Short dated option pricing under rough volatility

Christian Bayer
Peter Friz, Archil Gulisashvili, Blanka Horvath, Benjamin Stemper,
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1 Results

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Friz, Gerhold, Pinter (2016) study MOTM (moderately out of the money) options. For diffusion models, they find call option price asymptotics:

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<th>ATM</th>
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Friz, Gerhold, Pinter (2016) study MOTM (moderately out of the money) options. For diffusion models, they find call option price asymptotics:

\[
\begin{align*}
\text{ATM} & : \quad K = S_0 \\
& : \quad \log \frac{K}{S_0} \sim t^\beta, \quad \beta > \frac{1}{2}, \quad O(\sqrt{t})
\end{align*}
\]

\[
\begin{align*}
\text{AATM} & : \quad K = S_0 \\
& : \quad \log \frac{K}{S_0} \sim t^\beta, \quad \beta < \frac{1}{2}, \quad O(\sqrt{t})
\end{align*}
\]

\[
\begin{align*}
\text{MOTM} & : \quad \log \frac{K}{S_0} \sim t^\beta, \quad \beta < \frac{1}{2}, \quad \exp \left( -\frac{\text{const}}{t^{1-2\beta}} \right)
\end{align*}
\]

\[
\begin{align*}
\text{OTM} & : \quad \log \frac{K}{S_0} = \text{const} \quad \exp \left( -\frac{\text{const}}{t} \right)
\end{align*}
\]

- MOTM regime reflects the reality that the range of strikes of *liquidly traded* options decreases with maturity
- const in the OTM case is related to the energy \( \Lambda(k) \) of the underlying LDP, which may be hard to compute
- const in the MOTM case is, essentially, \( \Lambda''(0) \), which is often much easier to compute
\[ \frac{dS_t}{S_t} = \sigma(\widehat{B}_t) d(\rho B_t + \bar{\rho} W_t) \]

\[ \widehat{B}_t = \int_0^t K(t, s) dB_s \]

- \( K \) is a Volterra kernel with \( \int_0^1 \int_0^t K(t, s)^2 ds dt < \infty \)
- \( B, W \) are standard Brownian motions, \( \rho^2 + \bar{\rho}^2 = 1 \)
- \( \sigma : \mathbb{R} \rightarrow \mathbb{R}_{>0} \) “smooth”
- \( \widehat{B} \) is “small-time self-similar”: for any small \( t > 0 \) there is \( \varepsilon > 0 \) s.t.
  \[ \left. \widehat{B} \right|_{[0, t]} \overset{\text{law}}{=} \varepsilon \left. \widehat{B} \right|_{[0, 1]} \]
- For example: \( K(t, s) = |t - s|^{H-1/2}, 0 < H < \frac{1}{2} \)
Theorem

For $x \geq 0$ the call option price satisfies

\[
c \left( \frac{\varepsilon}{x}, t \right) := E \left[ \left( \exp (X_t) - \exp \left( \frac{\varepsilon}{x} x \right) \right)^+ \right]
= \exp \left( - \frac{I(x)}{\varepsilon^2} \right) \exp \left( \frac{\varepsilon}{x} x \right) J(\varepsilon, x), \quad x \geq 0,
\]

\[
J(\varepsilon, x) := E \left[ e^{-\frac{I'(x)}{\varepsilon^2}} \tilde{U}^\varepsilon \left( e^{\frac{\varepsilon}{x} \tilde{U}^\varepsilon} - 1 \right) e^{I'(x)R_2} 1_{\tilde{U}^\varepsilon \geq 0} \right],
\]

where $\tilde{U}^\varepsilon = \tilde{\varepsilon} g_1 + \tilde{\varepsilon}^2 R_2$ for a centered Gaussian r.v. $g_1$ and a remainder term $R_2$. Moreover, for any $\theta > 0$ and $0 < \beta < H$,

\[
\varepsilon^\theta \log J(\varepsilon, x \varepsilon^{2\beta}) \xrightarrow{\varepsilon \to 0} 0
\]

“uniformly in $x$ around $x = 0$”.
Consider the rough volatility regime $\tilde{\varepsilon} = \varepsilon^{2H}, \ 0 < H \leq \frac{1}{2}$ and moderate deviations $k_t = kt^{1/2-H+\beta}, \ 0 \leq \beta < H$

