Cryospheric IBVP's In Support of General Circulation Models Of the Earth's Climate Evolution

Kolumban Hutter, Prof. em TU Darmstadt
c/o Laboratory of Hydraulics Hydrology and Glaciology (VAW)
ETH Zurich

Lecture at WIAS, Berlin, 28 January 2013

Introduction

1.) Objects

We discuss large ice masses on our planet Earth:

- **Ice Sheets**: Large Ice masses sitting on solid ground (rock, soil) also called Inland Ices.
  Ex: Greenland, Antarctica, Fenno-Scandinavian, Laurentide Ice sheets

- **Ice Shelves**: (German Schelfeise) Large Ice masses floating on the ocean or lake water and being fed by inland ice, called Grand Barriers by explorers of the 19-20th centuries
  Ex: Ross, Ronne-Filchner Ice Shelves

- Related examples, Glaciers, Marine ice sheets, ice streams
- Not treated here is Sea-Ice = Frozen ocean water

2.) Material Behaviour

Cold Ice: This is a non-Newtonian fluid. Power law fluid,

$$\boldsymbol{D} = A(T)f(II_{\sigma'})\boldsymbol{\sigma'} \qquad f(x) = x^{(n-1)/2} \qquad n = 3$$

Temperate Ice : (T = melting temperature, everywhere). This is a mixture of class I of a viscous fluid with incorporated water bubbles, which can diffuse through the ice matrix. At the bubble walls melting/freezing processes can occur depending on the frictional heat available through viscous stress power.

$$\boldsymbol{D} = A(\omega)f(II_{\sigma'})\boldsymbol{\sigma'} \qquad f(x) = x^{(n-1)/2} \qquad n = 3$$

ω = Water content = $\rho_{water}/\rho_{mixture}$

+ Mass balance for the water, moisture balance equation

Cold and temperate ice often exist in separate domains of the same ice mass. They are then called **Polythermal Ice Masses.**

The cold and temperate domains are separated by a singular surface, across which the the Stefan condition between the jump of the water mass flow and jump in heat flow must be related by the latent heat

This surface is called the COLD TEMPERATE TRANSITION SURFACE (CTS)

The moisture flow in such a mixture of class I is diffusive with extremely small diffusivity. If this diffusivity in zero, then

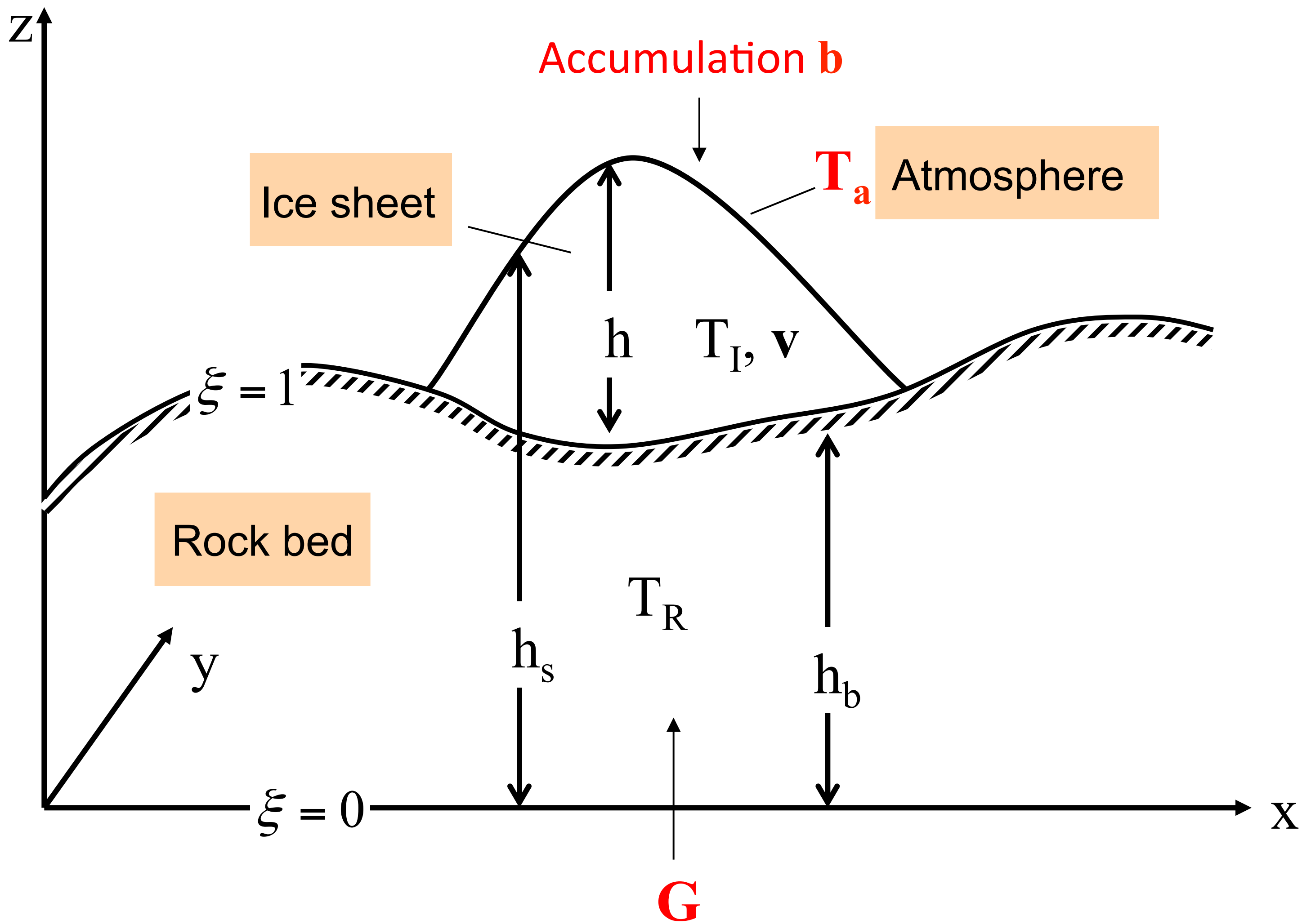
-Moisture balance equation is hyperbolic, not parabolic

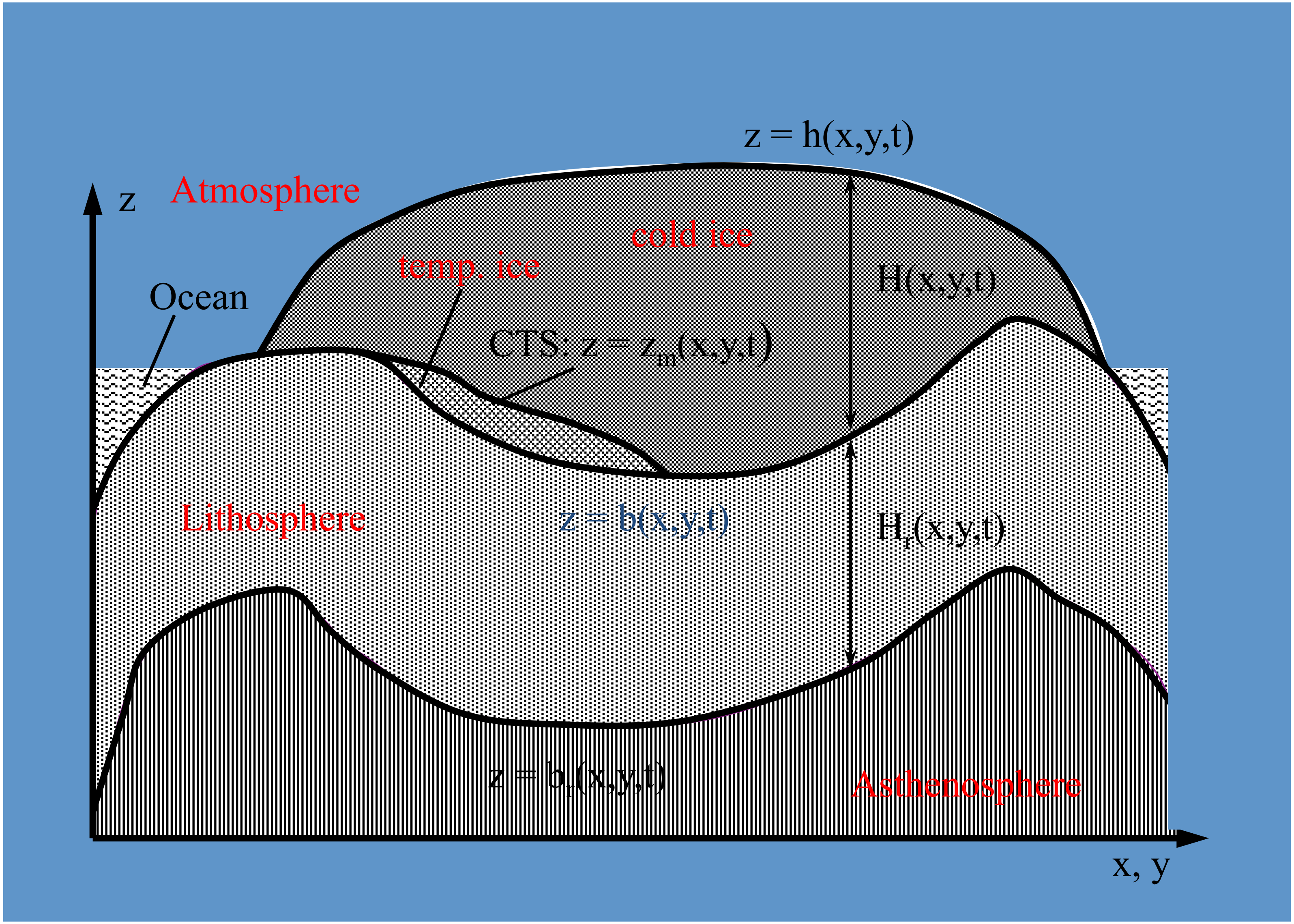
-No Stefan condition at CTS, heat flow across CTS is continuous,

