UTILITY MAXIMIZATION IN A BINOMIAL MODEL WITH TRANSACTION COSTS: A DUALITY APPROACH BASED ON THE SHADOW PRICE PROCESS

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Abstract. We consider the problem of optimizing the expected logarithmic utility of the value of a portfolio in a binomial model with proportional transaction costs with a long time horizon. By duality methods, we can find expressions for the boundaries of the no-trade-region and the asymptotic optimal growth rate, which can be made explicit for small transaction costs (in the sense of an asymptotic expansion). Here we find that, contrary to the classical results in continuous time, see Janeček and Shreve [Fin. Stoch. 8, 2004], the size of the no-trade-region as well as the asymptotic growth rate depend analytically on the level λ of transaction costs, implying a linear first order effect of perturbations of (small) transaction costs, in contrast to effects of order $\lambda^{1/3}$ and $\lambda^{2/3}$, respectively, as in continuous time models. Following the recent study by Gerhold, Muhle-Karbe and Schachermayer [Fin. Stoch. 2011 (online first)] we obtain the asymptotic expansion by an almost explicit construction of the shadow price process.

1. Introduction

In this paper we consider the problem of optimal investment in a market consisting of two assets, one risk-free asset, the bond, which, for simplicity, is assumed to be constant in time and one stock. More precisely, we assume that the investor wants to maximize her expected utility from final wealth, i.e.,

$$\mathbb{E}\left[U(V_T)\right] \to \max,$$

for a given finite horizon $T > 0$, a given utility function U and, certainly, a given initial wealth, say, x. Here, V_T denotes the value of the portfolio obtained by the investor at time T. In fact, we shall only consider the case of the most tractable utility function, $U(x) = \log(x)$.[1] In this framework, it is known since the seminal work of Merton in 1969 [Mer69] that in a frictionless market in which the price of the risky asset follows a geometrical Brownian motion (with drift μ and volatility σ), it is optimal for the investor to keep the *fraction* of wealth invested in the risky asset, $\varphi_t S_t$, w.r.t. the total portfolio wealth, $\varphi^0_t + \varphi_t S_t$ constant equal to μ/σ^2. In particular, this means that the portfolio has to be constantly re-balanced. Of course, this result fully deserves its fame, but nonetheless it mainly implies that the model of a frictionless financial market in continuous time is not an adequate model of reality in the context of portfolio optimization, since it gives an investment, which would lead to immediate bankruptcy if applied in practice due to the bid-ask spread. Consequently, it is essential to study the optimal investment problem

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[1] It is also possible to carry out our analysis for CRRA utility functions of the form $U(x) = \frac{x^\gamma}{\gamma}$.

under transaction costs, a work undertaken by many authors starting with Magill and Constantinides [MC76]. While actually treating the related problem of optimizing utility from consumption, in this work the main difference to the *Merton rule* has already been established in a heuristic way, namely that an investor optimizing his expected utility keeps the proportion of wealth invested in the stock to total wealth inside of a fixed *interval* instead of fixed single point. Consequently, the investor will not trade actively while the proportion remains inside the interval, suggesting the term "no-trade-region". On the other hand, when the proportion is about to leave the no-trade-region, then the investor will trade stocks for bonds (or conversely) so as to just keep the proportion inside the interval.

Since then, many papers in the finance and mathematical finance literature have treated the problem of portfolio optimization under proportional transaction costs, for instance [DN90], [SS94], [JS04] and [TKA88], to mention some of the most influential ones on the mathematical side. As usual for concave optimization problems, there are essentially two approaches for the analysis: the primal approach, which, in this case, is mostly based on the associated Hamilton-Jacobi-Bellman equation, and the dual approach. Representatives of the former method are the works [SS94] and [JS04], where the (asymptotic) first order effect of the transaction costs to the no-trade-region was found for the utility-from-consumption problem. An elegant formulation of the dual approach is based on the notion of shadow prices, see Kallsen and Muhle-Karbe [KMK10], and we especially mention the inspiring work of Gerhold, Muhle-Karbe and Schachermayer [GMKS11], where asymptotic expansions for the no-trade-region and the asymptotic growth rate were found in a utility-from-terminal-wealth problem. [JS04] and [GMKS11] found the characteristic result that the size of the no-trade-region is of order $\lambda^{1/3}$, where λ is the relative bid-ask-spread.

Almost all of the literature mentioned so far studied the effects of market-friction in the form of proportional transaction costs in the case of markets allowing continuous time trading, more specifically, in a Black-Scholes model. In the context of a discrete model, the problem seems to be less pressing, as infinite trading activities are anyway not possible, which implies that the optimal portfolio strategy of a friction-less, discrete-time model is, at least, admissible in a model with transaction costs. However, also in a discrete-time market, such a portfolio will be far from optimal. We refer to [GJ94] for numerical experiments on the effects of transaction costs in a utility-from-terminal-wealth problem. A thorough analytical and numerical study of the use of dynamic programming was done by Sass [Sas05] allowing for very general structures of transaction costs, including some numerical examples. [CSS06] use the dual approach for their analysis of the value function and the optimal strategy for the super-replication problem of a derivative. In particular, when the transaction costs are large enough, they show that buy-and-hold (or sell-and-hold) strategies are optimal. In the context of super-replication, one should also mention the recent [DS11]. Last but not least, we should also mention [Kus95], where the convergence of the super-replication cost in a binomial model with transaction costs was studied when the binomial model converges weakly to a geometrical Brownian motion.

A binomial model introduced in [CRR79] for the purpose of pricing options is not only a simple model of pedagogical worth but also widely used nowadays by practitioners. The valuation of American options, exotic options and options with dividends are some examples of applications.

The goal of this paper is to study the portfolio optimization problem with transaction costs in the binomial model. For this purpose, we are going to use the shadow price approach of [GMKS11], and, as common in this strand of research, we shall restrict our attention to the problem of a long investment horizon $T \to \infty$. Our contribution is twofold. First of all, we obtain the shadow price process explicitly. This permits us to find explicit terms for the optimal portfolios, the no-trade-region as well as the asymptotic optimal growth rate when the relative bid-ask-spread λ is small, in the sense of asymptotic expansions in terms of λ. Hence, in our approach we know more about the structure of the solution and do not face the curse of dimensionality compared to the backward induction solution of the Bellman equation performed numerically. Our second contribution is we show that, contrary to the continuous case, in a binomial model the first order effect of proportional transaction costs λ to both the no-trade-region and the optimal growth rate is of order λ.[2] Economically, this marked difference can be easily understood, as in a discrete-time model all-too-frequent trading is already hindered by the model itself, which does not allow infinite trading activities.

2. Setting

Let $(\Omega, \mathcal{F}, P)$ denote a probability space large enough that we can define a binomial model $(S_t)_{t\in\mathbb{N}}$ with infinite time horizon.[3] Throughout the paper, the filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ is generated by the process $(S_t)_{t\in\mathbb{N}}$. For simplicity, we assume interest rates $r = 0$. Consequently, the model is free of arbitrage when $u > 1 > d$. Here, we assume that we are given a *re-combining tree*, i.e., $d = 1/u < 1$, but allow for general $0 < p < 1$. (Recall that $S_{t+1} = uS_t$ with probability p and $S_{t+1} = dS_t$ with probability $1 - p$.) While we allow for binomial models with infinite time horizon, in general we shall consider the restriction to a finite time horizon, i.e., $(S_t)_{t=0,\dots,T}$. A portfolio is given by the number φ_t^0 of bonds held at time t (until time $t + 1$) and the number φ_t of stocks.

Moreover, we also have a proportional transaction cost λ, satisfying $0 < \lambda < 1$. That is, for each $t \geq 0$ the bid and ask prices are given by $(1 - \lambda)S_t$ and S_t, respectively.

Before we go to more details about the markets with transaction costs, we recall the log-optimal portfolio in a *generalized* binomial model without transaction costs.

Proposition 2.1. *Let w_t, $t = 1, \dots, T$, be a sequence of independent random variables taking the values ± 1 with positive probabilities each and define a stochastic process $(S_t)_{t=0,\dots,T}$ by some fixed value $S_0 > 0$ and by*

$$S_{t+1} := \begin{cases} u_{t+1} S_t, & w_{t+1} = 1, \\ d_{t+1} S_t, & w_{t+1} = -1, \end{cases}$$

where $u_{t+1} > 1 > d_{t+1} > 0$ are $\sigma(w_1, \dots, w_t)$-measurable random variables and $0 \leq t \leq T - 1$. Then the log*-optimizing portfolio for the stock-price is given in terms of the ratio π_t of wealth invested in stock and total wealth at time t by*

$$\pi_t := \frac{\varphi_t S_t}{\varphi_t^0 + \varphi_t S_t} = \frac{P(w_{t+1} = 1)u_{t+1} + P(w_{t+1} = -1)d_{t+1} - 1}{(u_{t+1} - 1)(1 - d_{t+1})}.$$

Proof. The usual proof in the normal binomial model (see, for instance, [Shr04]) goes through without modifications. □

Next we give a formal definition of a self-financing trading strategy in the binomial model with proportional transaction costs. Note that in a model with transaction costs the initial position of the portfolio, i.e., before the very first trading possibility, matters.

[2] In the continuous case, the first order effects are of order $\lambda^{1/3}$ and $\lambda^{2/3}$, respectively.

[3] In fact, it would be sufficient to consider a family of finite probability spaces $(\Omega_T, \mathcal{F}_T, P_T)$ carrying the binomial model with T periods for any $T \in \mathbb{N}$.