**Theorem**

\[-\log c(k_t, t) = \frac{I''(0)}{t^{2H-2\beta}} \frac{k^2}{2} (1 + o(1)), \ \ t \searrow 0\]

*with*

\[I''(0) = \frac{1}{\sigma(0)^2}.\]
Consider the rough volatility regime $\hat{\varepsilon} = \varepsilon^{2H}$, $0 < H \leq \frac{1}{2}$ and moderate deviations $k_t = kt^{1/2-H+\beta}$, $0 \leq \beta < \frac{2}{3}H$

**Theorem**

$$- \log c(k_t, t) = \frac{I''(0) k^2}{t^{2h-2\beta}} \frac{2}{2} + \frac{I'''(0) k^3}{t^{2h-3\beta}} \frac{6}{6} (1 + o(1)), \quad t \downarrow 0$$

with

$$I''(0) = \frac{1}{\sigma(0)^2}, \quad I'''(0) = -6\rho \frac{\sigma'(0)}{\sigma(0)^4} \int_0^1 \int_0^t K(t, s) ds dt.$$
Consider the rough volatility regime $\varepsilon = \varepsilon^{2H}, \ 0 < H \leq \frac{1}{2}$ and moderate deviations $k_t = kt^{1/2-H+\beta}, \ 0 \leq \beta < \frac{2}{3}H$

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$$-\log c(k_t, t) = \frac{I''(0)}{t^{2h-2\beta}} \frac{k^2}{2} + \frac{I'''(0)}{t^{2h-3\beta}} \frac{k^3}{6} (1 + o(1)), \quad t \searrow 0$$

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**Corollary**

*The implied volatility satisfies*

$$\sigma_{impl}(k_t, t) = \sigma(0) - \rho \frac{\sigma'(0)}{\sigma(0)} \int_0^1 \int_0^t K(t, s)dsdt \ k_t t^{H-1/2} (1 + o(1)).$$
Numerical evidence

\[ H = 0.3, \beta = 0.175 \]

Asymptotic formula

Cholesky Pricer

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Outline

1 Results

2 Proofs

3 Future work
Large deviations [Forde and Zhang 2016]

\[ dX_t = \sigma(\hat{B}_t) d(\rho W_t + \rho B_t) \季度 \text{drift}. \]

Short time asymptotics: \( X_t \overset{\text{law}}{=} X_1^\varepsilon, \varepsilon = \sqrt{t}, \hat{\varepsilon} = \varepsilon^{2H}, \) with

\[ dX_t^\varepsilon = \sigma(\hat{\varepsilon} B_t) \varepsilon d(\rho W_t + \rho B_t) \]

**Theorem**

\( \hat{X}_1^\varepsilon \define \frac{\varepsilon}{\hat{\varepsilon}} X_1^\varepsilon \) satisfies LDP with speed \( \hat{\varepsilon} \) and rate function

\[
I(x) \define \inf_{f \in H^1_0} \left\{ \frac{1}{2} \left(x - \rho \langle \sigma(K\dot{f}), f \rangle \right)^2 + \frac{1}{2} \|f\|_{H^1_0}^2 \right\}.
\]

**Notation:**

- \( \|f\|_{H^1_0} = \|\dot{f}\|_{L^2[0,1]} \)
- \( (K\dot{f})(t) = \int_0^t K(t, s)\dot{f}(s)ds, f \in H^1_0 \)
Large deviations [Forde and Zhang 2016]

\[ dX_t = \sigma(\hat{B}_t) d(\bar{\rho}W_t + \rho B_t) + \text{drift}. \]

Short time asymptotics: \( X_t \overset{\text{law}}{=} X^\varepsilon_1, \varepsilon = \sqrt{t}, \hat{\varepsilon} = \varepsilon^{2H}, \) with

\[ dX_t^\varepsilon = \sigma(\hat{\varepsilon}\hat{B}_t) \varepsilon d(\bar{\rho}W_t + \rho B_t) \]

**Theorem**

\( \hat{X}_1^\varepsilon := \frac{\varepsilon}{\hat{\varepsilon}} X_1^\varepsilon \) satisfies LDP with speed \( \hat{\varepsilon} \) and rate function

\[
I(x) := \inf_{f \in H^1_0} \left\{ \frac{1}{2} \left( x - \rho \left( \sigma(Kf), f \right) \right)^2 + \frac{1}{2} \| f \|^2_{H^1_0} \right\}.
\]

