Numerical Analysis of Elliptic PDEs with Random Coefficients

(Lecture I)

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University of Bath

Workshop on PDEs with Random Coefficients
Weierstrass Institute, Berlin, 13–15 November, 2013

Outline – Lecture 1

Numerical Analysis of Elliptic PDEs with Random Coefficients

- **Motivation:** uncertainty/lack of data & stochastic modelling
  Examples of PDEs with random data
- **Model problem:** groundwater flow and radwaste disposal
  Elliptic PDEs with **rough** stochastic coefficients
- What are the **computational/analytical challenges**?
- **Numerical Analysis**
  - Assumptions, existence, uniqueness, regularity
  - FE analysis: Cea Lemma, interpolation error, functionals
  - Variational crimes (truncation error, quadrature)
  - Mixed finite element methods

Outline – Lecture 2

Novel Monte Carlo Methods and Uncertainty Quantification

- Stochastic Uncertainty Quantification (in PDEs)
- *The Curse of Dimensionality* & the **Monte Carlo Method**
- **Multilevel Monte Carlo** methods
- Analysis of multilevel MC for the elliptic model problem
- **Quasi–Monte Carlo** methods
- Analysis of QMC for the elliptic model problem
- Bayesian Inference (stochastic inverse problems):
  Multilevel Markov Chain Monte Carlo

Stochastic Modelling

- Many reasons for stochastic modelling:
  - **lack of data** (e.g. data assimilation for weather prediction)
  - **data uncertainty** (e.g. uncertainty quantification in subsurface flow)
  - **parameter identification** (e.g. Bayesian inference in structural engineering)
  - **unresolvable scales** (e.g. atmospheric dispersion modelling)

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- **Input:** best knowledge about system (PDE), statistics of input parameters, measured ouput data with error statistics,...

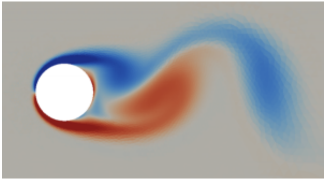
- **Output:** stats of quantities of interest or entire state space
  often very sparse (or no) output data $\rightarrow$ need a good physical model!
  - Data assimilation in NWP: data misfit, rainfall at some location
  - Radioactive waste disposal: flow at repository, 'breakthrough' time
  - Oil reservoir simulation: production rate
  - Certification of carbon fibre composites in aeronautical engineering

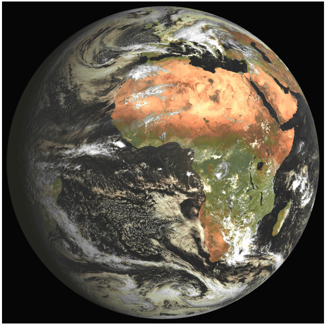
Examples

- **Navier–Stokes** (e.g. modelling plane or forecasting weather):

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \;=\; -\nabla p + \mu \nabla^2 \mathbf{v} + f \quad \text{in} \quad \Omega$$

$$\text{subject to IC} \quad v(x,0) \;=\; v_0(x) \quad + \quad \text{BCs}$$





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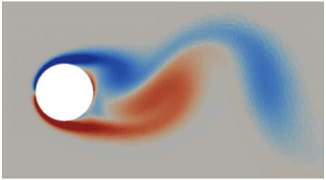

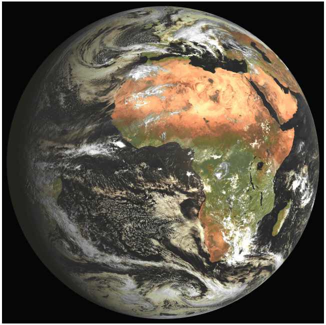

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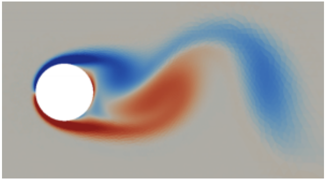

