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Short dated option pricing under rough volatility

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Outline

1. **Results**
2. Proofs
3. Future work

Friz, Gerhold, Pinter (2016) study MOTM (moderately out of the money) options. For diffusion models, they find call option price asymptotics:

ATM	AATM	MOTM	OTM
$K = S_0$	$\log \frac{K}{S_0} \sim t^\beta,\ \beta > \frac{1}{2}$	$\log \frac{K}{S_0} \sim t^\beta,\ \beta < \frac{1}{2}$	$\log \frac{K}{S_0} = \text{const}$
$\mathcal{O}(\sqrt{t})$	$\mathcal{O}(\sqrt{t})$	$\exp\left(-\frac{\text{const}}{t^{1-2\beta}}\right)$	$\exp\left(-\frac{\text{const}}{t}\right)$

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- MOTM regime reflects the reality that the range of strikes of *liquidly traded* options decreases with maturity
- const in the OTM case is related to the energy $\Lambda(k)$ of the underlying LDP, which may be hard to compute
- const in the MOTM case is, essentially, $\Lambda''(0)$, which is often much easier to compute

Setting

$$\frac{dS_t}{S_t} = \sigma\left(\widehat{B}_t\right) d\left(\rho B_t + \overline{\rho} W_t\right)$$

$$\widehat{B}_t = \int_0^t K(t,s) dB_s$$

- K is a *Volterra* kernel with $\int_0^1 \int_0^t K(t,s)^2 ds dt < \infty$
- B, W are standard Brownian motions, $\rho^2 + \overline{\rho}^2 = 1$
- $\sigma : \mathbb{R} \to \mathbb{R}_{>0}$ "*smooth*"
- $\widehat{B}$ is "small-time self-similar": for any small $t > 0$ there is $\widehat{\varepsilon} > 0$ s.t.

$$\widehat{B}\Big|_{[0,t]} \overset{\text{law}}{=} \widehat{\varepsilon}\, \widehat{B}\Big|_{[0,1]}$$

- For example: $K(t,s) = |t-s|^{H-1/2}$, $0 < H < \frac{1}{2}$

Theorem

For $x \geq 0$ the call option price satisfies

$$c\left(\frac{\varepsilon}{\widehat{\varepsilon}}x, t\right) := E\left[\left(\exp\left(X_t\right) - \exp\left(\frac{\varepsilon}{\widehat{\varepsilon}}x\right)\right)^+\right]$$
$$= \exp\left(-\frac{I(x)}{\widehat{\varepsilon}^2}\right)\exp\left(\frac{\varepsilon}{\widehat{\varepsilon}}x\right)J(\varepsilon, x), \quad x \geq 0,$$
$$J(\varepsilon, x) := E\left[e^{-\frac{I'(x)}{\widehat{\varepsilon}^2}\widehat{U}^\varepsilon}\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{U}^\varepsilon} - 1\right)e^{I'(x)R_2}\mathbf{1}_{\widehat{U}^\varepsilon \geq 0}\right],$$

where $\widehat{U}^\varepsilon = \widehat{\varepsilon} g_1 + \widehat{\varepsilon}^2 R_2$ for a centered Gaussian r.v. g_1 and a remainder term R_2. Moreover, for any $\theta > 0$ and $0 < \beta < H$,

$$\varepsilon^\theta \log J(\varepsilon, x\varepsilon^{2\beta}) \xrightarrow{\varepsilon \to 0} 0$$

"uniformly in x around $x = 0$".

Consider the rough volatility regime $\widehat{\varepsilon} = \varepsilon^{2H}$, $0 < H \leq \frac{1}{2}$ and moderate deviations $k_t = kt^{1/2-H+\beta}$, $0 \leq \beta < H$

Theorem

$$-\log c(k_t, t) = \frac{I''(0)}{t^{2h-2\beta}} \frac{k^2}{2}(1 + o(1)), \quad t \searrow 0$$

with

$$I''(0) = \frac{1}{\sigma(0)^2}.$$

Consider the rough volatility regime $\widehat{\varepsilon} = \varepsilon^{2H}$, $0 < H \leq \frac{1}{2}$ and moderate deviations $k_t = kt^{1/2-H+\beta}$, $0 \leq \beta < \frac{2}{3}H$