**Proof.**

- \( d\hat{X}_t^\varepsilon = \sigma(\hat{\varepsilon}\hat{B}_t) \hat{\varepsilon} d(\bar{\rho}W_t + \rho B_t) \). Hence, \( \hat{X}_1^\varepsilon = \Phi_1(\varepsilon W, \varepsilon B, \varepsilon B) \).
- Use (extended) extension principle based on LDP for \((W, B, \hat{B})\).
Energy expansion

\[
I(x) = \inf_{f \in H^1_0} \left\{ \frac{1}{2} \left( x - \rho \langle \sigma(Kf), f \rangle \right)^2 + \frac{1}{2} \|f\|_{H^1_0}^2 \right\} = \inf_{f \in H^1_0} I_x(f).
\]

1) First order optimality condition

\[
I : \mathbb{R} \times H^1_0 \to \mathbb{R}_{\geq 0}, (x, f) \mapsto I_x(f) \text{ is smooth in Fréchet sense. Hence, any local minimizer } f \text{ satisfies}
\]

\[
H(x, f) := D_f I_x(f) \cdot f = 0.
\]
Energy expansion

\[ I(x) = \inf_{f \in H_0^1} \left\{ \frac{1}{2} \left( x - \rho \langle \sigma(Kf), f \rangle \right)^2 + \frac{1}{2} \| f \|_{H_0^1}^2 \right\} = \inf_{f \in H_0^1} \mathcal{I}_x(f). \]

1) First order optimality condition

2) Local uniqueness and smoothness of minimizer

By the implicit function theorem, there is a unique \( f = f^x \) satisfying the first order condition in a neighborhood of \( x = 0, \ f = 0. \ x \mapsto f^x \) is smooth.
Energy expansion

\[ I(x) = \inf_{f \in H^1_0} \left\{ \frac{1}{2} \left( x - \rho \langle \sigma(K\dot{f}), f \rangle \right)^2 + \frac{1}{2} \|f\|_{H^1_0}^2 \right\} = \inf_{f \in H^1_0} I_x(f). \]

1) First order optimality condition
2) Local uniqueness and smoothness of minimizer
3) Existence of a minimizer

“Local convexity”: \( D^2 f I_x(0) \cdot (g, g) > 0 \) for any \( g \in H^1_0 \).

Remark: This point is not completely obvious, see the following minimization problem:

\[ \mathcal{G}(f) := \int_0^1 \left[ (f'(s)^2 - 1)^2 + f(s)^2 \right] ds \to \min! \]
Energy expansion

\[ I(x) = \inf_{f \in H^1_0} \left\{ \frac{1}{2} \left( x - \rho \langle \sigma(K\dot{f}), f \rangle \right)^2 + \frac{1}{2} \|f\|_{H^1_0}^2 \right\} = \inf_{f \in H^1_0} I(x)(f). \]

1) First order optimality condition

2) Local uniqueness and smoothness of minimizer

3) Existence of a minimizer

4) Expansion of minimizer \( f^x \) in \( x \to 0 \).

Make ansatz \( f^x_t = \alpha_t x + \beta_t \frac{x^2}{2} + O(x^3) \) and plug into first order condition \( H(x, f^x) = 0 \), yields formulas for \( \alpha, \beta, \ldots \)
Pricing formula

\[ c \left( \frac{\varepsilon}{\bar{\varepsilon}} x, t \right) = E \left[ \left( e^{\frac{\varepsilon}{\bar{\varepsilon}} \bar{X}_1} - e^{\frac{\varepsilon}{\bar{\varepsilon}} x} \right)^+ \right], \quad \bar{X}_1 = \int_0^1 \sigma(\varepsilon \bar{B}) \bar{d}(\rho W + \rho B) \]