Definition 2.2. *A trading strategy is an adapted $\mathbb{R}^2$-valued process $(\varphi^0_t, \varphi_t)_{-1\le t\le T}$ such that $(\varphi^0_{-1}, \varphi_{-1}) = (x, 0)$. It is called self-financing, if*

$$\varphi^0_t - \varphi^0_{t-1} + (\varphi_t - \varphi_{t-1})S_t \le 0 \text{ and } \varphi^0_t - \varphi^0_{t-1} + (\varphi_t - \varphi_{t-1})(1-\lambda)S_t \le 0,\ 0 \le t \le T.$$

Moreover, it is called admissible, if the corresponding wealth process

$$V_t(\varphi^0, \varphi) := \varphi^0_t + \varphi^+_t(1-\lambda)S_t - \varphi^-_t S_t,\ 0 \le t \le T,$$

is a.s. non-negative.

In general, one would allow for portfolio process with negative values, as long as there is a deterministic lower bound for the wealth. In a setting of log-optimization, however, it makes sense to rule out such strategies as the logarithm assigns utility $-\infty$ to outcomes with negative wealth.

Definition 2.3. *An admissible trading strategy $(\varphi^0_t, \varphi_t)_{-1\le t\le T}$ is called log-optimal on $\{0, \dots, T\}$ for the bid-ask process $((1-\lambda)S, S)$, if*

$$\mathbb{E}[\log(V_T(\psi^0, \psi))] \le \mathbb{E}[\log(V_T(\varphi^0, \varphi))]$$

for all admissible trading strategies (ψ^0, ψ).

Due to technical reasons, it is not easy to solve the above problem for finite $T > 0$, as the optimal strategy will be time-inhomogeneous. As usual in the literature on models with transaction costs, we will instead modify it in Definition 2.5, essentially by letting $T \to \infty$. Here, we introduce the notion of a shadow price process, for which we refer to [KMK10].

Definition 2.4. *A shadow price process for S is an adapted process $\widetilde{S}$ such that $(1-\lambda)S_t \le \widetilde{S}_t \le S_t$ for any $0 \le t \le T$ and the log utility optimizing portfolio (φ^0, φ) for the frictionless market with stock price process $\widetilde{S}$ exists and satisfies*

$$\{\varphi_t - \varphi_{t-1} > 0\} \subseteq \left\{\widetilde{S}_t = S_t\right\},$$
$$\{\varphi_t - \varphi_{t-1} < 0\} \subseteq \left\{\widetilde{S}_t = (1-\lambda)S_t\right\},$$

for all $0 \le t \le T$.

By results from [KMK10], [KMK11], it is known that a shadow price process exists and that the optimal portfolio in the frictionless market given by the shadow price process is, in fact, also optimal in the model with transaction costs. Indeed, the shadow price process can be seen as a solution of the *dual* optimization problem and is intimately related to the notion of a consistent price system. For more background information on dual methods for utility optimization in markets with transaction costs we refer to the lecture notes [Sch11].

Definition 2.5. *Given a shadow price process $\widetilde{S} = (\widetilde{S}_t)_{0\le t\le T}$, an admissible trading strategy $(\varphi^0_t, \varphi_t)_{-1\le t\le T}$ is called log-optimal on $\{0, \dots, T\}$ for the* modified problem *if*

$$\mathbb{E}[\log(\widetilde{V}_T(\psi^0, \psi))] \le \mathbb{E}[\log(\widetilde{V}_T(\varphi^0, \varphi))].$$

for all admissible trading strategies (ψ^0, ψ), where

$$\widetilde{V}_t(\varphi^0, \varphi) := \varphi^0_t + \varphi_t \widetilde{S}_t,\ t \ge 0.$$

Proposition 2.6. *Let $\widetilde{S}$ be a shadow price process for the bid-ask price process $((1-\lambda)S, S)$ and let (φ^0, φ) be its log-optimal portfolio. If $V(\varphi^0, \varphi) \ge 0$, then (φ^0, φ) is also log-optimal for the modified problem.*

Proof. The proof is rather standard and simple. See [GMKS11]. □

Using the above proposition, we obtain that difference between the true and the modified problem is of order λ.

Corollary 2.7. *Let $\widetilde{S}$ be a shadow price process for the bid-ask price process* $((1-\lambda)S, S)$.
(i) If its log-optimal portfolio (φ^0, φ) satisfies $\varphi^0 \geq 0$ and $\varphi \geq 0$, then

$$\sup_{(\psi^0,\psi)} \mathbb{E}[\log(V_T(\psi^0, \psi))] + \log(1-\lambda) \leq \mathbb{E}[\log(V_T(\varphi^0, \varphi))] \leq \sup_{(\psi^0,\psi)} \mathbb{E}[\log(V_T(\psi^0, \psi))].$$

(ii) In general, we can find a positive, bounded random variable $Y = Y(\lambda)$ having a finite, deterministic limit $Y(0) = \lim_{\lambda\to 0} Y(\lambda)$ such that

$$\sup_{(\psi^0,\psi)} \mathbb{E}[\log(V_T(\psi^0, \psi))] + E[\log(1-\lambda Y(\lambda))] \leq \mathbb{E}[\log(V_T(\varphi^0, \varphi))] \leq \sup_{(\psi^0,\psi)} \mathbb{E}[\log(V_T(\psi^0, \psi))].$$

Proof. Here we only give the proof of (i). For the second part we refer to Lemma 5.6. Let (ψ^0, ψ) be any admissible strategy for $((1-\lambda)S, S)$. As $(1-\lambda)S \leq \widetilde{S} \leq S$, we get $V_T(\psi^0, \psi) \leq \widetilde{V}_T(\psi^0, \psi)$. If $\varphi^0 \geq 0$ and $\varphi \geq 0$, then by the same reason we obtain $V_T(\varphi^0, \varphi) \geq (1-\lambda)\widetilde{V}_T(\varphi^0, \varphi)$. Combining these with Proposition 2.6, we obtain

$$\begin{aligned}\mathbb{E}[\log(V_T(\varphi^0, \varphi))] &\geq \mathbb{E}[\log(\widetilde{V}_T(\varphi^0, \varphi))] + \log(1-\lambda)\\ &\geq \mathbb{E}[\log(\widetilde{V}_T(\psi^0, \psi))] + \log(1-\lambda)\\ &\geq \mathbb{E}[\log(V_T(\psi^0, \psi))] + \log(1-\lambda).\end{aligned}$$

□

In particular, Corollary 2.7 implies that both problems coincide in the limit when $T \to \infty$. Intuitively, this is clear, as an additional transaction at a final time T should not matter much when T is large and we have a proper time-rescaling. To make this statement precise, we need to introduce one more notion.

Definition 2.8. *The* optimal growth rate *is defined as*

$$R := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\log\left(V_T(\varphi^{0,T}, \varphi^T)\right)\right],$$

where $(\varphi^{0,T}, \varphi^T)$ denotes the log-optimal portfolio for the time-horizon T.

Intuitively, this means that by trading optimally, the value of the portfolio will grow like e^{RT} on average. Now, Corollary 2.7 obviously implies that we can replace V_T by $\widetilde{V}_T$ and the optimal portfolio by the optimal portfolio of the modified problem.

3. Heuristic construction of the shadow price process

In this section, we are going to construct the shadow price process $\widetilde{S}$ on a heuristic level, which will then be made rigorous in the next section. In particular, we want to stress that most of the assumptions made in this section will be justified in Section 4. Moreover, some rather heuristic and vague constructions shall be made more precise.

Following [GMKS11], we make a particular *ansatz* for the parametrization of the shadow price process.

Assumption 3.1. *The shadow price process $\widetilde{S}$ is a generalized binomial model as introduced in Proposition 2.1. For any* excursion *of the shadow price process $\widetilde{S}$ away from the boundaries given by the bid- and ask-price process, there is a deterministic function g such that $\widetilde{S} = g(S)$ during the excursion, i.e., whenever the shadow price process satisfies*

$\widetilde{S}_t \in \{(1-\lambda)S_t, S_t\}$, $\widetilde{S}_{t+k} \in \{(1-\lambda)S_{t+k}, S_{t+k}\}$ *but* $(1-\lambda)S_{t+h} < \widetilde{S}_{t+h} < S_{t+h}$ *for any* $1 \le h \le k-1$*, then there is a function g such that* $\widetilde{S}_{t+h} = g(S_{t+h})$*,* $1 \le h \le k-1$. [4]

We assume that we start by buying at $t = 0$, i.e., $\widetilde{S}_0 = S_0$. Hence, the relation

$$x = \varphi_0 \widetilde{S}_0 + \varphi_0^0$$

implies $\varphi_0^0 = \frac{cx}{c+1}$ and $\varphi_0 = \frac{x}{(c+1)S_0}$, where $c := \frac{\varphi_0^0}{\varphi_0 S_0}$. Let us note once more that c is treated as a known quantity for the moment.

In the frictionless case, Proposition 2.1 shows that the optimal portfolio is, indeed, determined by c via $\pi = \frac{1}{1+c}$. Here, we treat the market with transaction costs as a *perturbation* of the frictionless market. Therefore, this motivates a parametrization of the portfolio by the fraction c also in that case. Keeping c constant over time requires continuous trading, incurring prohibitive transaction costs. Consequently, we may expect that the optimal portfolio will only be re-balanced when c leaves a certain interval. Our first objective, therefore, is to compute the initial holdings in the optimal portfolio, i.e., the initial c. In what follows, we shall, however, assume that c is known and compute the given transaction costs λ as a function of the parameter — a relation, which is going to be inverted to obtain c.

Next, we construct the shadow price process $\widetilde{S}$ during an *excursion* away from the boundary. For this, we parametrize $\widetilde{S}$ not by time t but by the number n of "net upwards steps" of the underlying price process, i.e., for a given $t \ge 0$, we consider $n = n(t)$ such that $S_t = u^n S_0$, $n \in \mathbb{Z}$, which is possible by our choice of a re-combining binomial tree model, i.e., by $d = u^{-1}$. During the first excursion from the bid-ask boundary, Assumption 3.1 implies that $\widetilde{S}_t = g(S_t)$ for some function g. In particular, since $n(s) = n(t)$ implies that $S_s = S_t$, we have that $\widetilde{S}_t$ will only depend on n, but not on time t itself. Therefore, we may, during the first excursion away from the bid-ask prices, index the shadow price process by n instead of t.