Theorem

$$-\log c(k_t, t) = \frac{I''(0)}{t^{2h-2\beta}}\frac{k^2}{2} + \frac{I'''(0)}{t^{2h-3\beta}}\frac{k^3}{6}(1 + o(1)), \quad t \searrow 0$$

with

$$I''(0) = \frac{1}{\sigma(0)^2}, \quad I'''(0) = -6\rho\frac{\sigma'(0)}{\sigma(0)^4}\int_0^1\int_0^t K(t,s)\,ds\,dt.$$

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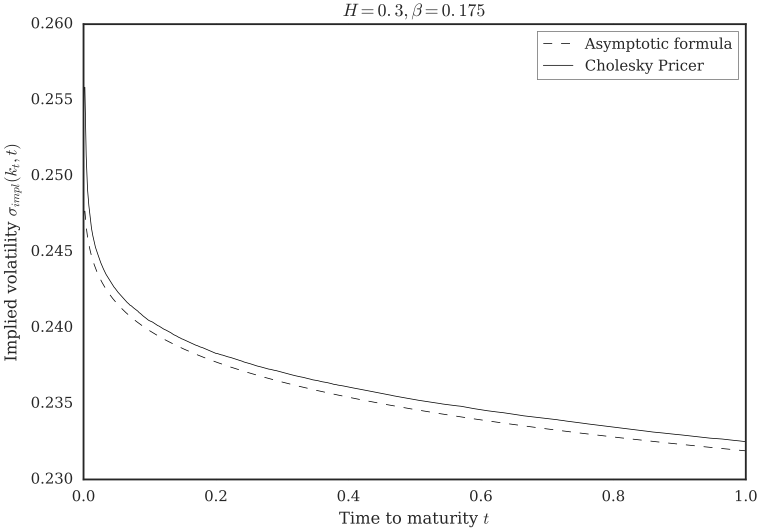
Corollary

The implied volatility satisfies

$$\sigma_{impl}(k_t, t) = \sigma(0) - \rho\frac{\sigma'(0)}{\sigma(0)}\int_0^1 \int_0^t K(t,s)dsdt\; k_t t^{H-1/2}(1 + o(1)).$$

Numerical evidence



Outline

1 Results

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Large deviations [Forde and Zhang 2016]

$$dX_t = \sigma(\widehat{B}_t)d\left(\overline{\rho}W_t + \rho B_t\right) \cancel{+\text{drift}}.$$

Short time asymptotics: $X_t \overset{\text{law}}{=} X_1^\varepsilon$, $\varepsilon = \sqrt{t}$, $\widehat{\varepsilon} = \varepsilon^{2H}$, with

$$dX_t^\varepsilon = \sigma\left(\widehat{\varepsilon}\widehat{B}_t\right)\varepsilon d\left(\overline{\rho}W_t + \rho B_t\right)$$

Theorem

$\widehat{X}_1^\varepsilon := \frac{\widehat{\varepsilon}}{\varepsilon}X_1^\varepsilon$ satisfies LDP with speed $\widehat{\varepsilon}$ and rate function

$$I(x) := \inf_{f \in H_0^1}\left\{\frac{1}{2}\frac{\left(x - \rho\left\langle\sigma(K\dot{f}),\, f\right\rangle\right)^2}{\overline{\rho}^2\left\langle\sigma^2(K\dot{f}),\, 1\right\rangle} + \frac{1}{2}\|f\|_{H_0^1}^2\right\}.$$

Notation:

- $\|f\|_{H_0^1} = \left\|\dot{f}\right\|_{L^2[0,1]}$
- $(K\dot{f})(t) = \int_0^t K(t,s)\dot{f}(s)ds$, $f \in H_0^1$

$$dX_t = \sigma(\widehat{B}_t)d\left(\overline{\rho}W_t + \rho B_t\right) \text{~~+drift~~}.$$

Short time asymptotics: $X_t \stackrel{\text{law}}{=} X_1^\varepsilon$, $\varepsilon = \sqrt{t}$, $\widehat{\varepsilon} = \varepsilon^{2H}$, with

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Proof.