-Zero moisture jump across CTS, $[[\rho\omega]] = 0$

-→ **Some solutions violate observations**

3.) Geometry and the variables, the boundary value problem




z = h(x,y,t)
z
Atmosphere
cold ice
temp. ice
H(x,y,t)
Ocean
CTS: z = z_m(x,y,t)
Lithosphere
z = b(x,y,t)
H_r(x,y,t)
z = b_r(x,y,t)
Asthenosphere
x, y

Field equations (for cold ice sheets)

- Within the ice

$$\operatorname{div} \boldsymbol{v} = 0,$$

$$\rho \dot{\boldsymbol{v}} = -\operatorname{grad} p + \operatorname{div} \boldsymbol{\sigma}' + \rho \mathrm{g},$$

$$\rho \dot{\varepsilon} = -\operatorname{div} \boldsymbol{q} + \operatorname{tr}\left(\boldsymbol{\sigma}' \boldsymbol{D}\right),$$

$$\varepsilon = \int_0^T C_p\left(\bar{T}\right) d\bar{T} + \varepsilon_0, \qquad A\left(T_I'\right) = A_0 \exp\left(-\frac{Q}{R\left(T_0 + T_I'\right)}\right),$$

$$\boldsymbol{q} = -\kappa_I\left(T\right) \operatorname{grad} T, \qquad T' = T_M - T_0 + ap,$$

Enhancement factor

$$\boldsymbol{D} = \operatorname{sym}\left(\operatorname{grad} \boldsymbol{v}\right) = E\, A\left(T'\right) f\left(II_{\boldsymbol{\sigma}'}\right) \boldsymbol{\sigma}', \qquad II_\sigma = \frac{1}{2} \operatorname{tr}\left(\boldsymbol{\sigma}'^2\right).$$

- In the rock bed

$$\rho c_R \dot{T}_R = \operatorname{div}\left(\kappa_R \operatorname{grad} T_R\right) = \kappa_R \Delta T_R.$$

- On the free surface and at the base

$$\frac{\partial h}{\partial t} + \nabla_H \cdot V_H = b_s + b_b, \quad V_H = \int_{h_b}^{h_s} \boldsymbol{v}_{\boldsymbol{H}} dz, \qquad \frac{\partial h_b}{\partial t} = -\frac{1}{\tau_l}\left(h_b - h_0 + \frac{\rho}{\rho_a} h\right).$$

Boundary conditions

- $z = h_s(x, y, t)$: $T = T_s(x, y, z, t), \quad b_s = b_s(x, y, z, t).$

- $z = h_b(x, y, t)$:

$$\kappa_I \frac{\partial T_I}{\partial n} = \kappa_R \frac{\partial T_R}{\partial n}, \quad T_I = T_R < T_M$$

$$b_b = \frac{1}{\rho_w L_s}\left(\kappa_I \frac{\partial T_I}{\partial n} - \kappa_R \frac{\partial T_R}{\partial n} + \boldsymbol{\tau}^* \cdot \boldsymbol{v}_{\|}\right), \quad T_I = T_R = T_M,$$

$$\boldsymbol{v}_{\|} = \begin{cases} C\left(\left|\boldsymbol{\tau}^*\right|, p^*\right)\boldsymbol{\tau}^* = c\left|\boldsymbol{\tau}^*\right|^{m-1} p^{*-l}\boldsymbol{\tau}^*, & if\ T = T_M, \\ 0, & if\ T < T_M, \end{cases}$$

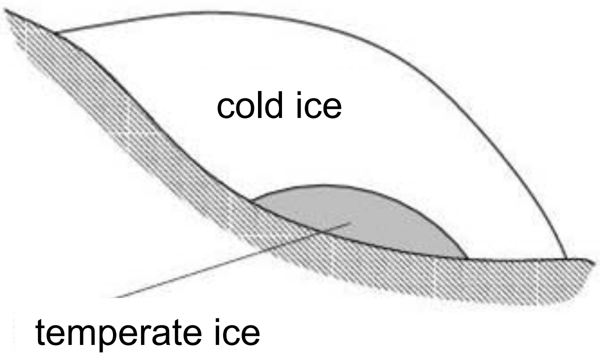
$$\boldsymbol{v}_{\|} = -\left(\boldsymbol{v} - \left(\boldsymbol{v}\cdot\boldsymbol{n}\right)\boldsymbol{n}\right)$$

$$\boldsymbol{\tau}^* = \left(\boldsymbol{\sigma} - p^* 1\right)\boldsymbol{n}.$$

- $z = h_b(x, y)$: $\kappa_R \frac{\partial T_R}{\partial z} = -G.$

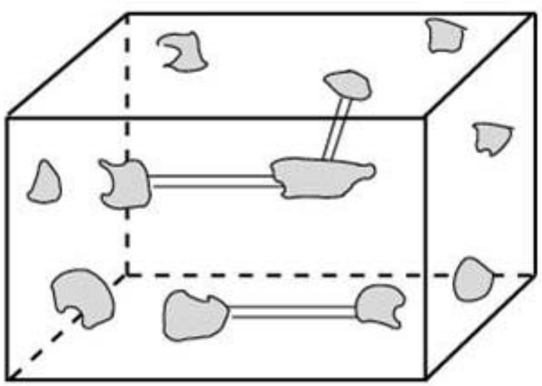
Constitutive relations of temperate ice in polythermal ice masses

Polythermal ice masses consist of disjoint regions with cold and temperate ice, respectively.



Definition: Temperate ice is a mixture of ice and water inclusions, where both ice and water are at the melting point.

The water inclusions are partly connected along the grain boundaries. Thus, the water can diffusively move, if, however, only very slowly.



Definition: Let ρ_w and ρ be the mass densities for the water and the ice-water mixture. The variable

$$\omega \equiv \rho_w / \rho$$

is called the moisture content of temperate ice.

The moisture content in the temperate ice of polythermal ice masses is 3–5 %, no more!

Postulate:

The temperate ice in polythermal ice masses is a binary mixture of ice + water with isolated and connected inclusions.

Remarks:

- The masses of the constituents ice and water are significant, since water diffuses through the „ice matrix“.
- Momentum is balanced for the mixture as a whole. So **v** is the barycentric velocity and ρ the mixture density.
- The mixture energy equation does not determine the temperature, which is at the melting point, but it rather serves as an equation that determines the production of water by the melting and freezing processes.

Assumption: The mixture can be treated as density preserving

Field equations

mixture mass: $div\,\mathbf{v} = 0$

water mass: (moisture content) $\dfrac{d\omega}{dt} = -div\mathbf{j}_\omega + \varsigma_\omega$

mixture momentum: $\rho \dfrac{d\mathbf{v}}{dt} = div\mathbf{T} + \rho\,\mathbf{g}$

mixture energy: $\rho \dfrac{d\varepsilon}{dt} = -div\mathbf{q} + tr(\mathbf{TD})$

Here ω = moisture content

$\mathbf{j}_\omega = \rho\omega(\mathbf{v}_{water} - \mathbf{v})$ moisture flux

$\mathbf{T}$ = Cauchy stress of mixture

ε = internal energy of mixture

$\mathbf{q}$ = heat flux of mixture

Postulate: The binary mixture of temperate ice in polythermal ice masses can be treated as a viscous heat conducting Newtonian-Fourier-Fick fluid with Glen-type constitutive relation for the stress.

Thermodynamics then show

$$\mathbf{j}_\omega = -\rho\, v_\omega \, grad\,\omega, \qquad \text{Fick's first law}$$
$$\mathbf{q} = -k(T)\, grad\, T, \qquad \text{Fourier's law}$$
$$\mathbf{D} = A(\omega)\, f\left(II_{\mathbf{T}_D}\right)\mathbf{T}_D, \qquad \text{Generalized Glen's law}$$

Remarks:

- The rate factor is now a function of the moisture content and not the temperature (which is given by the Clausius-Clapeyron equation)
- It is assumed that the free energy or enthalpy do not depend on the moisture content.Then the Gibbs relation reduces to

$$d\eta = \frac{1}{T}\left(d\varepsilon + p\,d\left(\frac{1}{\rho}\right)\right) \qquad \text{in the interior}$$
$$[[\eta]]\mathrm{T} = [[\varepsilon]] + \mathrm{p}[[1/\rho]] \qquad \text{at phase change surfaces}$$

and the Clausius-Clapeyron equation becomes

$$\frac{dT}{dp} = -c_M(T) \approx \text{constant}$$

where T is always the freezing/melting temperature. So $T - T_0 = -c_M(p - p_0)$

Expression for the internal energy of temperate ice

Entropy

$$\dot{\eta} = \underbrace{\left.\left(\eta_{water} - \eta_{ice}\right)\right|_{melting}}_{\equiv L/T \text{ (latent heat)}} \dot{\omega} = \frac{L}{T}\dot{\omega}$$