Before constructing the shadow prices in the interior of the bid-ask price interval, let us take a look at the expected behavior of the shadow price process when the stock price falls, i.e., when $n \le 0$. Intuitively, and following [GMKS11], when the stock price gets smaller than the initial price S_0, we have to continue buying stock, i.e., we have $\widetilde{S}_{-n} = S_{-n} = d^n S_0$ for $n \ge 0$, before the first instance of selling stock.[5] We formulate this extended ansatz as a second assumption.

Assumption 3.2. *Given a time $t \ge 0$ at which the number of bonds and stocks in the log-optimal portfolio for the frictionless market in the shadow price process $\widetilde{S}$ needs to be adjusted. Let $t + h$ be the (random) next time of an adjustment of the portfolio in the opposite direction. If $\widetilde{S}_t = S_t$ and $S_u < S_t$, then $\widetilde{S}_u = S_u$ and, conversely, if $\widetilde{S}_t = (1-\lambda)S_t$ and $S_u > S_t$, then $\widetilde{S}_u = (1-\lambda)S_u$, for $t \le u \le t+h$.*

What happens when S_t increases beyond S_0? Intuitively, it seems clear that we will not change the log-optimal portfolio at times t with $S_t > S_0$ except by selling stock, i.e., for positive n we expect to have $(1-\lambda)S_n \le \widetilde{S}_n < S_n$. Thus, during a positive excursion of the stock price process S from S_0, the excursion of the shadow price process away from the bid-ask price boundary will end at $\tau := \min\left\{t \ge 0 \,\middle|\, \widetilde{S}_t = (1-\lambda)S_t\right\}$, assuming that

[4] Note that for different excursions, the functions g are not assumed to be equal. Later on, we will, however, see that those functions can be easily transformed into each other, see Proposition 4.5 together with Proposition 4.4.

[5] Obviously, a positive excursion thereafter will be treated differently as a positive excursion immediately started at time 0, i.e., with different shadow price process.

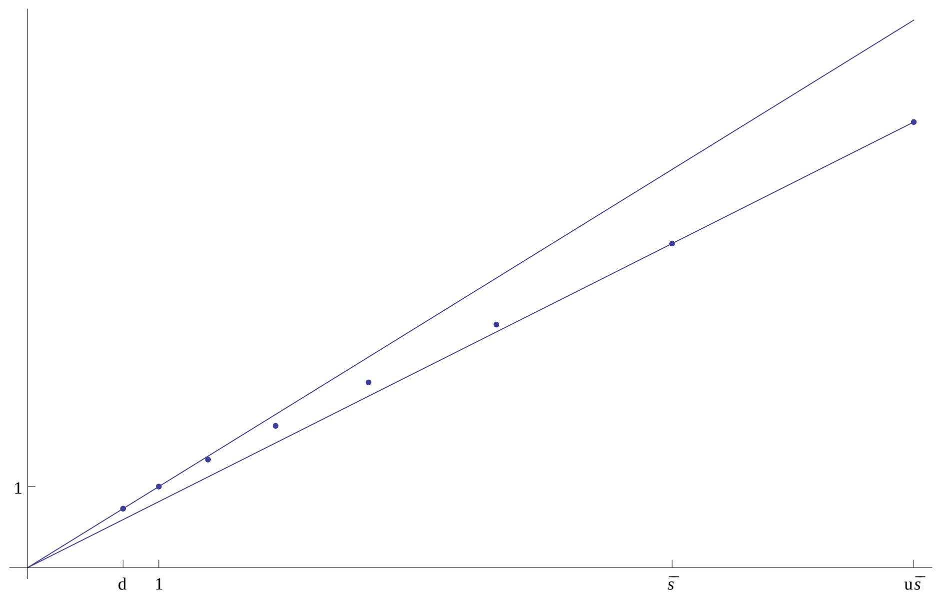

FIGURE 1. A path of the shadow price process

$\forall 0 \le t \le \tau : \ S_t \ge S_0$. We also let $k := n(\tau)$ be the corresponding net number of upwards steps. This means that

$$\frac{\varphi_n^0}{\varphi_n S_0} = c \text{ for } 0 \le n \le k-1$$

As for $0 \le n \le k-1$ the numbers of bonds and stock in the log-optimal portfolio for the market given by $\widetilde{S}$ may not change, Proposition 2.1 implies

$$\pi_n = \frac{\varphi_n \widetilde{S}_n}{\varphi_n^0 + \varphi_n \widetilde{S}_n} = \frac{\widetilde{S}_n}{cS_0 + \widetilde{S}_n} = \frac{p\widetilde{u}_{n+1} + (1-p)\widetilde{d}_{n+1} - 1}{(\widetilde{u}_{n+1} - 1)(1 - \widetilde{d}_{n+1})}. \tag{1}$$

where

$$\widetilde{u}_{n+1} = \frac{\widetilde{S}_{n+1}}{\widetilde{S}_n} \text{ and } \widetilde{d}_{n+1} = \frac{\widetilde{S}_{n-1}}{\widetilde{S}_n}.$$

Solving (1) gives the recursion

$$\widetilde{S}_{n+1} = \frac{\widetilde{S}_n cS_0 + p\widetilde{S}_n\widetilde{S}_{n-1} - cS_0(1-p)\widetilde{S}_{n-1}}{pcS_0 + \widetilde{S}_{n-1} - (1-p)\widetilde{S}_n}, \ \widetilde{S}_0 = S_0 \text{ and } \widetilde{S}_{-1} = S_0 d.$$

Fortunately, we can find an explicit solution for the above recursion. It is given by

$$\widetilde{S}_n = S_0 \frac{c(1 - (\frac{1-p}{p})^n) + \beta_p}{-(1 - (\frac{1-p}{p})^n) + \beta_p} \qquad \text{for } p \ne \frac{1}{2}, \tag{2}$$

$$\widetilde{S}_n = S_0 \frac{cn + \beta}{-n + \beta} \qquad \text{for } p = \frac{1}{2}, \tag{3}$$

where $\beta_p = \frac{(c+d)(2p-1)}{(1-d)(1-p)}$ and $\beta = \frac{c+d}{1-d}$.

When we do not want to parametrize the shadow price process in terms of n, we can still express $\widetilde{S}_t = S_0 g_c(S_t)$ for $0 \le t \le \tau$. Indeed, by $S_n = S_0 u^n$ we see that we can express

n in terms of the stock price s by $n = \frac{\log(s)}{\log(u)}$, and inserting into (2) gives

$$g_c(s) = \frac{c\left(1 - (\frac{1-p}{p})^{-\frac{\log(s)}{\log d}}\right) + \beta_p}{-\left(1 - (\frac{1-p}{p})^{-\frac{\log(s)}{\log d}}\right) + \beta_p} \qquad \text{for } p \neq \frac{1}{2},$$

$$g_c(s) = \frac{c\log(s) + \beta \log u}{-\log(s) + \beta \log u} \qquad \text{for } p = \frac{1}{2}.$$

Note that $g_c(s)$ is increasing, first concave and then convex.

Now we have constructed a candidate for the shadow price process $\widetilde{S}$ which is defined until the first time when it again hits either the bid or the ask price of the true stock. We have also, en passant, settled the case when the process first hits the ask price again: for $n = -1$, we have $\widetilde{S}_{-1} = S_{-1} = dS_0$, and we will buy additional stock and re-start the recursion, but at a different initial value, see the next section for a detailed account. However, when we actually consider the passage from ask to bid price, i.e., when $n = k$ and $\widetilde{S}_k = (1-\lambda)S_k$, we have to decide how to re-balance our portfolio. In practice, the situation will be a bit difficult: most likely, we are not able to follow our explicit formula (2), as it is quite possible that $\widetilde{S}_k < (1-\lambda)S_k$, i.e., that the recursion formula does not hold true anymore for the last step, because it would induce a violation of the first basic property of the shadow price process. In principle, it would be possible to handle this situation. However, it would lead to inherent non-continuities, which would not allow us to use the method of asymptotic expansions. Thus, we assume that the shadow price process *touches* the bid price at an integer point k. (Note that this is really an assumption on the model parameter, not just an ansatz! The assumption will be made more explicit in Assumption 4.3 in the subsequent section.)

Assumption 3.3. *The model parameters (u, d, p, S_0 and λ) are chosen such that* $\widetilde{S}_k = (1-\lambda)u^k S_0$ *and* $\widetilde{S}_{k+1} = (1-\lambda)u^{k+1}S_0$.

The second part of Assumption 3.3 requires some justification. In fact, it reflects a choice on the trading involved at the first opportunity of selling. More precisely, it means that we *do not re-balance the log-optimal portfolio when the shadow price process first hits the bid price*. Only when the stock price increases once more, the shadow price is again equal to the bid price and then we *do trade*. In the discrete time situation, this particular structure of the shadow price process seems arbitrary, but it reflects an important condition in the continuous problem as discussed in [GMKS11], namely the *smooth pasting condition* for the analogous function g in the Black-Scholes model with proportional transaction costs. This condition says that g is continuously differentiable at $\overline{s}$ with $g(\overline{s}) = (1-\lambda)\overline{s}$, i.e., in some sense the shadow price process "smoothly" merges with the bid price process. In continuous time, this assumption is very beneficial in, for instance, avoiding any reference to local times. In the discrete case, other choices are clearly also possible, which lead, inter alia, to different shadow price processes as the one studied by [GMKS11] in the Black-Scholes model seen as a limiting case of the binomial model. Since one of the main motivations for the present model is to study precisely this convergence, we impose the second part of Assumption 3.3.