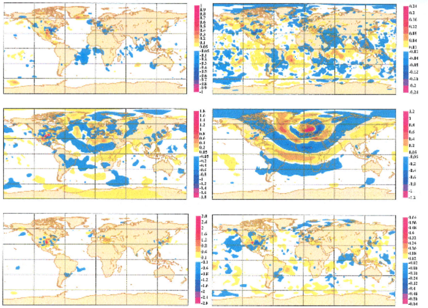

uncertain ICs → **data assimilation**

Examples

- **Structural Mechanics** (e.g. composites, tires or bone):

$$\nabla \cdot \left(\overline{\overline{C}} : \frac{1}{2} \left[\nabla \mathbf{u} + \nabla \mathbf{u}^T \right] \right) + \mathbf{F} \;=\; 0 \quad \text{in} \quad \Omega$$

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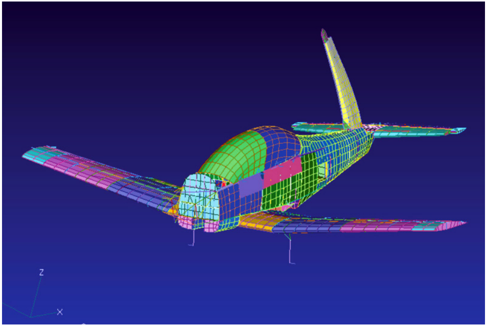

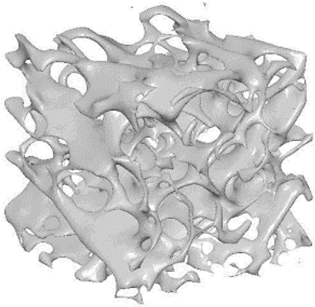

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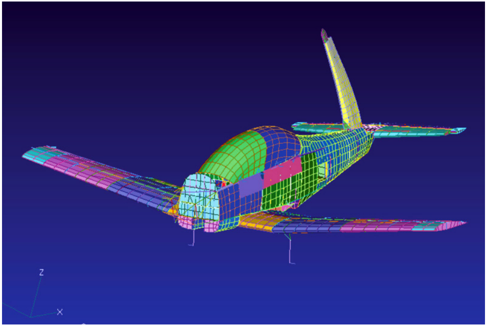

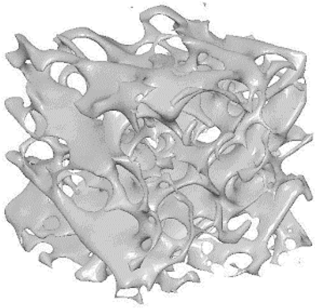

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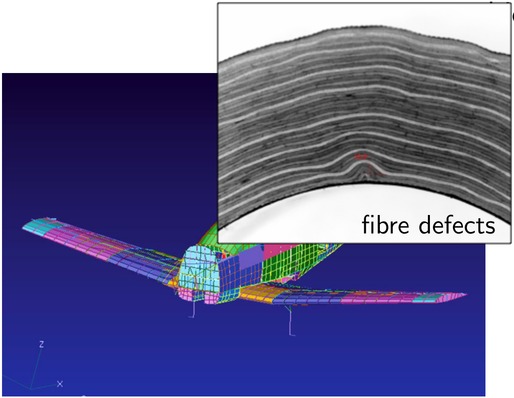
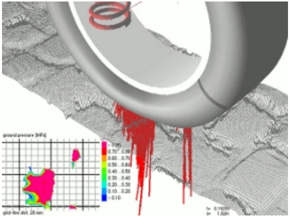


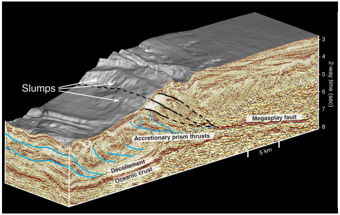
fibre defects

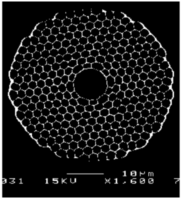
contact problems A. Chernov



Examples

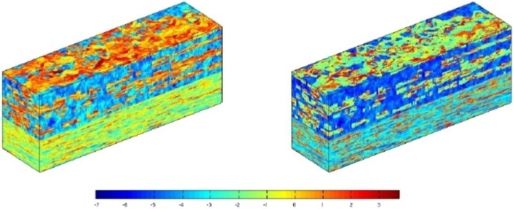
- **Waves in Heterogeneous Media** (e.g. seismic or optics):