Internal energy

$$\dot{\varepsilon} = T\dot{\eta} - \underbrace{p\left(\frac{1}{\rho}\right)^{\bullet}}_{\approx 0} = L\dot{\omega} = \dot{\varepsilon}$$

we also ignore $\rho c\dot{T}$

Thus, the energy equation becomes

$$\rho L\dot{\omega} = div\left(\kappa(T)\, grad T\right) + 2A(\omega) f(II_{TD}) II_{TD}$$

and mass balance reduces to

$$\rho\dot{\omega} = div\,(v_{\omega} grad\,\omega) + \varsigma_{\omega}$$

So: →

$$\varsigma_{\omega} = \frac{1}{L}\left\{2A(\omega) f(II_{TD}) II_{TD}\right\} + \underbrace{\frac{1}{L} div\left(k(T)\, grad T\right) - v_{\omega} div\ grad\,\omega}_{\text{negligible}}$$

$$\varsigma_{\omega} = \frac{1}{L} A(\omega) f(II_{TD}) II_{TD}$$

Jump conditions at the CTS

$[[\, T \,]] = 0$ (Def. of phase change surface)

$[[\mathbf{v}_{\parallel}]] = 0$ (Def. of phase change surface)

$[[\mathcal{M}]] = 0 \;\rightarrow\; \mathcal{M} = \rho(\mathbf{v} - \mathbf{w}).\mathbf{n}$ (Mass balance)

$[[\mathbf{v}]]\mathcal{M} - [[\mathbf{Tn}]] = \mathbf{0} \;\rightarrow\; [[\mathbf{Tn}]] \approx \mathbf{0},$ (Momentum balance)

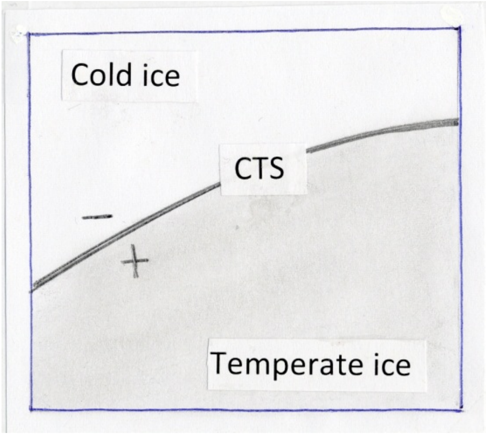
$[[\rho\omega]]\mathcal{M} + [[\mathbf{j}_{\omega}.\mathbf{n}]] = 0$ (mass bal. for water)

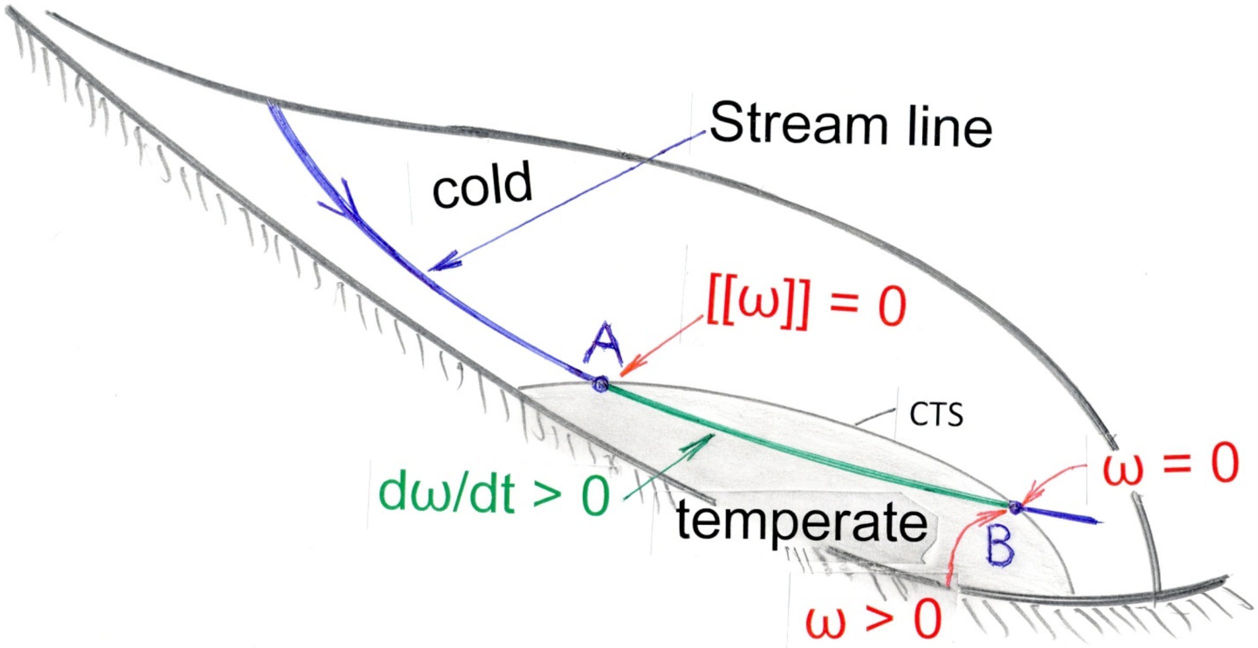
$[[\varepsilon + \underbrace{(\mathbf{v}.\mathbf{v})/2}_{\approx 0}]]\mathcal{M} + [[\mathbf{q}.\mathbf{n}]] - \underbrace{[[\mathbf{v}]].\mathbf{Tn}}_{\approx 0} = 0$ (balance of energy)

$[[\varepsilon]]\mathcal{M} + [[\mathbf{q}.\mathbf{n}]] \approx 0$ $\quad [[\varepsilon]] = L[[\omega]]$

$L[[\omega]]\,\mathcal{M} + [[\mathbf{q}.\mathbf{n}]] = 0 \quad \leftrightarrow \quad L[[\mathbf{j}_{\omega}.\mathbf{n}]] = [[\mathbf{q}.\mathbf{n}]]$

Stefan Condition





Fick's law: $j_\omega = - \rho \nu_\omega \, \mathrm{grad}\omega$

$\nu = 0 \rightarrow$ $[[\mathbf{q}.\mathbf{n}]] = 0$ at CTS
$[[\omega]] = 0$ at CTS

$d\omega/dt = \zeta_\omega = \mathrm{tr}(\boldsymbol{\sigma}'\mathbf{D})/L \geq 0$

At point A, $[[\omega]] = 0$, but along the streamline ω grows, so at B, $[[\omega]] \neq 0$ which Is a contradiction. $\nu \neq 0$ of necessity

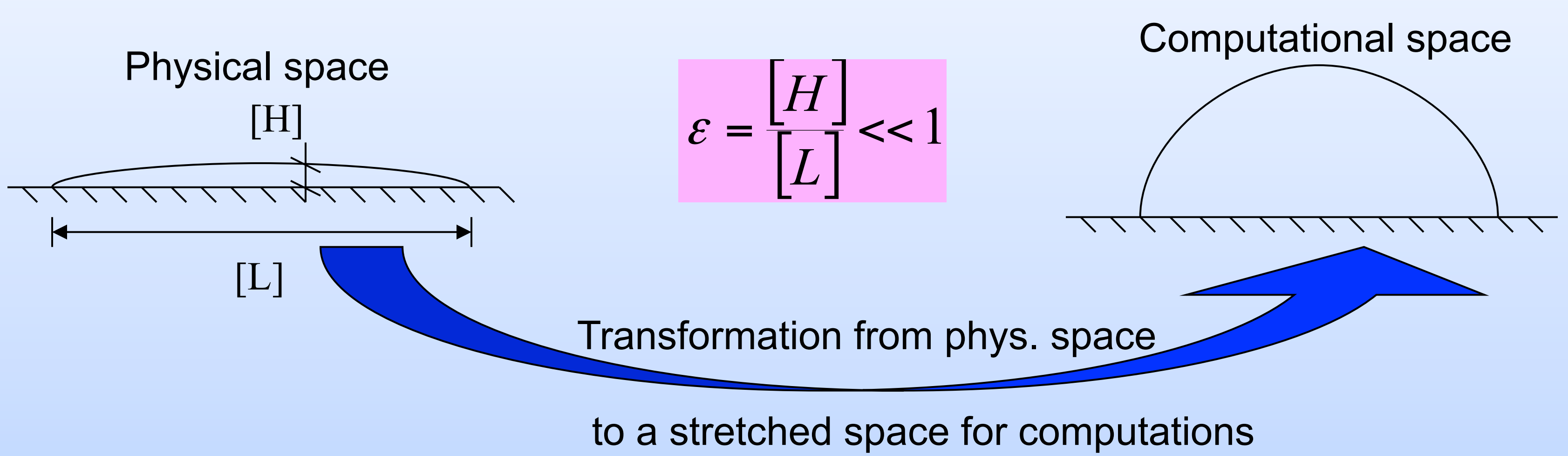
3.) Shallow Ice Approximation for Ice Sheets

3.1 Motivation

Goal: The boundary value problem governing the evolution of cold or temperate ice in polythermal ice sheets is in its general 3D formulation far too complicated that numerical computations can be performed economically over times spanning one or several Ice Ages. Simplifications are called for to achieve this.