1) Perturbation & Girsanov transform

Change measure \( \widehat{\varepsilon}(W, B) \rightarrow \varepsilon(W, B) + (h, f), \) \( h, f \in H^1_0 \) with Girsanov transform \( G_\varepsilon \) transforming \( \bar{X}_1 \rightarrow \bar{Z}_1 \) with

\[
G_\varepsilon = \exp \left( -\frac{1}{\varepsilon} \int_0^1 h dW - \frac{1}{\varepsilon} \int_0^1 h dB - \frac{1}{2\varepsilon^2} \int_0^1 (h^2 + f^2) dt \right)
\]

\[
\bar{Z}_1 = \int_0^1 \sigma(\varepsilon \bar{B} + f) \left[ \bar{d}(\rho W + \rho B) + d(\rho h + \rho f) \right]
\]
Pricing formula

\[ c \left( \frac{\varepsilon}{\varepsilon} x, t \right) = E \left[ \left( e^{\frac{\varepsilon}{\varepsilon} X_1} - e^{\frac{\varepsilon}{\varepsilon} x} \right)^+ \right], \quad \widehat{X}_1 = \int_0^1 \sigma(\widehat{B}) \varepsilon d(\rho W + \rho B) \]

1) Perturbation & Girsanov transform

2) Stochastic Taylor expansion \( \widehat{Z}_1 = x + \varepsilon g_1 + \varepsilon^2 R_2 \)

For \( h, f \) with \( \Phi_1(h, f) = x \) we have the above stochastic Taylor expansion with

\[
g_1 = \int_0^1 \left[ \sigma(\widehat{f}_t) d(\rho W_t + \rho B_t) + \sigma'(\widehat{f}_t) \widehat{B}_t d(\rho h_t + \rho f_t) \right].
\]
Pricing formula

\[ c \left( \frac{\varepsilon}{\varepsilon} x, t \right) = E \left[ \left( e^{\frac{\varepsilon}{\varepsilon} \tilde{X}_1} - e^{\frac{\varepsilon}{\varepsilon} x} \right)^+ \right], \quad \tilde{X}_1 = \int_0^1 \sigma(\varepsilon \tilde{B}) \varepsilon d(\tilde{\rho} W + \rho B) \]

1) Perturbation & Girsanov transform

2) Stochastic Taylor expansion \( \tilde{Z}_1^\varepsilon = x + \tilde{\varepsilon} g_1 + \tilde{\varepsilon}^2 R_2 \)

3) \( \int_0^1 \hat{h}^x dW + \int_0^1 \hat{f}^x dB = I'(x)g_1 \)

Following Ben Arous, we can show that

\[ \int_0^1 \hat{h}^x dW + \int_0^1 \hat{f}^x dB = I'(x)g_1 \]

when \( (h^x, f^x) \) is optimal configuration.
Pricing formula

\[ c \left( \frac{\varepsilon}{\varepsilon} x, t \right) = E \left[ \left( e^{\frac{\varepsilon}{\varepsilon} \tilde{X}_1} - e^{\frac{\varepsilon}{\varepsilon} x} \right)^+ \right], \quad \tilde{X}_1 = \int_0^1 \sigma(\varepsilon B) \varepsilon d(\varrho W + \rho B) \]

1) Perturbation & Girsanov transform

2) Stochastic Taylor expansion \( \tilde{Z}_1^\varepsilon = x + \varepsilon g_1 + \varepsilon^2 R_2 \)

3) \( \int_0^1 \dot{h}^x dW + \int_0^1 \dot{f}^x dB = I'(x) g_1 \)

4) Estimates for \( J(\varepsilon, x) \)

Steps 1-3 lead to the remainder term

\[ J(\varepsilon, x) = E \left[ e^{-\frac{I'(x)}{\varepsilon^2} \tilde{U}^\varepsilon} \left( e^{\frac{\varepsilon}{\varepsilon} \tilde{U}^\varepsilon} - 1 \right) e^{I'(x) R_2} 1_{\tilde{U}^\varepsilon \geq 0} \right], \quad \tilde{U}^\varepsilon = \tilde{Z}_1^\varepsilon - x, \]

which is then estimated from above and below.
Pricing formula

\[ c\left(\frac{\varepsilon}{\varepsilon}x, t\right) = E\left[\left(e^{\frac{\varepsilon}{\varepsilon}\tilde{X}_1} - e^{\frac{\varepsilon}{\varepsilon}x}\right)^+\right], \quad \tilde{X}_1 = \int_0^1 \sigma(\varepsilon\tilde{B})\varepsilon\tilde{d}(\rho W + \rho B) \]

1) Perturbation & Girsanov transform

2) Stochastic Taylor expansion \( \tilde{Z}_1^\varepsilon = x + \varepsilon g_1 + \varepsilon^2 R_2 \)