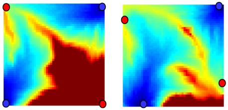




photonic crystal fibre design

- **Subsurface Fluid Flow** (e.g. oil reservoir simulation):





optimal well placement

Model Problem: Uncertainty in Groundwater Flow

(e.g. risk analysis of radwaste disposal or CO2 sequestration)

Darcy's Law: $\vec{q} + k \nabla p = \vec{f}$

Incompressibility: $\nabla \cdot \vec{q} = g$

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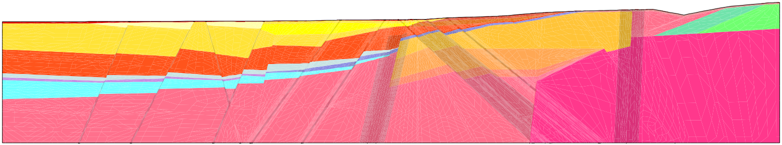

Geology at Sellafield (former potential UK radwaste site) ©NIREX UK Ltd.

Application: Longterm radioactive waste disposal

UK EPSRC funded research project with A Cliffe (Nottingham) & M Giles (Oxford)

- **Total current UK Waste** (intermediate & highly radioactive):

$$\boxed{\approx \mathbf{220{,}000\ m^3}}$$

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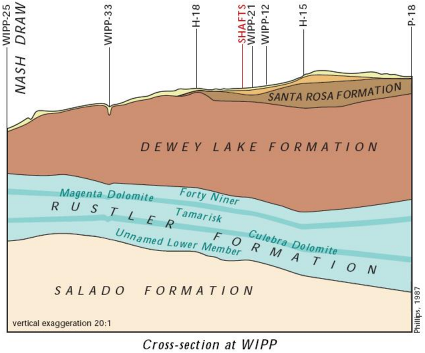
- **Key aspect:** How to quantify uncertainties in the models?

WIPP (Waste Isolation Pilot Plant) Test Problem

US Dept of Energy Radioactive Waste Repository in New Mexico



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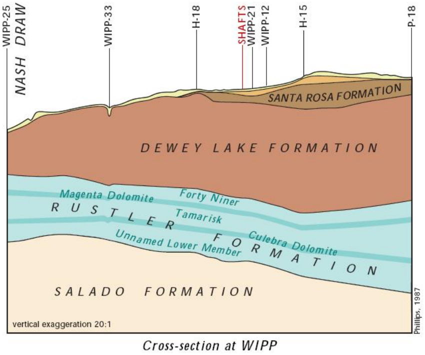

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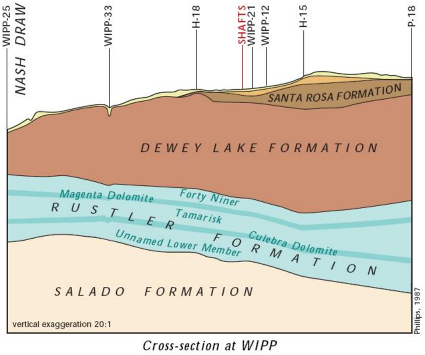

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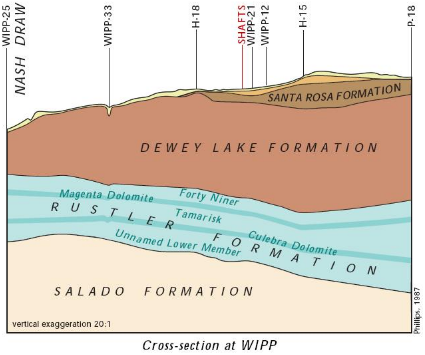