- $d\widehat{X}_t^\varepsilon = \sigma(\widehat{\varepsilon}\widehat{B}_t)\widehat{\varepsilon}d\left(\overline{\rho}W_t + \rho B_t\right)$. Hence, $\widehat{X}_1^\varepsilon = \Phi_1(\widehat{\varepsilon}W, \widehat{\varepsilon}B, \widehat{\varepsilon}\widehat{B})$.
- Use (extended) extension principle based on LDP for $(W, B, \widehat{B})$. □

$$I(x) = \inf_{f\in H_0^1} \left\{ \frac{1}{2} \frac{\left(x - \rho \left\langle \sigma(K\dot{f}),\ f \right\rangle\right)^2}{\overline{\rho}^2 \left\langle \sigma^2(K\dot{f}),\ 1 \right\rangle} + \frac{1}{2} \|f\|_{H_0^1}^2 \right\} = \inf_{f\in H_0^1} \mathcal{I}_x(f).$$

1) First order optimality condition

$\mathcal{I} : \mathbb{R} \times H_0^1 \to \mathbb{R}_{\geq 0}$, $(x, f) \mapsto \mathcal{I}_x(f)$ is smooth in Fréchet sense. Hence, any local minimizer f satisfies

$$H(x, f) := D_f \mathcal{I}_x(f) \cdot f = 0.$$

$$I(x) = \inf_{f \in H_0^1} \left\{ \frac{1}{2} \frac{\left(x - \rho \left\langle \sigma(K\dot{f}),\ f \right\rangle\right)^2}{\bar{\rho}^2 \left\langle \sigma^2(K\dot{f}),\ 1 \right\rangle} + \frac{1}{2} \|f\|_{H_0^1}^2 \right\} = \inf_{f \in H_0^1} \mathcal{I}_x(f).$$

1) First order optimality condition

2) Local uniqueness and smoothness of minimizer

By the implicit function theorem, there is a unique $f = f^x$ satisfying the first order condition in a neighborhood of $x = 0,\ f = 0$. $x \mapsto f^x$ is smooth.

$$I(x) = \inf_{f \in H_0^1} \left\{ \frac{1}{2} \frac{\left(x - \rho \left\langle \sigma(K\dot{f}),\ f \right\rangle\right)^2}{\bar{\rho}^2 \left\langle \sigma^2(K\dot{f}),\ 1 \right\rangle} + \frac{1}{2} \|f\|_{H_0^1}^2 \right\} = \inf_{f \in H_0^1} \mathcal{I}_x(f).$$

1) First order optimality condition

2) Local uniqueness and smoothness of minimizer

3) Existence of a minimizer

"Local convexity": $D_f^2 \mathcal{I}_x(0) \cdot (g, g) > 0$ for any $g \in H_0^1$.
Remark: This point is not completely obvious, see the following minimization problem:

$$\mathcal{G}(f) := \int_0^1 \left[\left(f'(s)^2 - 1\right)^2 + f(s)^2 \right] ds \to \min!$$

Energy expansion

$$I(x) = \inf_{f \in H_0^1} \left\{ \frac{1}{2} \frac{\left(x - \rho \left\langle \sigma(K\dot{f}),\ f \right\rangle\right)^2}{\bar{\rho}^2 \left\langle \sigma^2(K\dot{f}),\ 1 \right\rangle} + \frac{1}{2} \|f\|^2_{H_0^1} \right\} = \inf_{f \in H_0^1} \mathcal{I}_x(f).$$

1) First order optimality condition

2) Local uniqueness and smoothness of minimizer

3) Existence of a minimizer

4) Expansion of minimizer f^x in $x \to 0$.

