Obstacle Problems and Optimal Control

Exercise sheet 4

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1. Take $V = H_0^1(\Omega)$. We said that A is T-monotone if it satisfies

$$\langle Au^-, u^+\rangle \leq 0 \qquad \forall u \in V$$

(recall that $u = u^+ - u^-$). Show that this is equivalent to

$$\langle Au^+, u^-\rangle \leq 0 \qquad \forall u \in V.$$

2. Let $T\colon V \to V^*$ be a bounded linear and coercive operator. Prove that

$$u_n \rightharpoonup u \text{ in } V \implies \langle Tu, u\rangle \leq \liminf_{n\to\infty} \langle Tu_n, u_n\rangle.$$

3. Set $V := H_0^1(\Omega)$ and let $u = S(f, \psi)$. If $\{u = \psi\} = \emptyset$, what does this say about u (at least on an intuitive level)?

4. Let $T\colon X \to X^*$ be hemicontinuous and monotone. Prove that it is type (M).

5. Set $V := H_0^1(\Omega)$. Prove that the map $m_\rho\colon V \to V^*$ defined by $m_\rho(u) = u^+$ is monotone, hemicontinuous and satisfies the conditions

$$z_\rho \rightharpoonup z \text{ in } V \text{ and } m_\rho(z_\rho) \to 0 \text{ in } V^* \implies z \leq 0,$$
$$v \mapsto -\Delta v + \frac{1}{\rho} m_\rho(v - \psi) \text{ is coercive.}$$

Note that we have the interpretation

$$\langle m_\rho(u), v\rangle_{V^*,V} = \int_\Omega u^+ v.$$

6. Recall the penalised PDE (3.16):

$$Au_\rho + \frac{1}{\rho} m_\rho(u_\rho - \psi) = f.$$

Suppose $\psi \in H_0^1(\Omega)$. Make the substitution $\hat{u}_\rho := u_\rho - \psi$ and study well posedness of the resulting PDE. What is the advantage of this approach in comparison to what was presented in the lecture?

7. Let H be a Hilbert space and let a non-empty set $K \subset H$ satisfy

$$K = \{h \in H : (h, g)_H \geq 0 \text{ for all } g \in K\}.$$

(a) Show that K is closed and convex and verify that $0 \in K$.

(b) A *partial order* on an arbitrary set X is a relation $\leq$ which satisfies the following for all $x, y, z \in X$:

- $x \leq x$ (reflexivity)
- $x \leq y$ and $y \leq x$ implies $x = y$ (anti-symmetricity)
- $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity).

Show that the relation $\leq$ defined by

$$h_1 \leq h_2 \text{ if and only if } h_2 - h_1 \in K$$

induces a partial ordering in the space H.

(c) We write

$$H_+ := K$$

and $h^+ := P_{H_+}h$ to denote the orthogonal projection of $h \in H$ onto H_+ and define $h^- := P_{H_+}(-h)$. We have the decomposition

$$h = h^+ - h^-.$$

Prove that

$$(h^+, h^-) = 0.$$

(d) Define

$$C := \{f \in H^* : \langle f, u \rangle \geq 0 \text{ for all } u \in K\}.$$

Show that the relation $\leq$ defined by

$$f_1 \leq f_2 \text{ if and only if } f_2 - f_1 \in C$$

induces a partial ordering in the dual space H^*.

(e) In the case $H = L^2(\Omega)$, give an explicit example of K.
