Decay of Eigenfunctions of Magnetic Hamiltonians

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Based on the paper:
V. Iftimie, R. Purice: *Eigenfunctions decay for magnetic pseudodifferential operators*, **preprint arXiv:1005.1743, 10 pp.**

Plan of the talk

- The Result
- The Abstract Weighted Estimation
- Proof of the main Theorems

The Result

The Framework

We work in $\mathbb{R}^d$ (with $d \geq 2$) and consider:

- A magnetic field B described by a closed 2-form

$$B = \frac{1}{2} \sum_{1 \leq j,k \leq d} B_{jk}\, dx_j \wedge dx_k, \quad dB = 0, \quad B_{jk} = -B_{kj}.$$

- A classic Hamiltonian defined by a *real function* $h \in S^m(\mathbb{R}^d)$, i.e.:

$$h \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$$

and there exist $m \in \mathbb{R}$ such that $\forall \alpha \in \mathbb{N}^d$, $\forall \beta \in \mathbb{N}^d$:

$$\sup_{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d} < \xi >^{-m+|\beta|} \left| \left(\partial_x^\alpha \partial_\xi^\beta a \right)(x, \xi) \right| < \infty.$$

The Framework

Hypothesis B

$$B_{jk} \in BC^\infty(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \mid \partial^\alpha f \in L^\infty(\mathbb{R}^d) \forall \alpha \in \mathbb{N}^d \right\}.$$

Hypothesis h

For $m > 0$ the symbol $a \in S^m(\mathbb{R}^d)$ is elliptic, i.e. $\exists C > 0, \exists R > 0$ such that

$$|a(x,\xi)| \geq C < \xi >^m \;\; \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \text{ with } |\xi| \geq R.$$

The Framework

We can now use the *magnetic covariant quantization* to define the *quantum Hamiltonian* associated to h and B.

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Hypothesis A

We shall consider a smooth *vector potential* defining our magnetic field:

$$A = \sum_{1 \leq j \leq d} A_j \, dx_j, \quad A_j \in C^\infty_{\text{pol}}(\mathbb{R}^d), \quad B = dA$$

with $C^\infty_{\text{pol}}(\mathbb{R}^d)$ the space of infinitely differentiable functions with at most polynomial growth together with all their derivatives.

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(This always exist, the transverse gauge giving an explicit example).

The Framework

- For any $u \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz test functions), we define the operator:

$$\forall x \in \mathbb{R}^d, \qquad \left[\mathfrak{Op}^A(h)u\right](x) :=$$

$$(2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i<x-y,\eta>} \omega^A(x,y) h\left(\frac{x+y}{2}, \eta\right) u(y)\, dy\, d\eta,$$

where $\omega^A(x,y) := \exp\left(-i \int_{[x,y]} A\right)$, $\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$.

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- **Theorem** [IMP '07]: The operator $\mathfrak{Op}^A(h)$ is bounded for $m \leq 0$ and for $m > 0$ it is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^d)$; we denote by $H(h,A)$ its closure in $L^2(\mathbb{R}^d)$.

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- **Theorem** [IMP '07]: The operator $\mathfrak{Op}^A(h)$ is bounded for $m \leq 0$ and for $m > 0$ it is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^d)$; we denote by $H(h,A)$ its closure in $L^2(\mathbb{R}^d)$.
- **Theorem** [IMP '07]: For $m > 0$ the domain of $H(h,A)$ is

$$\mathcal{H}^s_A(\mathbb{R}^d) := \left\{u \in L^2(\mathbb{R}^d) \mid P_s u \in L^2(\mathbb{R}^d)\right\},$$

with $P_s := \mathfrak{Op}^A(p_s)$, $p_s(\eta) :=< \eta >^s$.

Polynomial decay

Theorem A - Polynomial Decay

Let us suppose that

- Hypothesis h is verified,
- the magnetic field verifies Hypothesis B,
- we fixed a vector potential for the magnetic field as in Hypothesis A.

Let $\lambda \in \sigma_{\text{disc}}(H(h, A))$ and $u \in \mathrm{Ker}\big(H(h, A) - \lambda\big)$.
Then

1. $< x >^p u \in \bigcap_{n \in \mathbb{N}} D\big(H(h, A)^n\big) \ \forall p \in \mathbb{N}$.
2. If $m > 0$ or if $m < 0$ and $\lambda \neq 0$ then $u \in \mathcal{S}(\mathbb{R}^d)$.

Notation: For $\delta > 0$

$$D_\delta := \left\{ \zeta \in \mathbb{C}^d \mid |\zeta_j| < \delta \, , \forall j \in \{1, \dots, d\} \right\}.$$

Hypothesis ω (of analytic extension)

Let $h \in S^m(\mathbb{R}^d)$ and suppose that there exists $\delta > 0$ and a function $\widetilde{h} : \mathbb{R}^d \times D_\delta \to \mathbb{C}$ such that:

- for any $x \in \mathbb{R}^d$ the function $\widetilde{h}(x, \cdot) : D_\delta \to \mathbb{C}$ is analytic;
- the map $\mathbb{R}^d \times \mathbb{R}^d \ni (x, \eta) \mapsto \widetilde{h}(x, \eta + i\xi) \in \mathbb{C}$ is of class $S^m(\mathbb{R}^d)$ uniformly (for the Fréchet topology) with respect to $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ for $|\xi_j| < \delta$, $1 \leq j \leq d$;
- we have: $h = \widetilde{h}\Big|_{\mathbb{R}^d \times \mathbb{R}^d}$.