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- Usually: $\sigma^2 > 1$, $\lambda < 1$ & $\nu < 1$
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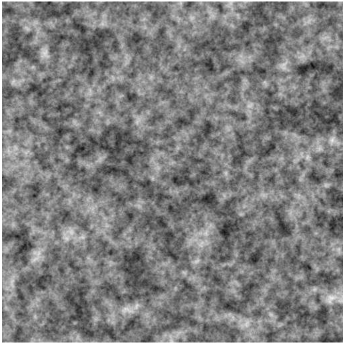

realisation (with $\lambda = \frac{1}{64}$, $\sigma^2 = 8$)

contrast: $\max_{x,y} \frac{k(x)}{k(y)} = \mathcal{O}(10^9)$

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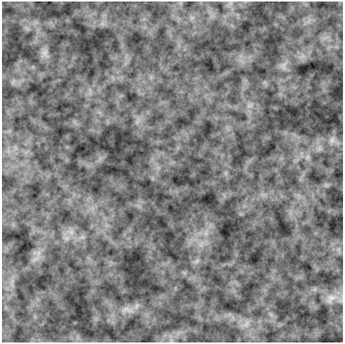
- Inverse problem ⇒ **MCMC**



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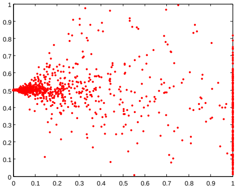

$\lambda = 0.3$ and $\sigma^2 = 4$

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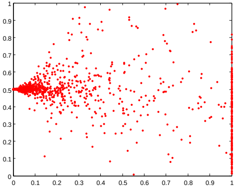

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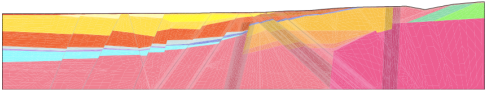
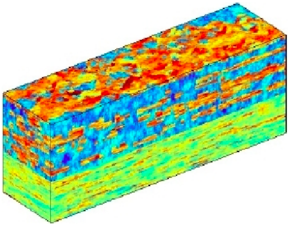
More Realistic Examples (e.g. Sellafield site or SPE10 Benchmark)





Key Computational Challenges

PDEs with Highly Heterogeneous Random Coefficients

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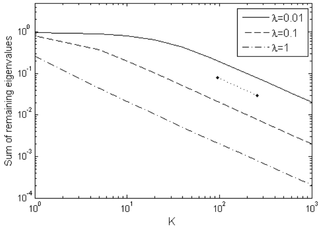
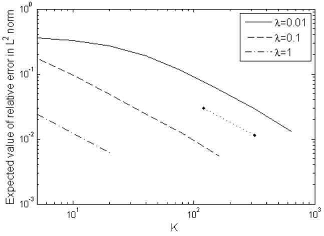
e.g. for truncated KL expansion $\log k(x,\omega) \approx \sum_{j=1}^{s} \sqrt{\mu_j}\phi_j(x)Y_j(\omega)$



Remainder $\sum_{j>J} \mu_j$ in 1D



Truncation error of $\mathbb{E}[\|p\|_{L_2(0,1)}]$ w.r.t. s

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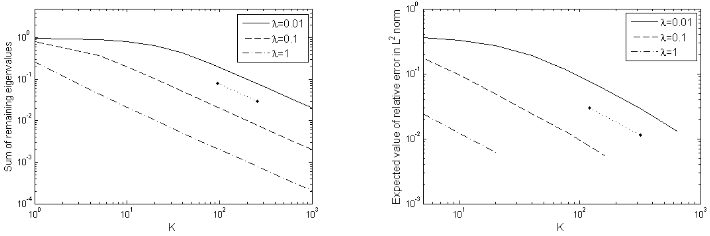
e.g. for truncated KL expansion $\log k(x,\omega) \approx \sum_{j=1}^{s} \sqrt{\mu_j}\phi_j(x)Y_j(\omega)$



Remainder $\sum_{j>J} \mu_j$ in 1D — Truncation error of $\mathbb{E}[\|p\|_{L_2(0,1)}]$ w.r.t. s

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Stochastic Finite Element Analysis

The Pioneers (in Stochastic Finite Elements)

- **Structural Engineers** (from the 1980s):
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- **Analysis for Lognormal Coefficients** (from ∼2009):
  - Galvis, Sarkis, Gittelson, Ullmann, Charrier, ...

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- AL Teckentrup (Florida State, previously Bath)
- J Charrier (Marseille)
- E Ullmann (Bath)
- IG Graham (Bath)
- F Kuo (UNSW Sydney)
- IH Sloan (UNSW Sydney)
- D Nuyens (Leuven)
- J Nicholls (UNSW Sydney)
- Ch Schwab (ETH Zürich)
- Ch Ketelsen (Boulder)