In the next step, we interpret the two equalities in Assumption 3.3 as a system of equations for the two unknowns k and λ.[6] For $p = \frac{1}{2}$, the solution is given by $k = \frac{(c+d)(c-1)}{c(1-d)}$ and $\lambda = 1 - c^2 d^k$.

For $p \neq \frac{1}{2}$, set $x = \frac{1-p}{p}$ and $y = x^k$. If we eliminate λ from equations, then we obtain

$$\frac{c(1-y)+\beta_p}{-(1-y)+\beta_p} = \frac{c(1-xy)+\beta_p}{-(1-xy)+\beta_p} d \tag{4}$$

which is second order polynomial equation for y. We obtain two solutions y_1 to be given in (5) and $y_2 = \frac{p}{1-p}$ which implies that the net number of upwards steps is $k = -1$. However, for $k = -1$, we indeed solve equation (4), but at the ask-price instead of the bid price. Therefore, the remaining solution must be the appropriate one,

$$y = \frac{[c(p+pd-d)+d(2p-1)][1-p-pd-c(2p-1)]}{c(1-d)^2(1-p)^2}. \tag{5}$$

Hence, $\bar{s} = d^{-k} = y^{-\log(d)/\log(x)}$. Inserting this, we obtain

$$\lambda = \frac{(cp((c+2)d+c)-c(c+1)d)\left(-\frac{c(d-1)^2(p-1)^2}{((c+2)dp-(c+1)d+cp)(c(2p-1)+dp+p-1)}\right)^{-\frac{\log(d)}{\log(x)}}}{c(2p-1)+dp+p-1} + 1 =: F(c). \tag{6}$$

4. Formal construction of the shadow price process

The proofs of most propositions in this section are found in Appendix A. From now on we fix $S_0 = 1, 0 < \lambda < 1$, $1 > d = 1/u > 0$ and $\frac{d}{1+d} < p < \frac{1}{1+d}$. (The last inequality translates to the condition $0 < \mu < \sigma^2$ in the Black-Scholes case. By modifying some of the functions, it is also possible to carry out the whole analysis for the other cases.) Moreover, we denote $\bar{c} = \frac{1-p-pd}{p+pd-d}$ and $b = \frac{\log(d)}{\log((1-p)/p)}$. Note that the optimal wealth fraction π_t in the frictionless binomial model is by Proposition 2.1 given by $\pi_t = \frac{1}{1+\bar{c}}$.

Proposition 4.1. *Define*

$$F(c) := \begin{cases} 1 - \left(\frac{c(p+pd-d)+d(2p-1)}{(1-d)(1-p)}\right)^2 \left(\frac{(c(p+pd-d)+d(2p-1))(1-p-pd-c(2p-1))}{c(1-d)^2(1-p)^2}\right)^{b-1} & \text{for } p \neq \frac{1}{2}, \\ 1 - c^2 d^{\frac{(c+d)(c-1)}{c(1-d)}} & \text{for } p = \frac{1}{2}. \end{cases}$$

Then, $F(c) = \lambda$ has a unique solution in $(\bar{c}, \infty)$ if $p \in (\frac{d}{1+d}, \frac{1}{2}]$, and a unique solution in $(\bar{c}, \frac{1-p-pd}{2p-1})$ if $p \in (\frac{1}{2}, \frac{1}{1+d})$.

As we have c, we can define k and $\bar{s}$. Denote

$$r(c) := \frac{[c(p+pd-d)+d(2p-1)][1-p-pd-c(2p-1)]}{c(1-d)^2(1-p)^2}.$$

Proposition 4.2. *Fix c and define*

$$k := \begin{cases} \frac{\log(r(c))}{\log\left(\frac{1-p}{p}\right)} & \text{for } p \neq \frac{1}{2}, \\ \frac{(c+d)(c-1)}{c(1-d)} & \text{for } p = \frac{1}{2}, \end{cases}$$

and $\bar{s} := u^k$. We have $k > 0$.

Assumption 4.3. *We assume that the model parameter d is given such that k is a positive integer in the above definition.*

---

[6]Recall that we treat λ as an unknown and c as a known quantity with the prospect of inverting the function for λ in terms of c at a later step.

Note that this is the only assumption left from the previous Section 3. A closer look at the definition of k shows the intuitively obvious fact that k converges to infinity when $d \to 1$. Consequently, at least when we are really interested in binomial models with $d \sim 1$, Assumption 4.3 is easy to fulfill by a slight modification of the model parameters.

Proposition 4.4. *Define the function g on $\{d, 1, \dots, \bar{s}, u\bar{s}\}$ by*

$$g(s) := \begin{cases} \dfrac{c\left(1-(\frac{1-p}{p})^{-\frac{\log(s)}{\log d}}\right)+\beta_p}{-\left(1-(\frac{1-p}{p})^{-\frac{\log(s)}{\log d}}\right)+\beta_p}, & \text{for } p \neq \frac{1}{2}, \\ \frac{c\log(s)+\beta\log u}{-\log(s)+\beta\log u}, & \text{for } p = \frac{1}{2}, \end{cases}$$

where $\beta_p = \frac{(c+d)(2p-1)}{(1-d)(1-p)}$ and $\beta = \frac{c+d}{1-d}$. Then g is increasing, maps $\{d, 1, \dots, \bar{s}, u\bar{s}\}$ onto $\{d, 1, \dots, (1-\lambda)\bar{s}, (1-\lambda)u\bar{s}\}$ and satisfies the "smooth pasting" conditions

$$g(d) = d,\ g(1) = 1,\ g(\bar{s}) = (1-\lambda)\bar{s},\ g(u\bar{s}) = (1-\lambda)u\bar{s}. \tag{7}$$

In addition,

$$(1-\lambda)s \leq g(s) \leq s \text{ for } 1 \leq s \leq \bar{s}.$$

Finally, we have

$$\frac{p\frac{g(us)}{g(s)} + (1-p)\frac{g(ds)}{g(s)} - 1}{(\frac{g(us)}{g(s)} - 1)(1 - \frac{g(ds)}{g(s)})} = \frac{g(s)}{c + g(s)} \text{ for } 1 \leq s \leq \bar{s}.$$

Define the sequence of stopping times $(\varrho_n)_{n=0}^{\infty}$, $(\sigma_n)_{n=1}^{\infty}$ and a process $(m_t)_{t\geq 0}$ by

$$\varrho_0 = 1 \text{ and } m_t = \min_{0\leq i\leq t} S_i,\ 0 \leq t \leq \sigma_1,$$

where σ_1 is defined as

$$\sigma_1 = \min\left\{t \geq \varrho_0 : \frac{S_t}{m_t} = \bar{s}\ \&\ \frac{S_{t-1}}{m_{t-1}} = \bar{s}\right\}.$$

Then, define the process $(M_t)_{t\geq 0}$ as

$$M_t = \max_{\sigma_1\leq i\leq t} S_i,\ \sigma_1 \leq t \leq \varrho_1,$$

where ϱ_1 is defined as

$$\varrho_1 = \min\left\{t \geq \sigma_1 : \frac{S_t}{M_t} = \frac{1}{\bar{s}}\ \&\ \frac{S_{t-1}}{M_{t-1}} = \frac{1}{\bar{s}}\right\}.$$

Afterwards, we again pass to the running minimum and define

$$m_t = \min_{\varrho_1\leq i\leq t} S_i,\ \varrho_1 \leq t \leq \sigma_2,$$

where

$$\sigma_2 = \min\left\{t \geq \varrho_1 : \frac{S_t}{m_t} = \bar{s}\ \&\ \frac{S_{t-1}}{m_{t-1}} = \bar{s}\right\}.$$

Then, for $t \geq \sigma_2$, we define

$$M_t = \max_{\sigma_2\leq i\leq t} S_i,\ \sigma_2 \leq t \leq \varrho_2,$$

where

$$\varrho_2 = \min\left\{t \geq \sigma_2 : \frac{S_t}{M_t} = \frac{1}{\bar{s}}\ \&\ \frac{S_{t-1}}{M_{t-1}} = \frac{1}{\bar{s}}\right\}.$$

Proceeding in a similar way, we get the stopping times $(\sigma_n)_{n=1}^{\infty}$, $(\varrho_n)_{n=1}^{\infty}$. Both σ_n and ϱ_n increase a.s. to infinity. Note that these stopping times are indeed attained because S

is a binomial model, $\bar{s} = u^k$ where $k \in \mathbb{N}$ and $\frac{S_0}{m_0} = 1, \frac{S_{\sigma_n}}{M_{\sigma_n}} = 1, \frac{S_{\varrho_n}}{m_{\varrho_n}} = 1$, for $n \geq 1$. Moreover, we see that m_t and M_t are only defined on stochastic intervals $[\![\varrho_{n-1}, \sigma_n]\!]$ and $[\![\sigma_n, \varrho_n]\!]$ respectively. Note that $\bar{s} m_{\sigma_n - 1} = S_{\sigma_n - 1}$ and $M_{\varrho_n - 1} = \bar{s} S_{\varrho_n - 1}$ for $n \geq 1$. Then, we extend the processes M and m to $\mathbb{N}$ by

$$M_t := \bar{s} m_t, \text{ for } t \in \bigcup_{n=1}^{\infty} [\![\varrho_{n-1}, \sigma_n[\![\text{ and } m_t := \frac{M_t}{\bar{s}}, \text{ for } t \in \bigcup_{n=1}^{\infty} [\![\sigma_n, \varrho_n[\![.$$

Therefore, we have

$$m_t \leq S_t \leq \bar{s} m_t \text{ for } t \geq 0.$$

Furthermore, by construction, m decreases only on $\{S_t = m_t\}$ and increases only on $\{S_t = M_t\} = \{S_t = \bar{s} m_t\}$.