Fact: Ice sheets and glaciers are shallow but long and wide, but not always

Method: Use the small aspect ratio ε = (typ. height)/(typ. length) << 1 to simplify the problem.



$$\varepsilon = \frac{[H]}{[L]} << 1$$

Shallow Ice Approximation (SIA)

3.3 The scaling analysis

• Length	horizontal	[L]	10^5-10^6	m
	vertical	[H]	10^2-10^3	m
• Velocities	horizontal	[U]	10^2-10^3	m a^{-1}
	vertical	[W]	1	m a^{-1}
• Stresses (pressure, frictional stresses)		[ρgH]	10^5-10^6	Pa
• Temperature		[T]	273.15°	K
(Temperature range)		[ΔT]	20-30°	K
• Typical stretching		[D]=[a]/[H]	10^{-3}-10^{-1} a^{-1} *	
• Typical material stress		[σ]=ε[ρgH]	10^4-10^5	Pa

*[a] is a typical accumulation rate

Dimensionless variables and characteristic parameters

$$\{x, y, z, t\} = \left\{[L]x^*, [L]y^*, [H]z^*, \frac{[W]}{[H]}t^*\right\},$$

$$\{u, v, w, b_s, b_b\} = \{[U]u^*, [U]v^*, [W]w^*, [W]b_s^*, [W]b_b^*\},$$

$$\{p, \sigma_x', \sigma_y', \sigma_z', \tau_{xy}, \tau_{xz}, \tau_{yz}\} = [\rho gH]\{p^*, \varepsilon^2\sigma_x^{*\prime}, \varepsilon^2\sigma_y^{*\prime}, \varepsilon^2\sigma_z^{*\prime}, \varepsilon^2\tau_{xy}^*, \varepsilon\tau_{xz}^*, \varepsilon\tau_{yz}^*\}$$

$$T = T_0 + [\Delta T]T^*,$$

$$A(T')f(II_{\sigma'}) = \frac{[D]}{[\sigma]}A^*(T^{*\prime})f^*(II_{\sigma'}^*).$$

$$\boxed{[H]/[L] = [W]/[U] = \varepsilon << 1,}$$

$$II_{\sigma'} = [\rho gH]\varepsilon^2\left\{\tau_{xz}^{*\,2} + \tau_{yz}^{*\,2} + \varepsilon^2\left[\frac{1}{2}\left(\sigma_x^{*\prime 2} + \sigma_y^{*\prime 2} + \sigma_z^{*\prime 2}\right) + \tau_{xy}^{*\,2}\right]\right\} = [\rho gH]\varepsilon^2 II^*,$$

$$F = \frac{[U^2]}{g[L]}, \quad G = \frac{\varepsilon^2}{S_\Sigma D_\Delta}, \quad S_\Sigma = \frac{[\sigma]}{[\rho gH]}, \quad D_\Delta = \frac{[W]/[H]}{[D]},$$

$$D = \frac{[\chi_I]}{\rho c_p}\frac{1}{[WH]}, \quad \mathcal{E} = \frac{A}{S_\Sigma D_\Delta}, \quad A = \frac{g[H]}{c_p[\Delta T]}$$

Dimensionless field equations

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}=0 \ ,$$

$$\frac{F}{\varepsilon}\frac{du^*}{dt^*}=-\frac{\partial p^*}{\partial x^*}+\varepsilon^2\frac{\partial \sigma_x^{*\prime}}{\partial x'}+\varepsilon^2\frac{\partial \tau_{xy}^*}{\partial y'}+\frac{\partial \tau_{xz}^*}{\partial z'} \ ,$$

$$\frac{F}{\varepsilon}\frac{dv^*}{dt^*}=\varepsilon^2\frac{\partial \tau_{xy}^*}{\partial x^*}-\frac{\partial p^*}{\partial y^*}+\varepsilon^2\frac{\partial \sigma_y^{*\prime}}{\partial y'}+\frac{\partial \tau_{yz}^*}{\partial z'} \ ,$$

$$F\varepsilon\frac{dw^*}{dt^*}=\varepsilon^2\left\{\frac{\partial \tau_{xz}^*}{\partial x^*}+\frac{\partial \tau_{yz}^*}{\partial y^*}\right\}-\frac{\partial p^*}{\partial z^*}+\varepsilon^2\frac{\partial \sigma_z^{*\prime}}{\partial z'}-1 \ ,$$

$$\frac{dT^*}{dt^*}=D\left\{\varepsilon^2\nabla^*{}_H\left(\chi_I^*\nabla^*{}_H T^*\right)+\frac{\partial}{\partial z^*}\left(\chi_*\frac{\partial T^*}{\partial z^*}\right)\right\}+\varepsilon\, 2A^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)II_{\boldsymbol{\sigma}'} \ ,$$

$$\frac{\partial u^*}{\partial z^*}+\varepsilon^2\frac{\partial w^*}{\partial x^*}=2GA^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)\tau_{xz}^* \ ,$$

$$\frac{\partial v^*}{\partial z^*}+\varepsilon^2\frac{\partial w^*}{\partial y^*}=2GA^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)\tau_{yz}^* \ ,$$

$$\frac{\partial u^*}{\partial x^*}=GA^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)\sigma_x^{*\prime},$$

$$\frac{\partial v^*}{\partial y^*}=GA^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)\sigma_y^{*\prime},$$

$$\frac{\partial w^*}{\partial z^*}=GA^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)\sigma_z^{*\prime},$$

$$\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*}=2GA^*\left(T^*\right)f^*\left(II_{\boldsymbol{\sigma}'}\right)\tau_{xy}^*,$$

$$A^*=\exp\left(-\frac{Q}{RT_0}\frac{1}{1+\frac{[\Delta T]}{T_0}T^*}\right),$$

$$f^*=S\,II^{*(n-1)/2},$$

$$S=\frac{[\sigma](\varepsilon[\rho gH])^{n-1}}{A_0}.$$

- Choice of N leads to the order of the approximation

- *Approximation of order zero (N=0)*
(All terms of order ε^λ, $\lambda \geq 1$ are ignored)

Classical „Shallow Ice Approximation“

$$\Phi = \Phi_0 \quad \text{(SIA)}$$

- *Approximation of order one (N=1)*
All terms of order ε^λ, $\lambda \geq 2$ are ignored. This approximation does not change the field equations, but only the boundary conditions under certain limited conditions. No new physics is obtained.

- *Approximation of second order (N=2)*
(All terms of order ε^λ, $\lambda \geq 3$ are dropped)

„Second Order Shallow Ice Approximation“

$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 \quad \text{(SOSIA)}$$

3.4 Limit Theory: Shallow Ice Approximation (SIA)

Consider the distinguished limit

$$F \to 0,\ F/\varepsilon \to 0,\ \varepsilon \to 0 \qquad \text{(N=0)}$$

Field equations

In the Ice

$$\nabla_H \cdot \boldsymbol{v}_{\boldsymbol{H}} + \frac{\partial w}{\partial z} = 0,$$

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0,$$

$$-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$-\frac{\partial p}{\partial z} = \rho g, \qquad \text{(cryostatic app.)}$$

$$\frac{dT}{dt} = \frac{\partial}{\partial z}\left(\kappa_I \frac{\partial T}{\partial z}\right) + 2EA(T')f\left(\boldsymbol{\tau}^2\right)\boldsymbol{\tau}^2,$$

$$\frac{\partial \boldsymbol{v}_{\boldsymbol{H}}}{\partial z} = 2EA(T')f\left(\boldsymbol{\tau}^2\right)\boldsymbol{\tau} \qquad \boldsymbol{\tau} = \left(\boldsymbol{\tau}_{xz}, \boldsymbol{\tau}_{yz}\right)$$

In the rock bed

$$\dot{T}_R = \frac{\partial}{\partial z}\left(\kappa_R \frac{\partial T_R}{\partial z}\right),$$

Note: E is the so-called **enhancement factor**. It accounts for the fact that pleistocene ice and holocene ice have different fluidities due to different contents of impurities (dust content) and induced anisotropies.

$$p(x,y,z,t) = \rho g\left(h_s(x,y,t) - z\right),$$
$$\boldsymbol{\tau} = \rho g\left(h_s(x,y,t) - z\right)\nabla_H h_s(x,y,t)$$

$$\boldsymbol{v_H}(x,y,z,t) = \boldsymbol{v_H}(h_b) + C\left(z,t,\|\nabla_H h_s\|\right)\nabla_H h_s,$$
$$C\left(z,t,\|\nabla h_s\|\right) = -2\rho g\int_{h_b}^{z} EA\left(T_1'(z')\right)f\left(\boldsymbol{\tau}^2(z')\right)\left(h_s - z'\right)dz',$$
$$w(x,y,z,t) = w(h_b) - \int_{h_b}^{z} \nabla_H \cdot \boldsymbol{v_H}(x,y,z')\,dz',$$
$$\boldsymbol{\tau}^2(x,y,z,t) = \left(\rho g\left(h_s(x,y,t) - z\right)\|\nabla_H h_s\|\right)^2,$$
$$\|\nabla_H h_s\| = \left(\frac{\partial h_s^{\,2}(x,y,t)}{\partial x} + \frac{\partial h_s^{\,2}(x,y,t)}{\partial y}\right),$$

$$\boldsymbol{v_H}(h_b) = \begin{cases} 0, & \text{(cold ice)} \\ -c(\rho g h)^{m-l}\|\nabla_H h_s\|\nabla_H h_s & \text{(temperate ice)} \end{cases}$$

$$w(h_b) = \boldsymbol{v_H}(h_b)\cdot\nabla_H h_b$$

Properties of the SIA-Solutions

- The horizontal projection of the velocity vector at any fixed *(x,y)*-position has the direction of steepest descent of the free surface

