Obstacle Problems and Optimal Control

Exercise sheet 5

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1. Define the indicator function of a set C as

$$I_C(x) = \begin{cases} 0 & : x \in C \\ \infty & : x \notin C. \end{cases}$$

Let $A \subset X$ be a non-empty, closed and convex subset of a Banach space and define the function $f : X \to \{0, \infty\}$ by $f(x) = I_A(x)$. If $x \in A$, show that f is directionally differentiable at x and characterise its derivative.

2. Let $K \subset V$ be closed and convex in a Hilbert space V and take $u \in K$.

(a) Prove that if $\mu \in V^*$ and $u \in K$ satisfy

$$\langle \mu, v - u \rangle \leq 0 \quad \forall v \in K$$

then $\mu \in T_K(u)^\circ$.

(b) Given $\lambda \in V^*$, prove that

$$\{w \in K - u : \langle \lambda, w \rangle = 0\}^\circ = (R_K(u) \cap \lambda^\perp)^\circ.$$

3. Let now K be a closed convex cone and define

$$\text{lin}(u) = \{tu : t \in \mathbb{R}\}$$

to be the linear space generated by u.

Show that $R_K(u) = K + \text{lin}(u)$.

4. Prove that $\text{cap}(\{a\}) > 0$ if $a \in \Omega \subset \mathbb{R}$.

5. Assuming sufficient regularity of the data and solution, justify heuristically why the critical cone associated to the obstacle problem

$$\mathcal{K}_K(u, \lambda) = \{z \in H_0^1(\Omega) : z \leq 0 \text{ q.e. on } \{u = \psi\} \text{ and } \langle Au - f, z \rangle = 0\}$$

becomes

$$\mathcal{K}_K(u, \lambda) = \{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } \{u = \psi\}\}$$

if we have strict complementarity of the obstacle problem in the sense that the biactive set is empty.

6. Let $u : \Omega \to \mathbb{R}$ and $u_n : \Omega \to \mathbb{R}$ (for $n \geq 1$) be quasi-continuous functions on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. We say that u_n **converges in capacity** to u if

$$\text{cap}(\{|u_n - u| \geq \epsilon\}) \to 0 \quad \text{as } n \to \infty$$

holds for every $\epsilon > 0$.

(a) If $u_n \to u$ in $H_0^1(\Omega)$, prove that u_n converges to u in capacity.
Hint: we have the identity

$$\text{cap}(O) = \inf \left\{ \|\nabla v\|^2_{L^2(\Omega)} : v \in H_0^1(\Omega) \text{ and } v \geq 1 \text{ q.e. on } O \right\}.$$

(b) If $\Omega \subset \mathbb{R}$ (i.e. $d = 1$) and $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, prove that u_n converges to u in capacity. That is, in 1D, weak convergence is sufficient for convergence in capacity.

Convergence in capacity is useful because it allows us to pass to the limit in statements such as

$$f_n \in \{v \in H^1_0(\Omega) : v \geq 0 \text{ q.e. on } \Omega \text{ and } v = 0 \text{ q.e. on } \{w_n = 0\}\}^\circ.$$

If $f_n \rightharpoonup f$ in $H^{-1}(\Omega)$ and $w_n \to w$ in capacity with $w_n, w \in H^1(\Omega)$, then the above implies

$$f \in \{v \in H^1_0(\Omega) : v \geq 0 \text{ q.e. on } \Omega \text{ and } v = 0 \text{ q.e. on } \{w = 0\}\}^\circ.$$

7. Recall again the penalised equation with m_ϵ chosen as the C^1 smoothing presented in the lectures:

$$Au_\epsilon + \frac{1}{\epsilon} m_\epsilon(u_\epsilon - \psi) = f.$$

Let us set $V := H^1_0(\Omega)$ and take $\psi \in V$. If we define $S_\epsilon \colon V^* \to V$ as the solution mapping $f \mapsto u_\epsilon$, the derivative $\alpha_\epsilon := S'_\epsilon(f)(d)$ satisfies

$$A\alpha_\epsilon + \frac{1}{\epsilon} m'_\epsilon(u_\epsilon - \psi)\alpha_\epsilon = d.$$

By testing with α_ϵ, we can show that for a subsequence (that we relabel), $\alpha_\epsilon \rightharpoonup \alpha$ in V to some element α.

(a) We have already shown that $S_\epsilon(f) \to S(f)$ in V where S is the solution mapping of the associated VI.

It is natural to wonder if $S'_\epsilon(f)(d)$ converges (weakly or strongly) to $S'(f)(d)$, at least for a subsequence. Explain why this cannot be true in general.

(b) Show that

$$\langle A\alpha_\epsilon - d, v\rangle = 0 \quad \forall v \in V : v = 0 \text{ a.e. on } \{(u_\epsilon - \psi)^- = 0\}.$$

(c) Deduce that

$$\langle A\alpha - d, v\rangle = 0 \quad \forall v \in V : v = 0 \text{ q.e. on } \{u = \psi\}.$$

This partially characterises the limit of the derivatives (and can be used to fully characterise the limit as a solution map associated to a PDE with measure data).

Hint: use the comment at the end of the previous question.