Now, we can define a candidate for a shadow price. The result shows that it is a generalized binomial model.

Proposition 4.5. *Define $\widetilde{S}_t = m_t g(\frac{S_t}{m_t}), t \geq 0$. Then, $\widetilde{S}$ is an adapted process which lies in the bid-ask interval $[(1-\lambda)S, S]$. Moreover, consider the multipliers $\widetilde{u}_t$ and $\widetilde{d}_t$ implicitly defined by*

$$\widetilde{S}_{t+1} = \begin{cases} \widetilde{u}_{t+1} \widetilde{S}_t, & S_{t+1} = u S_t, \\ \widetilde{d}_{t+1} \widetilde{S}_t, & S_{t+1} = d S_t, \end{cases}$$

then we have

$$\widetilde{u}_{t+1} = \frac{g(\frac{S_t u}{m_t})}{g(\frac{S_t}{m_t})} > 1 > \widetilde{d}_{t+1} = \frac{g(\frac{S_t d}{m_t})}{g(\frac{S_t}{m_t})}.$$

Proof. $\widetilde{S}$ is adapted because m is adapted. Moreover,

$$1 \leq \frac{S_t}{m_t} \leq \bar{s}, \text{ for } t \geq 0.$$

Also Proposition 4.4 implies that

$$(1-\lambda)s \leq g(s) \leq s \text{ for } 1 \leq s \leq \bar{s}.$$

Hence $\widetilde{S}$ lies in the bid-ask interval. The ratios in the last assertion easily follow in the case $m_t < S_t < \bar{s} m_t$ as m_t does not change. In the cases $S_t = m_t$ and $S_t = \bar{s} m_t$ they follow using $g(d) = d$ and $g(u\bar{s}) = (1-\lambda)u\bar{s}$ respectively. Finally, $\widetilde{u}_{t+1} > 1 > \widetilde{d}_{t+1}$, since g is increasing. □

The log-optimal portfolio can be given in closed form relative to the process m and the sequence of stopping times ϱ and σ.

Theorem 4.6. *Let $\widetilde{S}_t = m_t g\left(\frac{S_t}{m_t}\right)$. Then the* log*-optimizer (φ_t^0, φ_t) in the frictionless market with $\widetilde{S}$ exists and satisfies $(\varphi_{-1}^0, \varphi_{-1}) = (x, 0)$, $(\varphi_0^0, \varphi_0) = (\frac{cx}{c+1}, \frac{x}{c+1})$ and for $t > 0$*

$$\varphi_t^0 = \begin{cases} \varphi^0_{\varrho_{n-1}-1} \left(\frac{c+d}{c+1}\right)^{\frac{\log(m_t) - \log(m_{\varrho_{n-1}-1})}{\log(d)}}, & \text{on } \cup_{n=1}^{\infty} [\![\varrho_{n-1}, \sigma_n[\![, \\ \varphi^0_{\sigma_n - 1} \left(\frac{cd + (1-\lambda)\bar{s}}{c + (1-\lambda)\bar{s}}\right)^{\frac{\log(m_t) - \log(m_{\varrho_{n-1}-1})}{\log(d)}} \frac{m_t}{m_{\sigma_n - 1}}, & \text{on } \cup_{n=1}^{\infty} [\![\sigma_n, \varrho_n[\![, \end{cases}$$

together with

$$\varphi_t = \begin{cases} \varphi_{\varrho_{n-1}-1} \left(\frac{c+d}{c+1}\right)^{\frac{\log(m_t) - \log(m_{\varrho_{n-1}-1})}{\log(d)}} \frac{m_{\varrho_{n-1}-1}}{m_t}, & \text{on } \cup_{n=1}^{\infty} [\![\varrho_{n-1}, \sigma_n[\![, \\ \varphi_{\sigma_n - 1} \left(\frac{cd + (1-\lambda)\bar{s}}{c + (1-\lambda)\bar{s}}\right)^{\frac{\log(m_t) - \log(m_{\varrho_{n-1}-1})}{\log(d)}}, & \text{on } \cup_{n=1}^{\infty} [\![\sigma_n, \varrho_n[\![. \end{cases}$$

Furthermore, the optimal fraction of wealth invested in the stock satisfies

$$\widetilde{\pi}_t = \frac{\varphi_t \widetilde{S}_t}{\varphi_t^0 + \varphi_t \widetilde{S}_t} = \frac{g\left(\frac{S_t}{m_t}\right)}{c + g\left(\frac{S_t}{m_t}\right)}.$$

Proof. We will show that (φ_t^0, φ_t) given above is indeed the log-optimal portfolio. It is clear from the above definition that (φ_t^0, φ_t) is an adapted process. Inductively, we obtain that

$$\varphi_t^0 = c m_t \varphi_t, \text{ for } t \geq 0, \tag{8}$$

both on $\cup_{n=1}^{\infty} [\![\varrho_{n-1}, \sigma_n[\![$ and on $\cup_{n=1}^{\infty} [\![\sigma_n, \varrho_n[\![$. Therefore, the self-financing condition

$$\varphi_{t+1}^0 - \varphi_t^0 + \widetilde{S}_{t+1}(\varphi_{t+1} - \varphi_t) = 0,$$

follows easily when m_t does not change, as then φ_t^0 and φ_t do not change, either. If m_t changes and $t \in \cup_{n=1}^{\infty} [\![\varrho_{n-1}, \sigma_n[\![$, then the self-financing condition follows using (8) and the fact that $\widetilde{S}_t = m_t$ and $\widetilde{S}_{t+1} = m_{t+1} = d m_t$. It follows similarly for $t \in \cup_{n=1}^{\infty} [\![\sigma_n, \varrho_n[\![$. Therefore, (8) implies that the fraction of wealth in the stock is

$$\frac{\varphi_t \widetilde{S}_t}{\varphi_t^0 + \varphi_t \widetilde{S}_t} = \frac{g\left(\frac{S_t}{m_t}\right)}{c + g\left(\frac{S_t}{m_t}\right)}.$$

Now, we prove that the same holds for the log-optimizer and hence by uniqueness we are done. By Proposition 4.5, $\widetilde{S}$ is a generalized binomial model and hence Proposition 2.1 and Proposition 4.4 imply that the fraction of wealth invested in the stock is given by

$$\widetilde{\pi}_t = \frac{p\widetilde{u}_{t+1} + (1-p)\widetilde{d}_{t+1} - 1}{(\widetilde{u}_{t+1} - 1)(1 - \widetilde{d}_{t+1})} = \frac{p\frac{g(u\frac{S_t}{m_t})}{g(\frac{S_t}{m_t})} + (1-p)\frac{g(d\frac{S_t}{m_t})}{g(\frac{S_t}{m_t})} - 1}{(\frac{g(u\frac{S_t}{m_t})}{g(\frac{S_t}{m_t})} - 1)(1 - \frac{g(d\frac{S_t}{m_t})}{g(\frac{S_t}{m_t})})} = \frac{g\left(\frac{S_t}{m_t}\right)}{c + g\left(\frac{S_t}{m_t}\right)}. \qquad \square$$

Corollary 4.7. *Let $\widetilde{S}_t = m_t g\left(\frac{S_t}{m_t}\right)$. Then $\widetilde{S}_t$ is a shadow price.*

Proof. By definition, m decreases only on $\{S_t = m_t\}$ and increases only on $\{S_t = \bar{s} m_t\}$. Hence, by definition of φ in Theorem 4.6, we obtain

$$\{\varphi_t - \varphi_{t-1} > 0\} \subseteq \{S_t = m_t\} = \{\widetilde{S}_t = S_t\} \text{ and}$$
$$\{\varphi_t - \varphi_{t-1} < 0\} \subseteq \{S_t = \bar{s} m_t\} = \{\widetilde{S}_t = (1-\lambda) S_t\}. \qquad \square$$

5. Asymptotic expansions

Having constructed the shadow price process and the corresponding log-optimal portfolio process in Theorem 4.6, we can now start to reap the benefits. Note, however, that the almost explicit account of the log-optimal portfolio depends on the optimal ratio c between wealth invested in bonds and stocks, respectively. We have implicitly found c as solution of a non-linear equation $\lambda = F(c)$, see (6), but we need a better grip on it to facilitate further understanding of the optimal portfolio under proportional transaction costs λ, which can be gained by formal series expansions. In the following, denote $\eta := \frac{(2p-1)\log(d)}{(1-d)\log((1-p)/p)}$ if $p \neq \frac{1}{2}$ and $\eta := \frac{\log(d)}{-2(1-d)}$ if $p = \frac{1}{2}$.

Remark 5.1. Assuming that we know c, we can find the optimal portfolio and the value function by a simple iteration on the tree in *forward* direction, instead of the typical backward iteration. Thus, the shadow price method can be directly turned into an attractive numerical method by solving the equation for c numerically.

Proposition 5.2. *The optimal ratio of wealth invested in bonds and stocks c has the series expansion*

$$c = \bar{c} + \sum_{i=1}^{\infty} c_i \lambda^i,$$

where all the coefficients c_i can be computed by means of well-known symbolic algorithms. In particular, the first two coefficients are given by

$$c_1 = \frac{\bar{c}(1-p)}{(1+d)\eta - 1} \text{ and } c_2 = \frac{\bar{c}(1-p)\left[(1+d)^2\eta^2 + \frac{d^2+(2-2p)d-1-2p}{1-d}\eta + \frac{2p(p+pd-d)}{1-d}\right]}{2[(1+d)\eta-1]^3}.$$

Proof. We will try to formally invert the power series for λ as a function of c. Since we can only invert such a power series when the 0-order term vanishes, we expand the right hand side of equation (6) around the value $c = \bar{c} = \frac{1-p-pd}{p+pd-d}$, which is the optimal c in the frictionless binomial model.