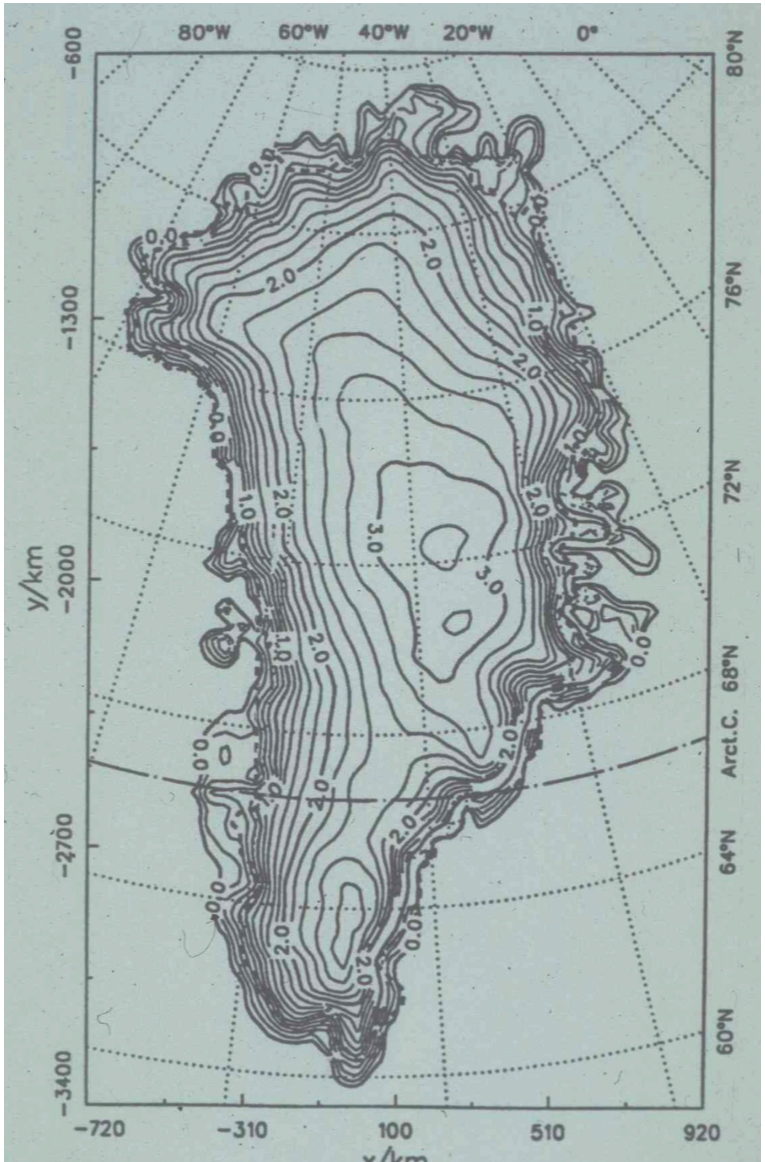
$$\boldsymbol{v}_{\boldsymbol{H}}(x,y,z,t) = -V_{\boldsymbol{H}}(x,y,z,t)\nabla_H h_s(x,y,t)$$ Isotropic sliding

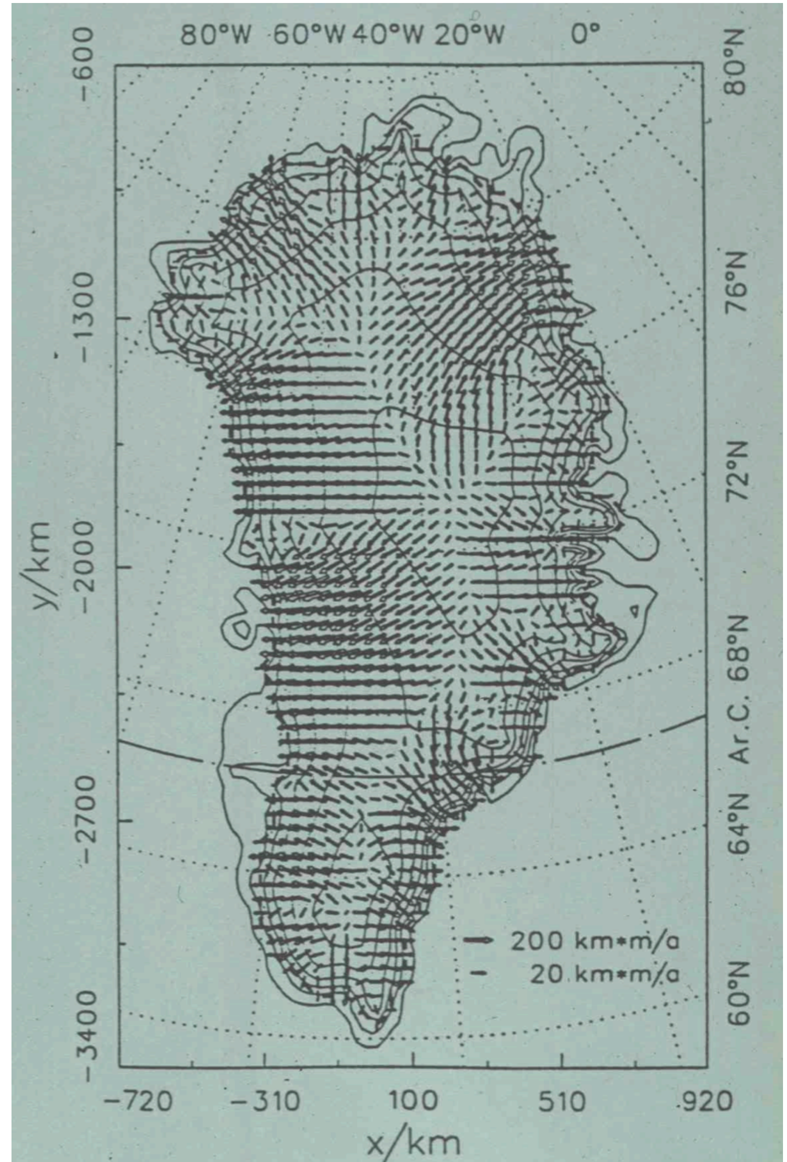
  This vanishes in the direction of the level lines.

- In any *(x,y)*-position the direction of the longitudinal velocity component does not change with depth. Its direction is that of the steepest descent, of the free surface, for all depths.

- A dome (or a trough) is the location at which the horizontal velocity component vanishes for all depths.
  Velocities are always away from a dome and towards a trough. Thus, a trough will eventually be filled and disappear.

- A ridge is a hydrological ice divide, since at no depth at a ridge, ice can flow transversely to the ridge.


80°W
60°W
40°W
20°W
0°
80°N
76°N
72°N
68°N
Arct.C.
64°N
60°N
y/km
-600
-1300
-2000
-2700
-3400
-720
-310
100
510
920
x/km


80°W
60°W
40°W
20°W
0°
80°N
76°N
72°N
68°N
Ar.C.
64°N
60°N
y/km
-600
-1300
-2000
-2700
-3400
-720
-310
100
510
920
x/km
200 km*m/a
20 km*m/a

Remarks:

- The SIA-equations are used in many numerical programs by computational specialists: Calov, Greve, Huybrechts, MacAyel, Ritz,...
  For whole-earth modeling there is also an SIA version in spherical coordinates, Calov, Greve
- The SIA equations have also been applied for polythermal ice masses. The software, due to Greve is known as SICOPOLIS (SImulation COde for POLythemal Ice Sheets)
- The horizontal projection of the trajectories of the ice particles can be determined in the SIA-approximation by constructing from topographic maps the orthogonal trajectories to the level lines.
- If observations contradict results of the SIA then certain assumptions of the SIA are not satisfied, e.g.,
  - Sliding at the base may be anisotropic,
  - Topography of the base may locally be steep,
  - ....

3.5 Some results from climate computations

<u>Model: SICOPOLIS</u>

- Simple parameterization of the climate input
  - Geothermal heat flow 5 km below the ice rock interface Q_{geoth}=42-55 mW m^{-2}
  - Relaxing Lithosophere-Asthenosphere system with relaxation time τ=3000 years
- Glen´s flow law with enhancement factor E
  - E=1, for ice younger than 11 ka (holocene, Eem-ice)
  - E=3, for ice older than 11 ka (pleistocene ice)
- Simplified climate input for the free surface, temperature and accumulation function.

