto the last d coordinates and let e_1, \ldots, e_n denote the standard basis of \mathbb{R}^n . Define the vector fields $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_{\mu}(\mathbf{x}) \coloneqq e_{m+\mu}, \qquad \mu = 1, \dots, d-1$$
$$f_{d}(\mathbf{x}) \coloneqq e_{m+d} + \sum_{\nu=1}^{m} \sigma_{\nu}(\pi(\mathbf{x}), \tilde{\pi}(\mathbf{x})) e_{\nu}.$$

Therefore, the corresponding solution to eq. (21) satisfies

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sum_{\mu=1}^d f_{\mu}(\mathbf{x}_k)(\mathbf{w}_{k+1}^{\mu} - \mathbf{w}_k^{\mu})$$

= $\mathbf{x}_k + \sum_{\nu=1}^m \sigma_{\nu}(\pi(\mathbf{x}_k), \tilde{\pi}(\mathbf{x}_k))e_{\nu} + \sum_{\mu=1}^d (\mathbf{w}_{k+1}^{\mu} - \mathbf{w}_k^{\mu})e_{m+\mu}$

In particular

$$\tilde{\pi}(\mathbf{x}_{k+1}) = \tilde{\pi}(\mathbf{x}_k) + \theta_{k+1} - \theta_k,$$

that is, $\tilde{\pi}(\mathbf{x}_k) = \theta_k$. Therefore,

$$\pi(\mathbf{x}_{k+1}) = \pi(\mathbf{x}_k) + \sigma(\pi(\mathbf{x}_k), \theta_k)$$

so that, if we pick an initial condition $\mathbf{x} \in \mathbb{R}^n$ such that $\pi(\mathbf{x}) = \mathbf{y} \in \mathbb{R}^m$ we immediately see that $\mathbf{x}_N = \mathbf{y}_N$.

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