We only consider the case $p \neq \frac{1}{2}$, the case $p = \frac{1}{2}$ being similar. Using Mathematica [Res10], we do a Taylor expansion

$$\lambda = F(c) = \lambda_1(c-\bar{c}) + \lambda_2(c-\bar{c})^2 + O((c-\bar{c})^3), \tag{9}$$

where

$$\lambda_1 = \frac{(1+d)\eta - 1}{\bar{c}(1-p)}, \quad \lambda_2 = -\frac{(1+d)^2\eta^2 + \frac{d^2+(2-2p)d-1-2p}{1-d}\eta + \frac{2p(p+pd-d)}{1-d}}{2\bar{c}^2(1-p)^2}.$$

Note that all coefficients of the series could, in principle, be found in symbolic form. As the first order term λ_1 does not vanish, the implicit function theorem implies the existence of an analytic local inverse function F^{-1}. The power series coefficients of the inverse function can be found using Lagrange's inversion theorem, see, for instance, [Knu98, p. 527]. Inverting the series (9), we thus obtain obtain a series for c in terms of λ

$$c = \bar{c} + c_1\lambda + c_2\lambda^2 + O(\lambda^3), \tag{10}$$

where

$$c_1 = \frac{1}{\lambda_1} = \frac{\bar{c}(1-p)}{(1+d)\eta - 1}, c_2 = -\frac{\lambda_2}{\lambda_1^3} = \frac{\bar{c}(1-p)\left[(1+d)^2\eta^2 + \frac{d^2+(2-2p)d-1-2p}{1-d}\eta + \frac{2p(p+pd-d)}{1-d}\right]}{2[(1+d)\eta-1]^3}.$$

Again, we note that higher order coefficients can be obtained explicitly using symbolical algorithms. □

Remark 5.3. When $p \geq 1/2$, Proposition 5.2 yields a nice economic interpretation. Indeed, c_1 is positive and increasing in d and decreasing in p. Hence, the investor becomes more conservative in the presence of transaction costs, as $c_1 \geq 0$, and this is more pronounced when d is large or p is small, as in these cases the potential average gains from investment in the risky asset are relatively small. For $p < 1/2$, the situation is less intuitive, as then the optimal fraction c can become negative, and it does so in a singular way – by a jump from $+\infty$ to $-\infty$.

When following the optimal strategy given in Theorem 4.6, the fraction π_t of the total wealth invested in the stock is kept in the interval $[(1+c)^{-1}, (1+c/\bar{s})^{-1}]$, the *no-trade region*.

Theorem 5.4. *The lower and upper boundaries $\underline{\theta}$ and $\overline{\theta}$ of the no-trade-region satisfy the asymptotic expansions*

$$\underline{\theta} := \frac{1}{1+c} = \frac{p+pd-d}{1-d} - \frac{(1-p-pd)(p+pd-d)(1-p)}{((1+d)\eta-1)(1-d)^2}\lambda + O(\lambda^2),$$

$$\overline{\theta} := \frac{1}{1+c/\bar{s}} = \frac{p+pd-d}{1-d} + \frac{(1-p-pd)(p+pd-d)((1+d)\eta-(1-p))}{((1+d)\eta-1)(1-d)^2}\lambda + O(\lambda^2)$$

for $p \neq 1/2$ and

$$\underline{\theta} := \frac{1}{2} - \frac{1}{4}\frac{1-d}{(1+d)\log(d^{-1})-2(1-d)}\lambda + O(\lambda^2),$$

$$\overline{\theta} := \frac{1}{2} + \frac{1}{4}\frac{1-d+(1+d)\log(d)}{2(1-d)+(1+d)\log(d)}\lambda + O(\lambda^2)$$

for $p = 1/2$. The width of the no-trade-region is therefore given by

$$\overline{\theta} - \underline{\theta} = \frac{(1-p-pd)(p+pd-d)(1+d)\eta}{((1+d)\eta-1)(1-d)^2}\lambda + O(\lambda^2)$$

for $p \neq 1/2$ and similarly for $p = 1/2$.

Proof. We again assume $p \neq \frac{1}{2}$, the case $p = \frac{1}{2}$ being similar. We first need to compute the expansion for $\bar{s} = u^k$. Inserting the expansion for c given in Proposition 5.2 into the formula for $\bar{s}$ given in Proposition 4.2, we obtain

$$\bar{s} = 1 + s_1\lambda + O(\lambda^2),$$

where $s_1 = \frac{(1+d)\eta}{(1+d)\eta-1}$ and the further coefficients can, as usually, be computed using symbolic algorithms. Then, again taking advantage of Mathematica [Res10], we find that the lower boundary and the upper boundaries of the no-trade region have the asymptotic series expansions

$$\underline{\theta} = \frac{1}{1+c} = \frac{p+pd-d}{1-d} - \frac{(1-p)(1-p-pd)(p+pd-d)}{((1+d)\eta-1)(1-d)^2}\lambda + O(\lambda^2),$$

$$\overline{\theta} = \frac{1}{1+c/\bar{s}} = \frac{p+pd-d}{1-d} + \frac{(1-p-pd)(p+pd-d)((1+d)\eta-(1-p))}{((1+d)\eta-1)(1-d)^2}\lambda + O(\lambda^2).$$

By subtracting, we get the desired formula for the width of the no-trade region. □

Remark 5.5. Note that the width of the no-trade-region is positive and increasing in d to first order. This makes sense economically as larger d means that the returns in the risky asset are smaller, so it makes sense to be more stringent about the transactions costs. Moreover, to first order the width of the no-trade-region is increasing in p for $p < 1/2$ and decreasing for $p > 1/2$. In other words, the size of the no-trade-regions increases with the "variability" of the stock returns.

Finally, we prove the second part of Corollary 2.7.

Lemma 5.6. *Let (φ^0, φ) be the log-optimal portfolio of the shadow-price process. For λ small enough we can find a positive, bounded random variable $Y = Y(\lambda)$ having a finite, deterministic limit $Y(0) = \lim_{\lambda\to 0} Y(\lambda)$ such that*

$$\sup_{(\psi^0,\psi)} \mathbb{E}[\log(V_T(\psi^0,\psi))] + E[\log(1-\lambda Y(\lambda))] \leq \mathbb{E}[\log(V_T(\varphi^0,\varphi))] \leq \sup_{(\psi^0,\psi)} \mathbb{E}[\log(V_T(\psi^0,\psi))].$$

Proof. It is easy to see that $(1-\xi)\widetilde{V}_T(\varphi^0,\varphi) \le V_T(\varphi^0,\varphi)$ provided that

$$\xi \ge \lambda \max\left(-\left(1-\lambda+\frac{\varphi_T^0}{\varphi_T S_t}\right)^{-1}, \left(1+\frac{\varphi_T^0}{\varphi_T S_T}\right)^{-1}\right) =: \lambda Y(\lambda).$$

Boundedness and positivity of Y now follows from Theorem 4.6 above, and we note that the limit for $\lambda \to 0$ is precisely given by the Merton proportion. The rest of the argument works just as for Corollary 2.7. □

6. The optimal growth rate

In the following, we are going to consider the optimal growth rate as given in Definition 2.8. In the frictionless binomial model, we recall from the proof of Proposition 2.1 that the value of the log-optimal strategy satisfies $V_T = \frac{V_0}{Z_T}$ and hence the expected utility is given by

$$\mathbb{E}[\log(V_T)] = \log(V_0) + T\log\left(\frac{(1+d)p^p(1-p)^{1-p}}{d^p}\right).$$

Therefore, the optimal growth rate satisfies

$$\lim_{T\to\infty}\frac{\mathbb{E}[\log(V_T)]}{T} = \log\left(\frac{(1+d)p^p(1-p)^{1-p}}{d^p}\right). \tag{11}$$

Theorem 6.1. *The optimal growth rate in a binomial model with proportional transaction costs satisfies*

$$R = \frac{c(1-d)}{c^2-d}\log\left(\frac{(c+d)}{\sqrt{d}(c+1)}\right)$$

when $p = \frac{1}{2}$ *and*

$$R = \frac{1-2p}{(1-p)(1-(\frac{p}{1-p})^{k+1})}\left[(1-p)\log\left(\frac{c+d}{c+1}\right) + p\left(\frac{p}{1-p}\right)^k \log\left(\frac{(c+d)p}{(c-1)(1-p)d}\right)\right]$$

otherwise.