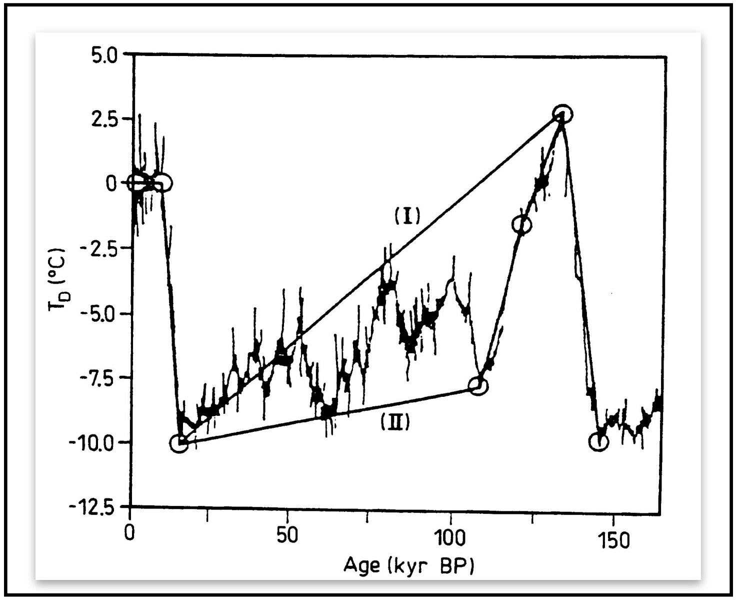
Atm. Temperature at free surface:

$$T_a(x,y,t) = T_a(x,y,t)\big|_{loc} + T_D(t) ,$$

Accumulation-Ablation rate:

$$b = S - M \qquad \left[m\ a^{-1},\ \text{ice equivalent}\right]$$

S=snow fall, M= melting rate



Climate Parameterization

- Temperature

  $T_a(x,y,t)$: from 20th century measurements,
  $T_D(t)$: from proxi-data of ice-core measurements,

- Snow fall

  $S_{today}(x,y)$: from 20th century measurements,
  $S(x,y,t)$: linear parameterization with $T_D(t)$,

- Melting: Positive Degree Day (PDD-M), or energy balance

$$M = \frac{\beta_2}{Y} \max\left(0, \sum_+ - \frac{P_{\max} Y}{\beta_1} S\right)$$

$$S(t) = S_{\min} + \left(S_{today} - S_{\min}\right) \frac{T_D(t) - T_D^{\min}}{T_D^{today} - T_D^{\min}},$$

$$S_{\min} = 0.5 S_{today}, \quad T_D^{\min} = -10^\circ C, \quad T_D^{today} = 0^\circ C$$

$$\sum_+ = \begin{cases} Y\, T_a & T_A \le T_{a,} \\ \dfrac{Y}{\pi}\left(T_a \arccos\left(-\dfrac{T_a}{T_A}\right) + \sqrt{T_A^2 - T_a^2}\right) & -T_A < T_a < T_A, \\ 0 & -T_A \ge T_a, \end{cases}$$

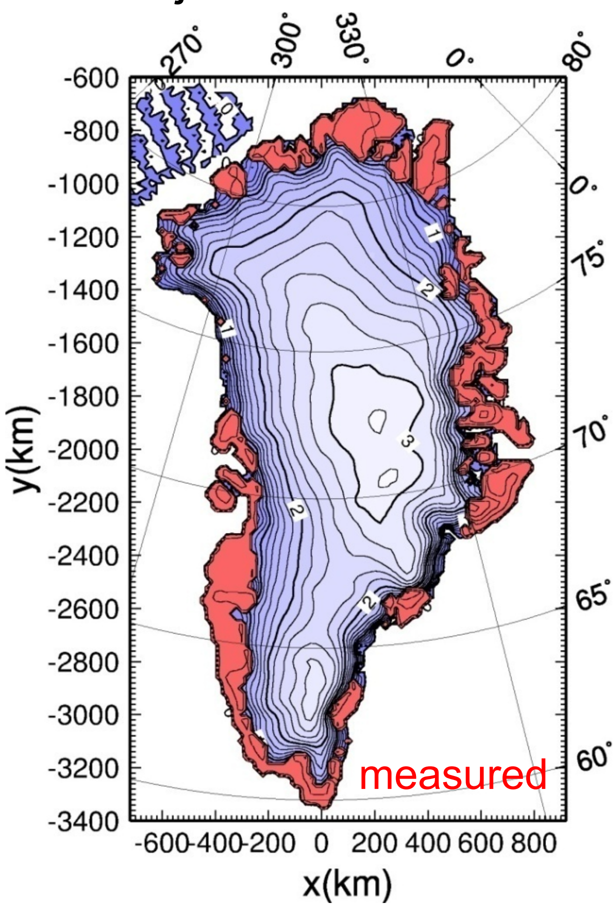
$$\beta_1 = 0.9, \quad \beta_2 = 2.6$$ [m water equivalent $a^{-1}C^{-1}$]

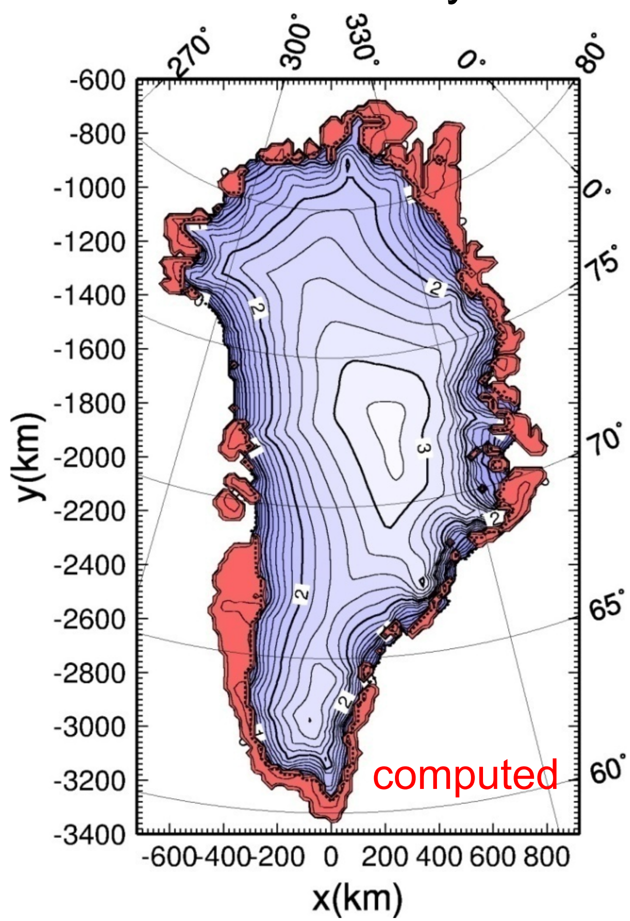
$$Y = 10a, \quad P_{\max} = 0.6$$

Example 1: Steady state presence Greenland Ice Sheet

Reproduction of today´s geometry of the Greenland Ice Sheet

- Start with today´s geometry, temperature and accumulation formulations and initial temperature of -10°C
- Computation subject to external boundary conditions for 100000 years



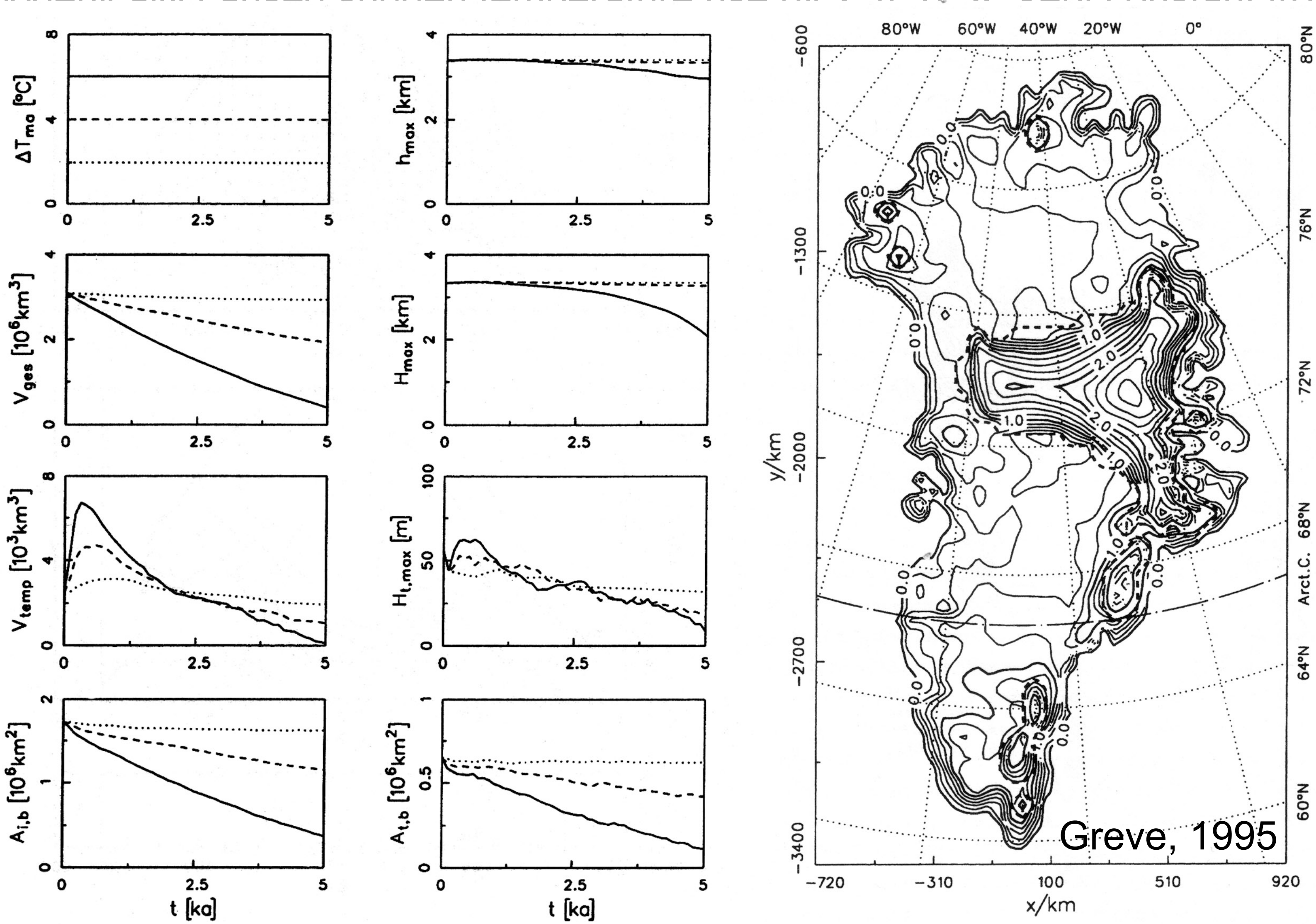


Example 2:

Antropogenically caused sudden temperature rise by 2, 4, 6° K, kept constant from

toda

Res



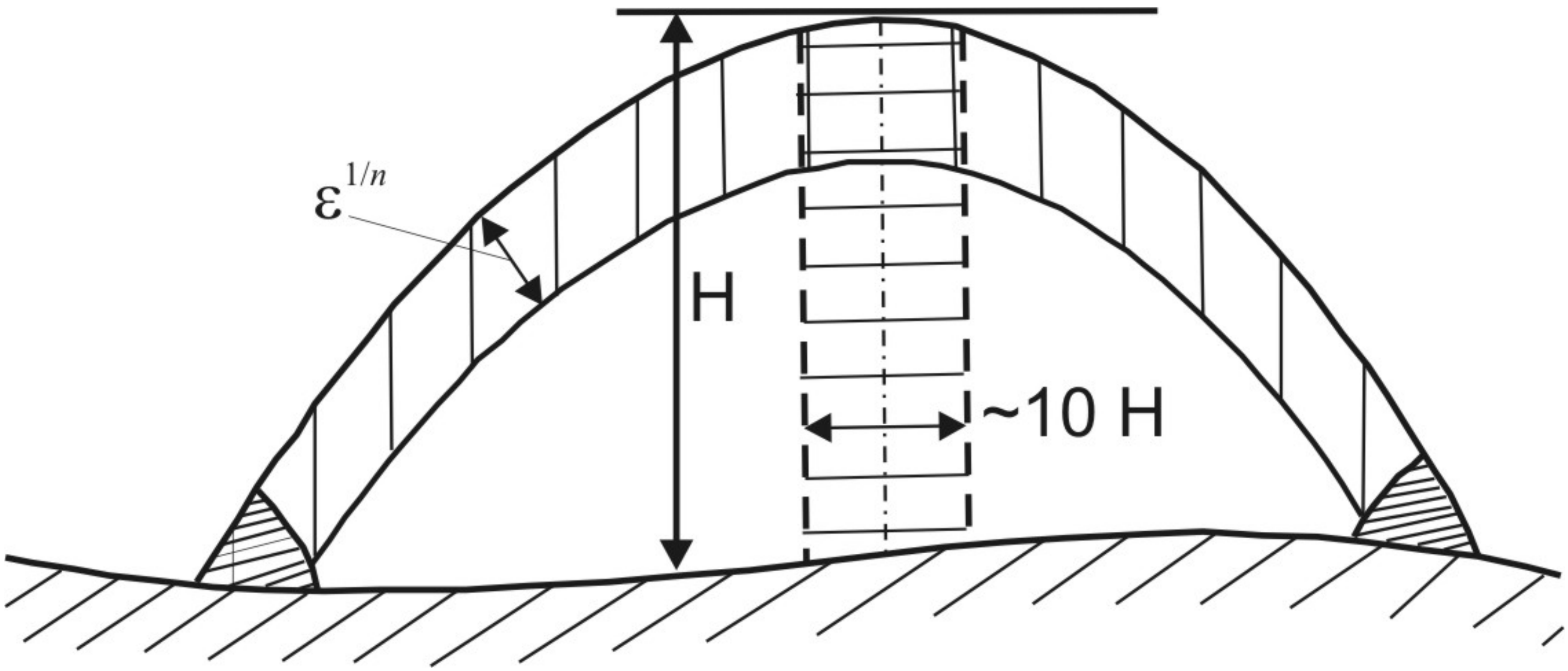
3.6 Limitations of the SIA

There are three regions in ice sheets, in which the SIA is physically not satisfactory and must be corrected.

(i) In the vicinity of domes and, in 2D or 3D, ice divides the SIA is violated since vertical velocities are much larger than horizontal velocities.

(ii) In a boundary layer, close to the free surface normal stresses of the (dissipative) stress deviator are not negligible. An infinite viscosity law generates a singularity and makes these stresses infinitely large. A finite viscosity law regularizes the problem, but normal stresses close to the surface are still large.

(iii) Close to the margin, where surface gradients are not small. This boundary region is a region typical of contact line problems. It is a passive boundary layer and in glacier and ice sheet problems of subgrid scale.

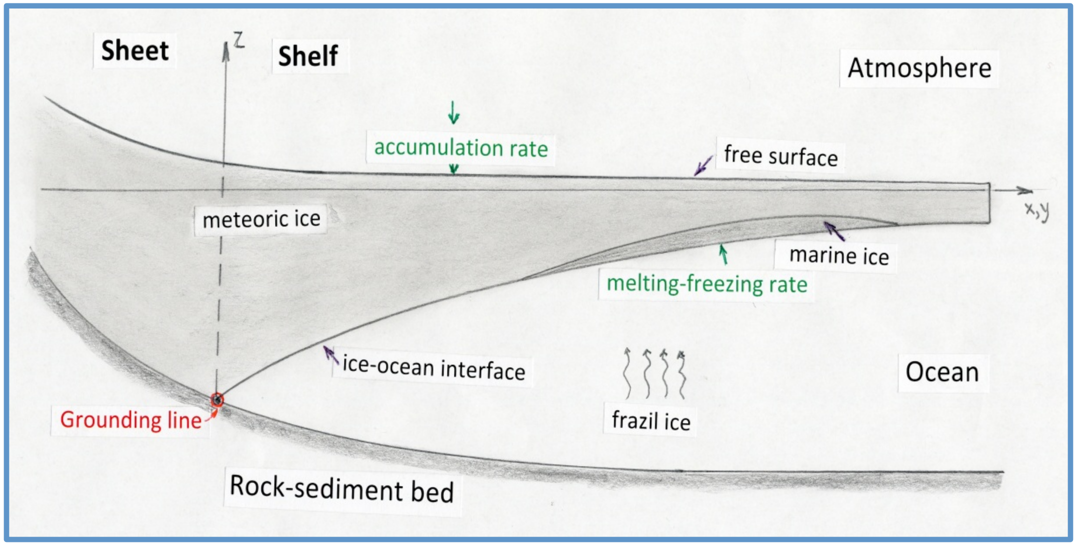


It is easy to see that

- The diffusion equation for the surface height has infinite viscosity at a dome
- the longitudinal stresses of the SIA-solution at the dome are infinitely large
- the longitudinal stresses at the free surface are infinitely large
  if an infinite viscosity law (power law) is used

Regularize the viscosity law → Polynomial law → Use second order SIA

Shallow Shelf Approximation (SSA)



Important
- Same class of viscous material as for SIA
- T_{xx}', T_{yy}', T_{zz}', T_{xy}' are significant
- T_{xz}' T_{yz}' are not significant
- Cryostatic pressure assumption
- Depth integration of mass and momentum balances

Aspect ratio $\varepsilon = [H]/[L] \ll 1$

Perturbation expansion in ε
Theory to $O(\varepsilon^\circ)$ ⟹ SSA
Theory to $O(\varepsilon)$ ⟹ SO-SSA

Paper:
Weis, M. Greve, R and Hutter, K. (1999)
Theory of Shallow ice shelves,
CMT, 11, 15-50

Shelf-Sheet Models

Equations

(SIA)

$$0 = -\frac{\partial p_{(0)}}{\partial x} + \frac{\partial T^E_{xz(0)}}{\partial z}$$

$$0 = -\frac{\partial p_{(0)}}{\partial y} + \frac{\partial T^E_{yz(0)}}{\partial z}$$