Make ansatz $f_t^x = \alpha_t x + \beta_t \frac{x^2}{2} + O(x^3)$ and plug into first order condition $H(x, f^x) = 0$, yields formulas for $\alpha, \beta, \ldots$

Pricing formula

$$c\left(\frac{\varepsilon}{\widehat{\varepsilon}}x, t\right) = E\left[\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{X}_1^\varepsilon} - e^{\frac{\varepsilon}{\widehat{\varepsilon}}x}\right)^+\right], \quad \widehat{X}_1^\varepsilon = \int_0^1 \sigma(\widehat{\varepsilon B})\widehat{\varepsilon}d(\overline{\rho}W + \rho B)$$

1) Perturbation & Girsanov transform

Change measure $\widehat{\varepsilon}(W, B) \to \widehat{\varepsilon}(W, B) + (h, f)$, $h, f \in H_0^1$ with Girsanov transform G_ε transforming $\widehat{X}_1^\varepsilon \to \widehat{Z}_1^\varepsilon$ with

$$G_\varepsilon = \exp\left(-\frac{1}{\widehat{\varepsilon}}\int_0^1 \dot{h}dW - \frac{1}{\widehat{\varepsilon}}\int_0^1 \dot{h}dB - \frac{1}{2\widehat{\varepsilon}^2}\int_0^1\left(\dot{h}^2 + \dot{f}^2\right)dt\right)$$

$$\widehat{Z}_1^\varepsilon = \int_0^1 \sigma\left(\widehat{\varepsilon B} + \widehat{f}\right)\left[\widehat{\varepsilon}d\left(\overline{\rho}W + \rho B\right) + d\left(\overline{\rho}h + \rho f\right)\right]$$

Pricing formula

$$c\left(\frac{\varepsilon}{\widehat{\varepsilon}}x, t\right) = E\left[\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{X}_1^\varepsilon} - e^{\frac{\varepsilon}{\widehat{\varepsilon}}x}\right)^+\right], \quad \widehat{X}_1^\varepsilon = \int_0^1 \sigma(\widehat{\varepsilon B})\widehat{\varepsilon} d(\overline{\rho}W + \rho B)$$

1) Perturbation & Girsanov transform
2) Stochastic Taylor expansion $\widehat{Z}_1^\varepsilon = x + \widehat{\varepsilon} g_1 + \widehat{\varepsilon}^2 R_2$

For h, f with $\Phi_1(h, f) = x$ we have the above stochastic Taylor expansion with

$$g_1 = \int_0^1 \left[\sigma(\widehat{f_t}) d\left(\overline{\rho}W_t + \rho B_t\right) + \sigma'(\widehat{f_t})\widehat{B}_t d\left(\overline{\rho}h_t + \rho f_t\right)\right].$$

Pricing formula

$$c\left(\frac{\varepsilon}{\widehat{\varepsilon}}x, t\right) = E\left[\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{X}_1^\varepsilon} - e^{\frac{\varepsilon}{\widehat{\varepsilon}}x}\right)^+\right], \quad \widehat{X}_1^\varepsilon = \int_0^1 \sigma(\widehat{\varepsilon B})\widehat{\varepsilon} d(\overline{\rho}W + \rho B)$$

1) Perturbation & Girsanov transform

2) Stochastic Taylor expansion $\widehat{Z}_1^\varepsilon = x + \widehat{\varepsilon} g_1 + \widehat{\varepsilon}^2 R_2$

3) $\int_0^1 \dot{h}^x dW + \int_0^1 \dot{f}^x dB = I'(x) g_1$

Following Ben Arous, we can show that

$$\int_0^1 \dot{h}^x dW + \int_0^1 \dot{f}^x dB = I'(x) g_1$$

when (h^x, f^x) is optimal configuration.

Pricing formula

$$c\left(\frac{\varepsilon}{\widehat{\varepsilon}}x, t\right) = E\left[\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{X}_1^\varepsilon} - e^{\frac{\varepsilon}{\widehat{\varepsilon}}x}\right)^+\right], \quad \widehat{X}_1^\varepsilon = \int_0^1 \sigma(\widehat{\varepsilon B})\widehat{\varepsilon} d(\overline{\rho}W + \rho B)$$

1) Perturbation & Girsanov transform

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4) Estimates for $J(\varepsilon, x)$

Steps 1-3 lead to the remainder term

$$J(\varepsilon, x) = E\left[e^{-\frac{I'(x)}{\widehat{\varepsilon}^2}\widehat{U}^\varepsilon}\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{U}^\varepsilon} - 1\right)e^{I'(x)R_2}\mathbf{1}_{\widehat{U}^\varepsilon \geq 0}\right], \quad \widehat{U}^\varepsilon = \widehat{Z}_1^\varepsilon - x,$$

which is then estimated from above and below.