3) \( \int_0^1 \dot{h}^x dW + \int_0^1 \dot{f}^x dB = I'(x)g_1 \)

4) Estimates for \( J(\varepsilon, x) \)

5) Example: Black-Scholes case

\[
J(\varepsilon, x) = M\left(-\frac{I'(x)\sigma}{\varepsilon} + \varepsilon\sigma\right) - M\left(-\frac{I'(x)\sigma}{\varepsilon}\right) \sim \frac{1}{\sqrt{2\pi}} \frac{\sigma^3\varepsilon^3}{x^2},
\]

with \( M(\alpha) := e^{\alpha^2/2} F(\alpha) \), \( F \) being the c.d.f. of \( \mathcal{N}(0, 1) \)
Estimating the remainder term

Stochastic Taylor expansion gives

\[ \epsilon^2 R_\epsilon^2(t) = \epsilon \int_0^t \left[ \sigma(\epsilon B_s + f_s) - \sigma(f_s) \right] d[\bar{\rho}W_s + \rho B_s] + \text{BV process} \]
Estimating the remainder term

Stochastic Taylor expansion gives

\[ \varepsilon^2 R_2^\varepsilon(t) = \varepsilon \int_0^t [\sigma(\varepsilon B_s + f_s) - \sigma(f_s)] d[\bar{\rho}W_s + \rho B_s] + \text{BV process} =: M_t^\varepsilon \]

For \( M^{\kappa,\varepsilon} := M^\tau \) with \( \tau := \inf \{ t \mid |\varepsilon B_t| \geq \kappa \} \), we have

\[ \frac{d[M^{\kappa,\varepsilon}]_t}{dt} = \varepsilon^2 [\sigma(\varepsilon B_t + f_t) - \sigma(f_t)]^2 \leq \varepsilon^4 \|\sigma'|_{\infty;K}^2 |B^\tau_t|^2 \]
Estimating the remainder term

Stochastic Taylor expansion gives

\[ \varepsilon^2 R_2^\varepsilon(t) = \varepsilon \int_0^t \left[ \sigma(\varepsilon B_s + f_s) - \sigma(f_s) \right] d \left[ \rho W_s + \rho B_s \right] + \text{BV process} =: M_t^\varepsilon \]

For \( M^{\kappa, \varepsilon} := M^\tau \) with \( \tau := \inf \{ t \mid |\varepsilon B_t| \geq \kappa \} \), we have

\[ \frac{d[M^{\kappa, \varepsilon}]}{dt} = \varepsilon^2 \left[ \sigma(\varepsilon B_t + f_t) - \sigma(f_t) \right]^2 \leq \varepsilon^4 \left\| \sigma' \right\|_\infty^2 \left| B^\tau_t \right|^2 \]

As \( \varepsilon^{-2} M^{\kappa, \varepsilon} = O \left( |B^{\kappa, \varepsilon}|^2_{\infty; [0,1]} \right) \), which has exponential tails, BDG inequality implies (for some \( c_1, c_2 > 0 \))

\[ P \left( \left| R_2^\varepsilon(t) \right| > r, \ |\varepsilon B|_{\infty; [0,1]} < \kappa \right) \leq c_1 \exp \left( -c_2 r \right) \]
Outline

1. Results

2. Proofs

3. Future work
Complete price expansion

\[ J(\varepsilon, x) = E \left[ e^{-\frac{I'(x)}{\varepsilon^2} \widehat{U}^\varepsilon} \left( e^{\frac{\varepsilon}{\varepsilon} \widehat{U}^\varepsilon} - 1 \right) e^{I'(x)R_2} 1_{\widehat{U}^\varepsilon \geq 0} \right], \]

- \( \widehat{U}^\varepsilon = \widehat{\varepsilon} g_1 + \widehat{\varepsilon}^2 R_2 \)
- \( g_1 \) given explicitly in terms of optimal configuration \( f^x \)
- \( R_2 \) remainder term in stochastic Taylor expansion; not given explicitly, but we have control of tail behaviour

**Goal**

Obtain precise asymptotics/expansion of \( J(\varepsilon, x), x = x(\varepsilon), \) as \( \varepsilon \downarrow 0. \)

- So far, we have polynomial upper and lower bounds.
- Advantage: no need for heat kernel asymptotics.