parabolic

$$1 = -\frac{\partial p_{(0)}}{\partial z}$$

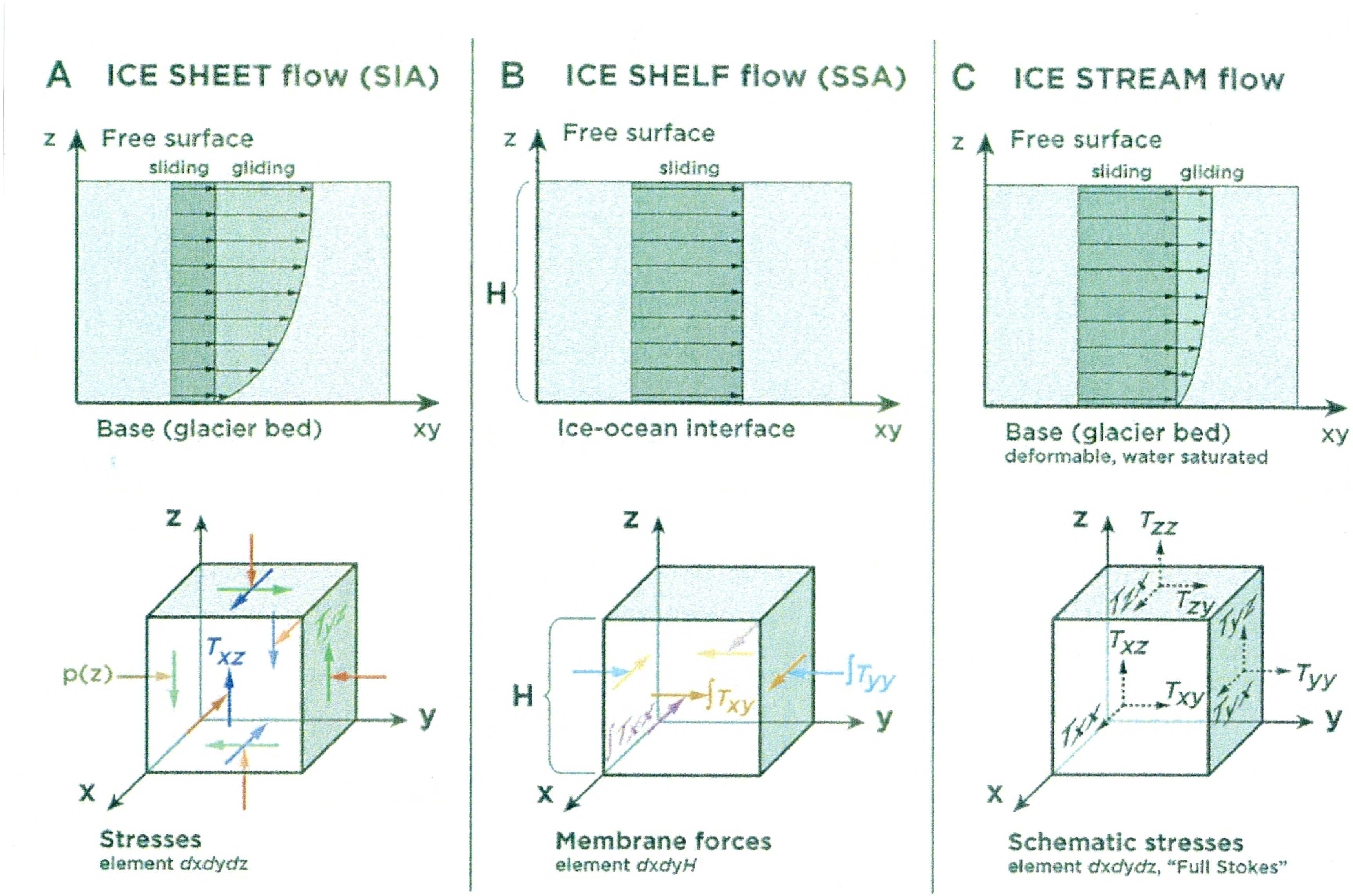
hyperbolic (SSA)

$$0 = -\frac{\partial p_{(0)}}{\partial x} + \frac{\partial T^E_{xx(0)}}{\partial x} + \frac{\partial T^E_{xy(0)}}{\partial y} + \frac{\partial T^E_{xz(0)}}{\partial z}$$

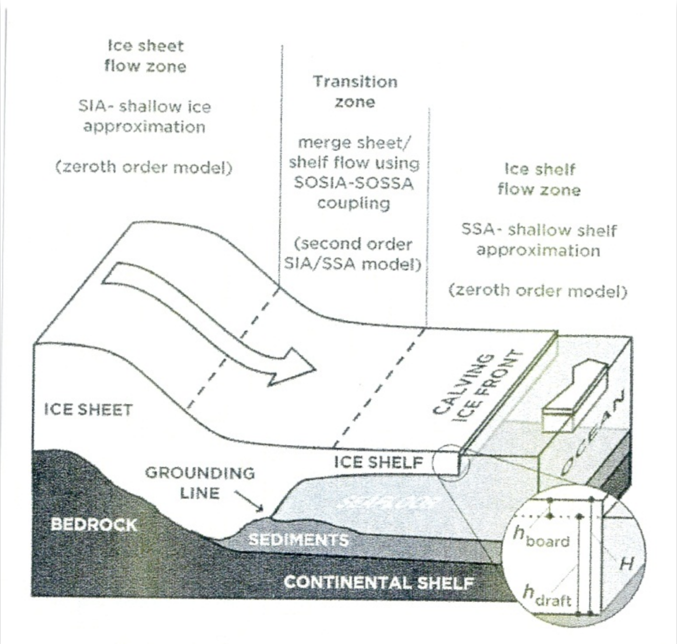
$$0 = -\frac{\partial p_{(0)}}{\partial y} + \frac{\partial T^E_{xy(0)}}{\partial x} + \frac{\partial T^E_{yy(0)}}{\partial y} + \frac{\partial T^E_{yz(0)}}{\partial z}$$

$$\frac{\rho_{\mathrm{sw}}}{\rho_{\mathrm{sw}} - \rho} = -\frac{\partial p_{(0)}}{\partial z} + \frac{\partial T^E_{zz(0)}}{\partial z}$$

Ice streams are not described neither by SIA nor by SSA


A ICE SHEET flow (SIA)
z
Free surface
sliding
gliding
Base (glacier bed)
xy
z
p(z)
T_{xz}
y
x
Stresses
element dxdydz
B ICE SHELF flow (SSA)
z
Free surface
sliding
H
Ice-ocean interface
xy
z
H
$\int T_{yy}$
$\int T_{xy}$
y
x
Membrane forces
element dxdyH
C ICE STREAM flow
z
Free surface
sliding
gliding
Base (glacier bed)
deformable, water saturated
xy
z
T_{zz}
T_{zy}
T_{xz}
T_{yy}
T_{xy}
y
x
Schematic stresses
element dxdydz, "Full Stokes"

Shelf-Sheet Coupling



The two sets of zeroth order equations canot be couple

=> Matching procedure

Ref:Chugunov and Wilshinsky
J. Glaciol **23**, 1996

==> Second order equations

Refs:

Baral Dambaru, Dissertation TU Darmstadt 1999.

Full derivation for shallow ice sheets and shallow iice shelves SO-SIA, SO-SSA

Baral. D. Greve, R. and Hutter, K. 2001 Appll. Mech. Reviews 54, 215-250

Short version for SO-SIA

SO-SIA – SO-SSA Ccoupling

$$
(\mathrm{SO-SIA})\quad
\begin{aligned}
0 &= -\frac{\partial p_{(2)}}{\partial x} + \frac{\partial T^E_{xx(0)}}{\partial x} + \frac{\partial T^E_{xy(0)}}{\partial y} + \frac{\partial T^E_{xz(2)}}{\partial z}, \\
0 &= -\frac{\partial p_{(2)}}{\partial y} + \frac{\partial T^E_{xy(0)}}{\partial x} + \frac{\partial T^E_{yy(0)}}{\partial y} + \frac{\partial T^E_{yz(2)}}{\partial z}, \\
0 &= -\frac{\partial p_{(2)}}{\partial z} + \frac{\partial T^E_{xz(0)}}{\partial x} + \frac{\partial T^E_{yz(0)}}{\partial y} + \frac{\partial T^E_{zz(0)}}{\partial z}.
\end{aligned}
$$

$$
(\mathrm{SO-SSA})\quad
\begin{aligned}
0 &= -\frac{\partial p_{(2)}}{\partial x} + \frac{\partial T^E_{xx(2)}}{\partial x} + \frac{\partial T^E_{xy(2)}}{\partial y} + \frac{\partial T^E_{xz(2)}}{\partial z}, \\
0 &= -\frac{\partial p_{(2)}}{\partial y} + \frac{\partial T^E_{xy(2)}}{\partial x} + \frac{\partial T^E_{yy(2)}}{\partial y} + \frac{\partial T^E_{yz(2)}}{\partial z}, \\
0 &= -\frac{\partial p_{(2)}}{\partial z} + \frac{\partial T^E_{xz(0)}}{\partial x} + \frac{\partial T^E_{yz(0)}}{\partial y} + \frac{\partial T^E_{zz(2)}}{\partial z},
\end{aligned}
$$

Proposal for procedure:

Construct SIA and SSA solutions
Construct SO-SIA and SO-SSA solutions
and patch them together at the grounding line

Quaternary Science Reviews 30 (2011) 3691–3704

ELSEVIER

Contents lists available at SciVerse ScienceDirect

Quaternary Science Reviews

journal homepage: www.elsevier.com/locate/quascirev

QUATERNARY SCIENCE REVIEWS

Capabilities and limitations of numerical ice sheet models: a discussion for Earth-scientists and modelers

Nina Kirchner[a,*], Kolumban Hutter[b], Martin Jakobsson[c], Richard Gyllencreutz[c]

[a] Department of Physical Geography and Quaternary Geology, Stockholm University, 10691 Stockholm, Sweden
[b] c/o Laboratory of Hydraulics, Hydrology and Glaciology, Swiss Federal Institute of Technology, 8092 Zürich, Switzerland
[c] Department of Geological Sciences, Stockholm University, 10691 Stockholm, Sweden

Ahlkrona, J, Kirchner, N. and Lötstedt, P.:
A numerical study of scaling relations for non-Newtonian thin film flows with applications in ice sheet modeling.
Q. Jl. Mech. Appl. Math. (under review)

Thank you

Go from SIA, SSA
to SOSIA, SOSSA