Obstacle Problems and Optimal Control

Exercise sheet 2

Dr. Amal Alphonse (amal.alphonse@wias-berlin.de)

1. Assuming the result of the Stampacchia theorem, deduce Lax–Milgram.

2. For a given non-negative function $\psi \in L^2(\Omega)$, define the set with gradient constraints

$$K := \{v \in H^1_0(\Omega) : |\nabla v| \leq \psi \text{ a.e. in } \Omega\}.$$

Given a source term $f \in H^{-1}(\Omega)$ and the bilinear form

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v,$$

explain if the VI

$$u \in K : a(u, u-v) \leq \langle f, u-v \rangle \qquad \forall v \in K$$

is well posed via the Stampacchia theorem or not. If it is not, can you see a way to make it well posed by strengthening or adding an extra assumption?

3. The same set up and question as above, except now we are given $\psi_1, \psi_2 \in C^0(\bar{\Omega})$ and we define the set

$$K := \{v \in H^1_0(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ a.e. in } \Omega\}.$$

4. Let $V := H^1_0(\Omega)$. We have a bounded, linear and coercive operator $A\colon V \to V^*$, and suppose we have $f_n \to f$ in V^* and $\psi_n \to \psi$ in V.

For each n, define

$$K_n := \{v \in V : v \leq \psi_n \text{ a.e. in } \Omega\}$$

and define u_n as the solution of the VI

$$u_n \in K_n : \langle Au_n - f_n, u_n - v \rangle \leq 0 \qquad \forall v \in K_n.$$

(a) Prove that there exists some $u \in V$ such that at least for a subsequence (that we relabel),

$$u_n \rightharpoonup u \text{ in } V$$

(b) Prove that in fact u is the solution of the VI

$$u \in K : \langle Au - f, u - v \rangle \leq 0 \qquad \forall v \in K$$

where

$$K := \{v \in V : v \leq \psi \text{ a.e. in } \Omega\}.$$

(c) Can we say that the entire sequence $\{u_n\}$ converges to u (and not just that a subsequence converges)?

5. Suppose we are given functions $u\colon (0,T) \times \Omega \to \mathbb{R}$ and $v\colon (0,T) \times \partial\Omega \to \mathbb{R}$ satisfying

$$\begin{aligned}
-\Delta u(t) &= 0 && \text{in } \Omega\\
\nabla u(t) \cdot \nu + \partial_t v(t) &= 0 && \text{on } \partial\Omega\\
u(t)v(t) &= 0 && \text{on } \partial\Omega\\
v(0) &= v_0 && \text{on } \partial\Omega\\
u(t) \geq 0 \text{ in } \Omega, &\quad v(t) \leq 0 \text{ on } \partial\Omega,
\end{aligned}$$

where $v_0 \leq 0$ is a given initial condition and ν is the unit outward normal.

This kind of problem appears as a reaction/singular limit in a system of PDEs modelling reactions between biological species in a cell. The functions u and $-v$ represent concentrations of the biological species.

For this question you may argue informally and you may just assume everything is smooth enough.

(a) Show that there exists a function z such that

$$\begin{aligned} -\Delta z(t) &= 0 && \text{in } \Omega \\ \nabla z(t)\cdot\nu + v(t) &= v_0 && \text{on } \partial\Omega \\ z(t)v(t) &= 0 && \text{on } \partial\Omega \\ z(t) \geq 0 \text{ in } \Omega, &\quad v(t) \leq 0 \text{ on } \partial\Omega, \end{aligned}$$

You may assume $\chi_{\{z>0\}} = \chi_{\{u>0\}}$.

Hint: define z as a suitable transformation of u.

(b) By constructing an appropriate constraint set K, show that z satisfies a VI.
