Obstacle Problems and Optimal Control

Exercise sheet 1

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1. On $\Omega = (1, \infty)$, consider the function

$$u(x) = \frac{1}{x}.$$

For which $p \geq 1$ does $u \in L^p(\Omega)$?

2. Let $x^* \in \Omega$ be given. If $k > 0$, prove that there is no $g \in L^2(\Omega)$ such that

$$\int_\Omega g\varphi = k\varphi(x^*) \qquad \forall \varphi \in C_c^\infty(\Omega).$$

3. Consider on the domain $\Omega = (0, 2)$ the two functions

$$u(x) = \begin{cases} x & : x \in (0,1) \\ 1 & : x \in [1,2) \end{cases}$$

and

$$v(x) = \begin{cases} x & : x \in (0,1) \\ 10 & : x \in [1,2). \end{cases}$$

(a) What is the best L^p space that u belongs to? That is, what is the largest p such that $u \in L^p(\Omega)$?

(b) Same question for v.

(c) Are u and v weakly differentiable? Prove your claims.

4. On $\Omega = (0, 1)$, consider the function $u(x) = \sqrt{x}$.

(a) Show that $u \in L^2(\Omega)$.

(b) Which (if any) Sobolev space does u belong to?

5. For some given number c and a function $u \colon \mathbb{R} \to \mathbb{R}$ which is defined at the point c, define the Dirac delta functional

$$\delta_c(u) := u(c).$$

With $\Omega = (0, 1)$, prove that $\delta_c \in H^{-1}(\Omega) \equiv H_0^1(\Omega)^*$.

Hint: in 1D, we have that $H^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is a continuous embedding.

6. Show that the norms given by the expressions

$$\|u\|_{H^1(\Omega)}^2 = \int_\Omega |u|^2 + |\nabla u|^2$$

and

$$\|u\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla u|^2$$

(which are equivalent on $H_0^1(\Omega)$) are not equivalent on $H^1(\Omega)$.

7. Define $a \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$.

Explain if this is a bounded and/or coercive bilinear form and if so, derive the boundedness and/or coercivity constants.

8. Define the space

$$X := \left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\}.$$

Prove that there exists a constant C such that

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in X.$$

Hence we have the Poincaré inequality for functions in X too.

Hint: argue by contradiction and use that

(a) $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ is a compact embedding

(b) if $\nabla u = 0$ a.e., u is constant.

9. Let $f \in L^2(\Omega)$. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Using Green's first identity

$$\int_\Omega (\Delta \eta)\varphi = -\int_\Omega \nabla\eta \cdot \nabla\varphi + \int_{\partial\Omega} \varphi \nabla\eta \cdot \nu,$$

derive the weak form and argue well posedness by applying Lax–Milgram (state what the bilinear form and the linear functional are, etc.).

10. Consider the problem

$$\begin{aligned} \Delta^2 u + ku &= f \quad \text{in } \Omega, \\ u = \frac{\partial \Delta u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where k is a constant and $f \in H_0^1(\Omega)$ is given. Here, $\Delta^2 u = \Delta(\Delta u)$. Derive a weak form and discuss well posedness.

Hint: The Hilbert space

$$H := \left\{ u \in H_0^1(\Omega) : \frac{\partial}{\partial x_j} \Delta u \in L^2(\Omega),\ j = 1, \ldots, n \right\}$$

with norm given by

$$\|u\|_H^2 := \|u\|_{H_0^1(\Omega)}^2 + \sum_{j=1}^n \left\| \frac{\partial}{\partial x_j} \Delta u \right\|_{L^2(\Omega)}^2$$

may be useful.