Obstacle Problems and Optimal Control

Exercise sheet 6

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1. Let $J\colon X \to \mathbb{R}$ be Gateaux differentiable and convex, and let $K \subset X$ be a non-empty convex subset. If $x^* \in K$ satisfies

$$J'(x^*)(x^* - x) \leq 0 \qquad \text{for all } x \in K$$

prove that

$$J(x^*) \leq J(x) \text{ for all } x \in K.$$

2. Let U and H be Hilbert spaces. Let $S\colon U \to H$ be a bounded linear map and $y_d \in H$ be given. Fix $\nu \in \mathbb{R}_+$ and define

$$J(u) := \frac{1}{2}\|S(u) - y_d\|_H^2 + \frac{\nu}{2}\|u\|_U^2.$$

Prove that if $\nu > 0$ or S is injective, then J is strictly convex.

(By Theorem 6.3, this gives uniqueness to the optimal control problem (6.1).)

3. Let $V \overset{d}{\hookrightarrow} H \overset{c}{\hookrightarrow} V^*$ be a Gelfand triple and define $S\colon V^* \to V$ as the VI solution mapping: $y = S(u)$ solves

$$y \in K : \langle Ay - u, y - v\rangle \leq 0 \qquad \forall v \in K$$

under all the usual assumptions guaranteeing well posedness. Suppose that

(a) $J\colon V \times H \to \mathbb{R}$ is bounded from below.

(b) If $y_n \to y$ in $V \times V$ and $u_n \rightharpoonup u$ in H, then

$$J(y,u) \leq \liminf_{n\to\infty} J(y_n, u_n).$$

(c) If $\{J(y_n, u_n)\}$ is bounded for a sequence $\{(y_n, u_n)\} \subset V \times U_{ad}$, then $\{u_n\}$ is bounded in H.

Consider the problem

$$\min_{u \in U_{ad}} J(y,u) \quad \text{where} \quad y = S(u) \tag{1}$$

where $U_{ad} \subset H$ is closed, convex and bounded.

Prove that there exists an optimal pair (y^*, u^*) of (1).

4. In this question we will clarify the final part of the proof of Proposition 6.7 where the constraint of belonging to $B_\gamma^H(u^*)$ is removed.

Recall

$$\bar{J}(y,u) := J(y,u) + \frac{1}{2}\|u - u^*\|_H^2$$

and that $(\bar{y}_\epsilon, \bar{u}_\epsilon)$ denotes the optimal point of

$$\min_{u \in U_{ad} \cap B_\gamma^H(u^*)} \bar{J}(S_\epsilon(u), u).$$

(a) Take an arbitrary $w \in B_{\gamma/2}^H(\bar{u}_\epsilon)$. Show that $w \in B_\gamma^H(u^*)$ if ϵ is sufficiently small.

(b) Deduce that if ϵ is sufficiently small then $\bar{u}_\epsilon$ is a local minimiser of

$$\min_{u \in U_{ad}} \bar{J}(S_\epsilon(u), u).$$

5. In the situation of the weak C-stationarity result of Proposition 6.9, let J be of tracking type and let

$$U_{ad} = \{u \in H : u_a \leq u \leq u_b \text{ a.e. in } \Omega\}$$

(such constraints are known as box constraints) for given functions $u_a, u_b \in L^2(\Omega)$.

Show that $u^* \in V$ if u_a and u_b also belong to V.

Hint: use the VI (6.11d) relating u^* and p^*.

6. Set $V := H_0^1(\Omega)$. Show that if $g_n \rightharpoonup g$ in V^* and $s_n \to s$ in V with $s_n \geq 0$ and

$$\langle g_n, v \rangle = 0 \qquad \forall v \in V, \;\; 0 \leq v \leq s_n,$$

then

$$\langle g, v \rangle = 0 \qquad \forall v \in V, \;\; 0 \leq v \leq s.$$

We will use this in the next lecture.
