## **Obstacle Problems and Optimal Control**

## Exercise sheet 4

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1. Let u be a solution of the obstacle problem such that the strong complementarity system

$$\begin{split} \lambda &:= f - Au, \\ \lambda &\geq 0, \\ \lambda(u - \psi) &= 0, \\ u &\leq \psi, \end{split}$$

holds. Show that the complementarity condition above is equivalent to

$$\lambda = (\lambda + \gamma(u - \psi))^+$$
 for any constant  $\gamma > 0$ .

**2**. Let  $T: V \to V^*$  be a bounded linear and coercive operator. Prove that

$$u_n \rightharpoonup u \text{ in } V \implies \langle Tu, u \rangle \leq \liminf_{n \to \infty} \langle Tu_n, u_n \rangle.$$

- **3**. Let  $T: X \to X^*$  be hemicontinuous and monotone. Prove that it is type (M).
- 4. In the setting of §3.7.2 with  $V := H_0^1(\Omega)$ , prove that the map  $m_{\epsilon} \colon V \to V^*$  defined by

$$m_{\epsilon}(u) = u^+$$

is hemicontinuous and satisfies the conditions

$$z_{\epsilon} \rightarrow z \text{ in } V \text{ and } m_{\epsilon}(z_{\epsilon}) \rightarrow 0 \text{ in } V^* \implies z \leq 0,$$
  
 $v \mapsto -\Delta v + \frac{1}{\epsilon} m_{\epsilon}(v - \psi) \text{ is coercive.}$ 

Note that we have the interpretation

$$\langle m_{\epsilon}(u), v \rangle_{V^*, V} = \int_{\Omega} u^+ v.$$

**5**. Recall the penalised PDE (3.16):

$$Au_{\epsilon} + \frac{1}{\epsilon}m_{\epsilon}(u_{\epsilon} - \psi) = f.$$

Suppose  $\psi \in H_0^1(\Omega)$ . Make the substitution  $\hat{u}_{\epsilon} := u_{\epsilon} - \psi$  and study well posedness of the resulting PDE. What is the advantage of this approach in comparison to what was presented in the lecture?

**6**. Let *H* be a Hilbert space and let a non-empty set  $K \subset H$  satisfy

$$K = \{h \in H : (h, g)_H \ge 0 \text{ for all } g \in K\}.$$

- (a) Show that K is closed and convex and verify that  $0 \in K$ .
- (b) A *partial order* on an arbitrary set X is a relation  $\leq$  which satisfies the following for all  $x, y, z \in X$ :
  - $x \le x$  (reflexivity)
  - $x \leq y$  and  $y \leq x$  implies x = y (anti-symmetricity)
  - $x \leq y$  and  $y \leq z$  implies  $x \leq z$  (transitivity).

Show that the relation  $\leq$  defined by

$$h_1 \leq h_2$$
 if and only if  $h_2 - h_1 \in K$ 

induces a partial ordering in the space H.

(c) We write

$$H_+ := K$$

and  $h^+ := P_{H_+}h$  to denote the orthogonal projection of  $h \in H$  onto  $H_+$  and define  $h^- := P_{H_+}(-h)$ . We have the decomposition

$$h = h^+ - h^-.$$

Prove that

$$(h^+, h^-) = 0.$$

(d) Define

$$C := \{ f \in H^* : \langle f, u \rangle \ge 0 \text{ for all } u \in K \}.$$

Show that the relation  $\leq$  defined by

$$f_1 \leq f_2$$
 if and only if  $f_2 - f_1 \in C$ 

induces a partial ordering in the dual space  $H^*$ .

- (e) In the case  $H = L^2(\Omega)$ , give an explicit example of K.
- 7. Let  $F: X \to Y$  be a map between Banach spaces.
  - (a) If  ${\cal F}$  is directionally differentiable and Lipschitz, prove that it is Hadamard differentiable.
  - (b) If F is Hadamard differentiable, prove that  $F'(x) \colon X \to Y$  is continuous.
  - (c) If F is Hadamard differentiable, prove that the limit

$$\lim_{t \to 0^+} \frac{F(x+th) - F(x)}{t} = F'(x)(h)$$

is uniform in h whenever  $h \in C$  belongs to a compact set C.