Obstacle Problems and Optimal Control

Exercise sheet 3

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We use $V := H_0^1(\Omega)$ throughout.

1. Define $M(u) := u^+$.

(a) Show that $M\colon L^2(\Omega) \to L^2(\Omega)$ is Lipschitz.

(b) If $u_n \rightharpoonup u$ in V, prove that $u_n^+ \rightharpoonup u^+$ in V, i.e., that M is weakly sequentially continuous in V.

Hint: it may be useful to note that

$$\partial_{x_i} u^+ = \begin{cases} \partial_{x_i} u & : \text{ in } \{u > 0\} \\ 0 & : \text{ in } \{u \leq 0\}. \end{cases}$$

2. We said that A is T-monotone if it satisfies

$$\langle Au^-, u^+ \rangle \leq 0 \qquad \forall u \in V$$

(recall that $u = u^+ - u^-$). Show that this is equivalent to

$$\langle Au^+, u^- \rangle \leq 0 \qquad \forall u \in V.$$

3. Recall the solution map $S\colon V^* \times V \to V$ defined by

$$S(f, \psi) = u$$

where u solves

$$u \in V,\ u \leq \psi : \langle Au - f, u - v \rangle \leq 0 \quad \forall v \in V,\ v \leq \psi$$

and take $A = -\Delta$ to be the Laplacian.

Let $u_0 \in V$ be given. Define the sequence of elements

$$u_n = S(f, u_{n-1}) \quad \text{for } n \geq 1.$$

(a) What can you say about $\{u_n\}$?

(b) We say that $x \in V$ is a *subsolution* of a map $T\colon V \to V$ if $x \leq T(x)$.
Suppose now that u_0 is a subsolution of $S(f, \cdot)$. What does this imply about $\{u_n\}$?

We will use similar kinds of iterative sequences later on.

4. Let $V \subset H \subset V^*$ be a Gelfand triple where $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

(a) Suppose $f, g \in V^*$ and

$$g \leq f \leq g \text{ in } V^*.$$

Using the definition of the dual space inequality, prove that $f = g$.

(b) Suppose we have $u, v, f \in V^*$ such that

$$u \leq f \leq v \quad \text{in } V^*.$$

If $u, v \in H$, prove that $f \in H$ too.

5. Prove the following, which was used in the last lecture.

(a) If $u \in V$ solves

$$u \leq \psi : \int_\Omega \nabla u \cdot \nabla(u - v) \leq \int_\Omega f(u - v) \quad \forall v \in V,\ v \leq \psi,$$

then for $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$, u also solves

$$u \leq \psi : \int_\Omega \nabla u \cdot \nabla(\varphi(u - v)) \leq \int_\Omega f\varphi(u - v) \quad \forall v \in V,\ v \leq \psi.$$

Hint: consider the two cases $\varphi \not\equiv 0$ and $\varphi \equiv 0$.

(b) Taking further $\varphi \leq 1$, the function $\tilde{u} := \varphi u$ satisfies

$$\tilde{u} \leq \varphi\psi : \int_\Omega \nabla \tilde{u} \cdot \nabla(\tilde{u} - v) \leq \int_\Omega \tilde{f}(\tilde{u} - v) \quad \forall v \in V,\ v \leq \varphi\psi$$

with source term

$$\tilde{f} = \varphi f - u\Delta\varphi - 2\nabla u \nabla\varphi.$$