Proof. We recall from Proposition 4.5 that up and down factors for $\widetilde{S}$ are $\widetilde{u}_{t+1} = \frac{g(Z_t u)}{g(Z_t)}$ and $\widetilde{d}_{t+1} = \frac{g(Z_t d)}{g(Z_t)}$, where $Z_t := \frac{S_t}{m_t}$. Hence, using Proposition 2.1, we compute the expected log-utility as

$$\begin{aligned}
\mathbb{E}[\log(\widetilde{V}_T)] &= \log(\widetilde{V}_0) - \sum_{t=1}^T \mathbb{E}\left[\log\left(\frac{\widetilde{p}_t}{p}\mathbf{1}_{\{1\}}(w_t) + \frac{\widetilde{q}_t}{1-p}\mathbf{1}_{\{-1\}}(w_t)\right)\right] \\
&= \log(\widetilde{V}_0) - p\sum_{t=1}^T \mathbb{E}\left[\log\left(\frac{1-\widetilde{d}_t}{p\left(\widetilde{u}_t - \widetilde{d}_t\right)}\right)\right] - (1-p)\sum_{t=1}^T \mathbb{E}\left[\log\left(\frac{\widetilde{u}_t - 1}{(1-p)\left(\widetilde{u}_t-\widetilde{d}_t\right)}\right)\right] \\
&= \log(\widetilde{V}_0) - p\sum_{t=1}^T \mathbb{E}\left[\log\left(\frac{g(Z_{t-1}) - g(Z_{t-1}d)}{p\left(g(Z_{t-1}u) - g(Z_{t-1}d)\right)}\right)\right] \\
&\quad - (1-p)\sum_{t=1}^T \mathbb{E}\left[\log\left(\frac{g(Z_{t-1}u) - g(Z_{t-1})}{(1-p)\left(g(Z_{t-1}u) - g(Z_{t-1}d)\right)}\right)\right].
\end{aligned}$$

Now, we know from Proposition 4.4 that $\frac{pg(us)+(1-p)g(ds)-1}{(g(us)-g(s))(g(s)-g(ds))} = \frac{1}{c+g(s)}$ for $1 \le s \le \bar{s}$. Then, an elementary calculation implies

$$\frac{g(s)-g(ds)}{p(g(us)-g(ds))} = \frac{c+g(s)}{c+g(us)}, \qquad \frac{g(us)-g(s)}{(1-p)(g(us)-g(ds))} = \frac{c+g(s)}{c+g(ds)}.$$

Thus, using these identities we obtain that

$$\begin{aligned}
R &= \lim_{T\to\infty}\left\{-p\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\log\left(\frac{c+g(Z_{t-1})}{c+g(Z_{t-1}u)}\right)\right]-(1-p)\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\log\left(\frac{c+g(Z_{t-1})}{c+g(Z_{t-1}d)}\right)\right]\right\}\\
&= -p\mathbb{E}^*\left[\log\left(\frac{c+g(Z_t)}{c+g(uZ_t)}\right)\right]-(1-p)\mathbb{E}^*\left[\log\left(\frac{c+g(Z_t)}{c+g(dZ_t)}\right)\right],
\end{aligned}$$

where the last step is due to the ergodic theorem and $\mathbb{E}^*$ denotes the expectation with respect to the invariant distribution of Z_t. Note that Z_t is a Markov chain with state space $\{1,u,u^2,\dots,u^k\}$ and transition matrix

$$P_{i,j} := \mathbb{P}[Z_{t+1}=u^j|Z_t=u^i] = \begin{cases} p, & j=i+1,\ 0\le i\le k-1,\\ 1-p, & j=i-1,\ 1\le i\le k,\\ p, & j=i=k,\\ 1-p, & j=i=0,\\ 0, & \text{else.}\end{cases}$$

Then the invariant distribution is the solution of $\alpha^T P = \alpha^T$ normalized to $\sum_n \alpha_n = 1$. If $p=\frac{1}{2}$, the solution satisfies $\alpha_n = \frac{1}{k+1}$, for $0\le n\le k$. If $p\neq\frac{1}{2}$, we get $\alpha_n = \frac{1-2p}{(1-p)\left(1-\left(\frac{p}{1-p}\right)^{k+1}\right)}\left(\frac{p}{1-p}\right)^n$, for $0\le n\le k$.

For the remainder of the proof, we assume $p\neq\frac{1}{2}$, the other case being similar. Then, the optimal growth rate becomes

$$\begin{aligned}
R =& -p\mathbb{E}^*\left[\log\left(\frac{c+g(Z_t)}{c+g(uZ_t)}\right)\right]-(1-p)\mathbb{E}^*\left[\log\left(\frac{c+g(Z_t)}{c+g(dZ_t)}\right)\right]\\
=&\mathbb{E}^*\left[\log\left(-\left(1-(\frac{1-p}{p})^{\frac{\log(Z_t)}{\log(u)}}\right)+\beta_p\right)\right]-p\mathbb{E}^*\left[\log\left(-\left(1-(\frac{1-p}{p})^{\frac{\log(uZ_t)}{\log(u)}}\right)+\beta_p\right)\right]\\
&-(1-p)\mathbb{E}^*\left[\log\left(-\left(1-(\frac{1-p}{p})^{\frac{\log(dZ_t)}{\log(u)}}\right)+\beta_p\right)\right]\\
=&\frac{1-2p}{(1-p)(1-(\frac{p}{1-p})^{k+1})}\Bigg[(1-p)\left(\log(\beta_p)-\log\left(\left(\frac{1-p}{p}\right)^{-1}+\beta_p-1\right)\right)\\
&+p\left(\frac{p}{1-p}\right)^k\left(\log\left(\left(\frac{1-p}{p}\right)^k+\beta_p-1\right)-\log\left(\left(\frac{1-p}{p}\right)^{k+1}+\beta_p-1\right)\right)\Bigg]\\
=&\frac{1-2p}{(1-p)(1-(\frac{p}{1-p})^{k+1})}\left[(1-p)\log\left(\frac{c+d}{c+1}\right)+p\left(\frac{p}{1-p}\right)^k\log\left(\frac{(c+d)p}{(c-1)(1-p)d}\right)\right]. \qquad \square
\end{aligned}$$

Writing k in terms of c and plugging in the series expansion for c, we get

Corollary 6.2. *The optimal growth rate has the expansion*

$$\begin{aligned}
R =& \log\left(\frac{(1+d)p^p(1-p)^{1-p}}{d^p}\right)\\
&+\frac{(p+pd-d)(1-p-pd)-(1+d)^2(1-p)p\log\left(\frac{(1+d)^2(1-p)p}{d}\right)}{(1-d^2)\left[(1+d)\eta-1\right]}\lambda+O(\lambda^2).
\end{aligned}$$

Remark 6.3. The first order correction term in Corollary 6.2 is negative, reflecting the trivial observation that transaction costs reduce the optimal growth rate. Moreover, contrary to the width of the no-trade-region, the term is decreasing in d and increasing in p for $p > 1/2$ and decreasing for $p < 1/2$. Thus, the optimal growth rate is most effected by transactions costs, when the model is close to the Black-Scholes model.

7. Conclusions

Using the shadow price approach, we compute the optimal trading strategy, the no-trade region and the optimal growth rate for a binomial model under proportional transaction costs λ for λ small, in the sense of an asymptotic expansion. (In fact, the results are fully explicit up to the solution of a rather complicated, non-linear equation (6).) Comparing these results to the corresponding results for the Black-Scholes model, we see a markedly different effect of small transaction costs in discrete time compared to continuous time: to first order, the size of the no-trade region as well as the optimal growth rate depend only linearly on λ, instead of dependence of the order $\lambda^{1/3}$ and $\lambda^{2/3}$, respectively, as in the continuos time case. This result is intuitive, as the punishment for all-too-frequent trading in discrete time is less severe than in continuous time – where blindly following the Merton rule would lead to negative infinite utility under transaction costs.

Despite the very different asymptotic expansions, it is not very difficult to see that both the no-trade regions (and thus the optimal trading strategies) and the optimal growth rate will finally converge to the ones obtained in [GMKS11] for the Black-Scholes model, if one approximates a Black-Scholes model by binomial models on finer and finer grids in the usual way.

Nevertheless, we think that the connections between continuous and discrete time modelling of financial markets under transaction costs should be further studied. In particular, it would be highly desirable to study the effects of transaction costs on a continuous time model, when trading is restricted to a discrete grid of trading times. Indeed, portfolio managers would often only check individual positions and portfolios, say, once per day or even once per week, implying that such discrete trading times seem to have high practical relevance. Unfortunately, an extension of the techniques of this paper to such a model is not immediate.

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Appendix A. Proofs of Some Theorems

Proof of Proposition 4.1. For $p = \frac{1}{2}$, we have $\bar{c} = 1$ and $F(\bar{c}) = F(1) = 0$. Moreover,

$$F'(c) = -\frac{\log(d)}{1-d} d^{\frac{(c+d)(c-1)}{c(1-d)}} \left[c^2 + d + 2\frac{1-d}{\log(d)} c \right].$$

We see that F is increasing on $[x_1, \infty)$ where $x_1 = \frac{1-d}{-\log(d)} + \sqrt{(\frac{1-d}{-\log(d)})^2 - d}$ is the larger root of the parabola $c^2 + d + 2\frac{1-d}{\log(d)} c$. Elementary calculus shows that $1 > x_2$. Hence, we conclude that there exists a unique $c > 1$ s.t. $F(c) = \lambda$.

Now, let $p \neq \frac{1}{2}$. Denote $c_1 = \frac{d(1-2p)}{p+pd-d}$ and $c_2 = \frac{1-p-pd}{2p-1}$ which are the roots of $c(p + pd - d) + d(2p - 1)$ and $1 - p - dp - c(2p - 1)$, respectively. Moreover, denote

$$r(c) = \frac{[c(p - d + pd) + d(2p - 1)][1 - p - dp - c(2p - 1)]}{c(1 - d)^2(1 - p)^2}. \tag{12}$$

We see that $F(\bar{c}) = 0$ and

$$F'(c) = (2p - 1)(p + pd - d)\left(\frac{c^2(1 + b) - 2c_2 c + (b - 1)d\bar{c}}{\left[1 - p - dp - c(2p - 1)\right]^2} \right) r(c)^b.$$

If $\frac{1}{2} < p < \frac{1}{1+d}$, then $c_1 < 0 < \bar{c} < c_2$ and $b > 1$. Note that $r(c) > 0$ for $\bar{c} < c < c_2$ and $r(c) \to 0$ for $c \uparrow c_2$. Hence, we obtain $F(c) \to 1$ for $c \uparrow c_2$. Intermediate value theorem implies that there is a c on $(\bar{c}, c_2)$. s.t. $F(c) = \lambda$.