Pricing formula

$$c\left(\frac{\varepsilon}{\widehat{\varepsilon}}x, t\right) = E\left[\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{X}_1^\varepsilon} - e^{\frac{\varepsilon}{\widehat{\varepsilon}}x}\right)^+\right], \quad \widehat{X}_1^\varepsilon = \int_0^1 \sigma(\widehat{\varepsilon}\widehat{B})\widehat{\varepsilon}d(\overline{\rho}W + \rho B)$$

1) Perturbation & Girsanov transform

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4) Estimates for $J(\varepsilon, x)$

5) Example: Black-Scholes case

$$J(\varepsilon, x) = M\left(-\frac{I'(x)\sigma}{\varepsilon} + \varepsilon\sigma\right) - M\left(-\frac{I'(x)\sigma}{\varepsilon}\right) \sim \frac{1}{\sqrt{2\pi}}\frac{\sigma^3\varepsilon^3}{x^2},$$

with $M(\alpha) := e^{\alpha^2/2}F(\alpha)$, F being the c.d.f. of $\mathcal{N}(0, 1)$

Estimating the remainder term

- Stochastic Taylor expansion gives

$$\varepsilon^2 R_2^{\varepsilon}(t) = \varepsilon \int_0^t \left[\sigma(\varepsilon B_s + f_s) - \sigma(f_s)\right] d\left[\overline{\rho} W_s + \rho B_s\right] + \text{BV process}$$

- Stochastic Taylor expansion gives

$$\varepsilon^2 R_2^\varepsilon(t) = \varepsilon \int_0^t \left[\sigma(\varepsilon B_s + f_s) - \sigma(f_s)\right] d\left[\overline{\rho} W_s + \rho B_s\right] + \text{BV process} =: M_t^\varepsilon$$

- For $M^{\kappa,\varepsilon} := M^\tau$ with $\tau := \inf\{ t \mid |\varepsilon B_t| \geq \kappa \}$, we have

$$\frac{d[M^{\kappa,\varepsilon}]_t}{dt} = \varepsilon^2 \left[\sigma(\varepsilon B_t + f_t) - \sigma(f_t)\right]^2 \leq \varepsilon^4 \left\|\sigma'\right\|_{\infty;K}^2 \left|B_t^\tau\right|^2$$

- Stochastic Taylor expansion gives

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- As $\varepsilon^{-2} M^{\kappa,\varepsilon} = \mathcal{O}\left(|B^{\kappa,\varepsilon}|_{\infty;[0,1]}^2\right)$, which has exponential tails, BDG inequality implies (for some $c_1, c_2 > 0$)

$$P\left(\left|R_2^\varepsilon(t)\right| > r,\ |\varepsilon B|_{\infty;[0,1]} < \kappa\right) \leq c_1 \exp\left(-c_2 r\right)$$

Outline

$$J(\varepsilon, x) = E\left[e^{-\frac{I'(x)}{\widehat{\varepsilon}^2}\widehat{U}^\varepsilon}\left(e^{\frac{\varepsilon}{\widehat{\varepsilon}}\widehat{U}^\varepsilon} - 1\right)e^{I'(x)R_2}\mathbf{1}_{\widehat{U}^\varepsilon \geq 0}\right],$$

- $\widehat{U}^\varepsilon = \widehat{\varepsilon} g_1 + \widehat{\varepsilon}^2 R_2$
- g_1 given explicitly in terms of optimal configuration f^x
- R_2 remainder term in stochastic Taylor expansion; not given explicitly, but we have control of tail behaviour

Goal

Obtain precise asymptotics/expansion of $J(\varepsilon, x)$, $x = x(\varepsilon)$, as $\varepsilon \searrow 0$.

- So far, we have polynomial upper and lower bounds.
- Advantage: no need for heat kernel asymptotics.