We see that if $\bar{c} < c < c_2$, then the sign of the parabola $c^2(1 + b) - 2c_2 c + (b - 1)d\bar{c}$ determines the sign of F'. If the parabola has no root, then $F'(c) > 0$ for $\bar{c} < c < c_2$. Recalling $F(\bar{c}) = 0$, we conclude that there exists a unique c on $(\bar{c}, c_2)$ s.t. $F(c) = \lambda$. If the parabola has a root, then the smaller root x_1 satisfies

$$x_1 \leq \frac{c_2}{1 + b} < \frac{c_2}{c_2 + 2} \leq \bar{c}.$$

Hence, depending on whether $\bar{c} < x_2$ or not, F decreases on $(\bar{c}, x_2)$ and increases on (x_2, c_2) or only increases on $(\bar{c}, c_2)$. Due to $F(\bar{c}) = 0$, in both cases, we get that there exists a unique c on $(\bar{c}, c_2)$ s.t. $F(c) = \lambda$.

If $\frac{d}{1+d} < p < \frac{1}{2}$, then $c_2 < 0 < c_1 < \bar{c}$ and $b < -1$. Note that $r(c) > 0$ for $c > \bar{c}$ and

$$\frac{r(c)}{\frac{c(p+pd-d)+d(2p-1)}{(1-d)(1-p)}} \to \frac{1-2p}{(1-d)(1-p)} > 0 \text{ for } c \uparrow \infty.$$

Since $b - 1 < -2$, we get $F(c) \to 1$ for $c \uparrow \infty$. Now, intermediate value theorem implies that there is a $c \in (\bar{c}, \infty)$ s.t. $F(c) = \lambda$.

If $c > \bar{c}$, then the sign of F' is the opposite of the sign of the parabola $c^2(1+b) - 2c_2c + (b-1)d\bar{c}$ due to $2p - 1 < 0$. The leading coefficient of the parabola, $1 + b$, is negative. Hence, if the parabola has no root, then $F'(c) > 0$ for $c > \bar{c}$. Hence, there exists a unique c on $(\bar{c}, \infty)$ s.t. $F(c) = \lambda$. If the parabola has a root, then the smaller root x_1 satisfies

$$x_1 \leq \frac{c_2}{1+b} \leq \bar{c},$$

where the last inequality follows due to the fact that the function $w(z) = \frac{z\log(z)}{z-1}$ is increasing on $(1, \infty)$ and hence $w(\frac{1-p}{p}) \leq w(u)$. Hence, by the same argument as in the previous case, we obtain that there exist a unique root on $(\bar{c}, \infty)$. □

Proof of Proposition 4.2. For $p = \frac{1}{2}$, we know from Proposition 4.1 that $c > 1$. As $\frac{(c+d)(c-1)}{c(1-d)}$ is strictly increasing for $c > 0$, we get $k > 0$.

Now let $p \neq \frac{1}{2}$. Recall, how we defined $r(c)$. Differentiation yields

$$r'(c) = \frac{(1-2p)}{(1-d)^2(1-p)^2}\frac{(p+pd-d)c^2 + d(1-p-pd)}{c^2}.$$

If $\frac{1}{2} < p < \frac{1}{1+d}$, then $c_1 < 0 < \bar{c} < c < c_2$. We see that r is strictly decreasing and positive function on $(0, c_2)$. This implies $0 < r(c) < r(\bar{c}) = 1$. As $p > \frac{1}{2}$, we get $k > 0$.

If $\frac{d}{1+d} < p < \frac{1}{2}$, then we note that $c_2 < 0 < c_1 < \bar{c} < c$. Since r is strictly increasing and positive on $[\bar{c}, \infty)$ we get $r(c) > r(\overline{c}) = 1$. Due to $p < \frac{1}{2}$, we obtain $k > 0$. □

Proof of Proposition 4.4. We assume $p \neq \frac{1}{2}$, the other case being similar. To start with, we shall prove that g is well-defined on $\{d, 1, \ldots, \bar{s}, u\bar{s}\}$. If $\frac{d}{1+d} < p < \frac{1}{2}$, then $\frac{1-p}{p} > 1$ and hence the numerator satisfies

$$\begin{aligned} -\left(1 - (\frac{1-p}{p})^{-\frac{\log(s)}{\log d}}\right) + \beta_p &\leq (\frac{1-p}{p})^{k+1} + \beta_p - 1 = \frac{1-p}{p}r(c) + \frac{c(2p-1)+p+pd-1}{(1-d)(1-p)} \\ &= \frac{1-p-pd-c(2p-1)}{(1-d)(1-p)}\frac{(c+1)d(2p-1)}{cp(1-d)} < 0, \end{aligned}$$

which shows that g is well-defined. If $\frac{1}{2} < p < \frac{1}{1+d}$, then $\frac{1-p}{p} < 1$ and so we get

$$\begin{aligned} -\left(1 - (\frac{1-p}{p})^{-\frac{\log(s)}{\log d}}\right) + \beta_p &\geq (\frac{1-p}{p})^{k+1} + \beta_p - 1 = \frac{1-p}{p}r(c) + \frac{c(2p-1)+p+pd-1}{(1-d)(1-p)} \\ &= \frac{1-p-pd-c(2p-1)}{(1-d)(1-p)}\frac{(c+1)d(2p-1)}{cp(1-d)} > 0, \end{aligned}$$

where $1 - p - pd - c(2p-1) > 0$ is due to $c < \frac{1-p-pd}{2p-1}$ (recall Proposition 4.1). As a result, we obtain that g is well-defined.

To show that g is increasing, we calculate

$$g'(s) = \frac{(c+1)\beta_p(\frac{1-p}{p})^{\frac{-\log(s)}{\log(d)}}\log\left(\frac{1-p}{p}\right)}{s\log(d)\left[-\left(1-(\frac{1-p}{p})^{\frac{\log(s)}{\log(u)}}\right)+\beta_p\right]^2}.$$

Here, the denominator is negative since $\log(d) < 0$. The sign of the numerator depends on the signs of β_p and $\log\left(\frac{1-p}{p}\right)$. We easily check that $\beta_p \log\left(\frac{1-p}{p}\right)$ is negative for both cases $\frac{d}{1+d} < p < \frac{1}{2}$ and $\frac{1}{2} < p < \frac{1}{1+d}$. Therefore, we conclude that g is increasing.

Moreover, we observe that elementary calculation shows that g indeed satisfies the "smooth pasting" conditions after plugging the values for c, $\bar{s}$ and β_p.

Now, we show that $(1-\lambda)s \leq g(s) \leq s$ for $1 \leq s \leq \bar{s}$. Define $H(s) = \frac{g(s)}{s}$. Since $H(1) = 1$ and $H(\bar{s}) = g(\bar{s})/\bar{s} = 1 - \lambda$, it is enough to prove that H is decreasing. Calculation yields

$$H'(s) = \frac{c(\frac{1-p}{p})^{\frac{-2\log(s)}{\log(d)}} + \left[(c+1)\beta_p\left(\frac{\log(\frac{1-p}{p})}{\log(d)}\right) + \beta_p(c-1) - 2c\right](\frac{1-p}{p})^{\frac{-\log(s)}{\log(d)}} - (\beta+c)(\beta-1)}{s^2\left[-\left(1-(\frac{1-p}{p})^{\frac{\log(s)}{\log(u)}}\right)+\beta_p\right]^2}.$$

The denominator is positive, hence it suffices to show that the numerator is negative. We observe that the numerator is a parabola in $(\frac{1-p}{p})^{\frac{-\log(s)}{\log(d)}}$ with positive leading coefficient c. Thus, the numerator attains its maximum value at the boundaries of $[1, \bar{s}]$. Denoting the numerator by $N(s)$, we obtain that $N(\bar{s}) = r(c)N(1)$ where

$$N(1) = \beta_p\left((c+1)\left(\frac{\log(\frac{1-p}{p})}{\log(d)} - \frac{2p-1}{(1-p)(1-d)}\right) + \frac{2p-1}{1-p}\right).$$

Since $r(c) > 0$, we are done if we show that $N(1) < 0$. If $\frac{1}{2} < p < \frac{1}{1+d}$, then we obtain $\frac{\log(\frac{1-p}{p})}{\log(d)} < \frac{2p-1}{p(1-d)}$ since the function $\frac{\log(z)}{z-1}$ is decreasing on $(0,1)$ and $0 < d < \frac{1-p}{p} < 1$. Combining this with $c < \frac{1-p-pd}{2p-1}$ (recall Proposition 4.1), we obtain $N(1) < 0$. If $\frac{d}{1+d} < p < \frac{1}{2}$, then by similar arguments we get $\frac{\log(\frac{1-p}{p})}{\log(d)} > \frac{(2p-1)d}{p(1-d)}$. Recalling from Proposition 4.1 that $c > \bar{c}$, we again obtain $N(1) < 0$.

Lastly, denoting $n = \frac{\log(s)}{\log(u)}$ and $x = \frac{1-p}{p}$, we obtain

$$\begin{aligned}
\frac{p\frac{g(us)}{g(s)} + (1-p)\frac{g(ds)}{g(s)} - 1}{(\frac{g(us)}{g(s)} - 1)(1 - \frac{g(ds)}{g(s)})} &= \frac{[p(g(us)-g(s)) + (1-p)(g(ds)-g(s))]g(s)}{[g(us)-g(s)][g(s)-g(ds)]} \\
&= \frac{[p\frac{(c+1)\beta_p(x^n-x^{n+1})}{(x^{n+1}+\beta_p-1)(x^n+\beta_p-1)} + (1-p)\frac{(c+1)\beta_p(x^n-x^{n-1})}{(x^{n-1}+\beta_p-1)(x^n+\beta_p-1)}]g(s)}{\frac{(c+1)\beta_p(x^n-x^{n+1})}{(x^{n+1}+\beta_p-1)(x^n+\beta_p-1)}\frac{-(c+1)\beta_p(x^n-x^{n-1})}{(x^{n-1}+\beta_p-1)(x^n+\beta_p-1)}} \\
&= \frac{(x^n+\beta_p-1)g(s)}{(c+1)\beta_p} = \frac{g(s)}{c+g(s)}.
\end{aligned}$$

□

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