Theorem B - Exponential Decay

Let us suppose that

- Hypothesis h and ω are verified,
- the magnetic field verifies Hypothesis B,
- we fixed a vector potential for the magnetic field as in Hypothesis A.

Let $\lambda \in \sigma_{\text{disc}}(H(h,A))$ and $u \in \mathrm{Ker}\big(H(h,A) - \lambda\big)$.
Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ we have that

1. $e^{\epsilon <x>} u \in \bigcap_{n \in \mathbb{N}} D(H^n)$.
2. If $m > 0$ or if $m < 0$ and $\lambda \neq 0$ then $e^{\epsilon <x>} u \in \mathcal{S}(\mathbb{R}^d)$.

The Abstract Weighted Estimation

Abstract Weighted Estimation

- Let $f : \mathbb{R}^d \to [1, \infty)$ be a measurable function.
- Let $f_\epsilon(x) := f(\epsilon x)$ for any $\epsilon > 0$.

Hypothesis W

Let H be a self-adjoint operator in $L^2(\mathbb{R}^d)$. We suppose that:

- for any $u \in D(H)$ and $\epsilon \in (0, 1]$ we have that $f_\epsilon^{-1} u \in D(H)$;
- on $D(H)$ we define $H_\epsilon := f_\epsilon H f_\epsilon^{-1} = H + \epsilon R_\epsilon$ for any $\epsilon \in (0, 1]$ and suppose that the operators R_ϵ are H-relatively bounded uniformly in $\epsilon \in (0, 1]$.

Proposition

Let H be a self-adjoint operator in $L^2(\mathbb{R}^d)$ that verifies Hypothesis W with respect to a weight function f and suppose $\lambda \in \sigma_{\text{disc}}(H)$. Then there exists $\epsilon_0 \in (0,1]$ such that for any $\epsilon \in (0, \epsilon_0]$ we have that $f_\epsilon u \in D(H^n)$ for any $n \geq 1$ and any $u \in \mathrm{Ker}(H - \lambda)$.

The proof is based on basic perturbation results and the following observation: $\forall v_\epsilon \in D(H)$

$$u_\epsilon := f_\epsilon^{-1} v_\epsilon \in D(H), \text{ and } (H - \lambda_\epsilon) u_\epsilon = f_\epsilon^{-1} (H_\epsilon - \lambda_\epsilon) v_\epsilon.$$

Proof of the Main Theorems

Proofs

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- The behaviour with respect to the small parameter $\epsilon \in (0, \epsilon_0]$ is achieved by using some asymptotic expansions near $\epsilon = 0$
- The control of the remainders puts into evidence in a natural way a class of symbols with three variables.
  In fact it seems easier to sistematically work with a pseudodifferential calculus associated to such symbols.

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- The control of the remainders puts into evidence in a natural way a class of symbols with three variables.
  In fact it seems easier to sistematically work with a pseudodifferential calculus associated to such symbols.

Definition

For $a \in S^m(\mathbb{R}^{2d} \times \mathbb{R}^d)$ we consider the linear operator defined by the following oscillatory integral:

$$[\mathfrak{E}(a)u](x) := (2\pi)^{-1} \int_{\mathbb{R}^{2d}} e^{i<x-y,\eta>} \omega^A(x,y) a(x,y,\eta) u(y)\, dy\, d\eta,$$

$$\forall u \in \mathcal{S}(\mathbb{R}^d), \quad \forall x \in \mathbb{R}^d.$$

Proposition

Let $a \in S^m(\mathbb{R}^{2d} \times \mathbb{R}^d)$ for some $m \in \mathbb{R}$. Then there exists a unique symbol $\mathring{a} \in S^m(\mathbb{R}^d)$ such that $\mathfrak{E}(a) = \mathfrak{Op}^A(\mathring{a})$ and the following map is continuous

$$S^m(\mathbb{R}^{2d} \times \mathbb{R}^d) \ni a \mapsto \mathring{a} \in S^m(\mathbb{R}^d).$$

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Proof:
With $L : \mathbb{R}^{2d} \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$; $L(x, y, \eta) := ((x + y)/2, \eta)$
and $\mathbb{R}^{2d} \ni (u, v) \mapsto S(u, v) := ((u + v)/2, (u - v)) \in \mathbb{R}^{2d}$,
for $b \in S^m(\mathbb{R}^d)$ we can write the distribution kernel of $\mathfrak{E}(a)$ as

$$K_{b \circ L} = \omega^A S^{-1} \mathcal{F}_2^{-1} b.$$

But $\omega^A S^{-1} \mathcal{F}_2^{-1}$ is an invertible operator.

Proof of Theorem A - Polynomial Decay

- We shall apply our abstract weighted estimation to the self-adjoint operator H associated to $\mathfrak{Op}^A(h)$.

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- Let $f(x) := < x >^p$ for some fixed $p \in \mathbb{N}$.
- Let $b \in S^m(\mathbb{R}^{2d} \times \mathbb{R}^d)$ for some $m \in \mathbb{R}$.
- Then there exists a bounded family of symbols $\{s_\epsilon\}_{\epsilon \in (0,1]}$ in $S^{m-1}(\mathbb{R}^{2d} \times \mathbb{R}^d)$ that verify the following equality as linear operators on $\mathcal{S}(\mathbb{R}^d)$:

$$f_\epsilon \mathfrak{E}(b) f_\epsilon^{-1} = \mathfrak{E}(b) + \epsilon \mathfrak{E}(s_\epsilon), \quad \forall \epsilon \in (0,1].$$

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- We know that $D(H) = \mathcal{H}^m_A(\mathbb{R}^d)$ if $m > 0$ and $D(H) = L^2(\mathbb{R}^d)$ if $m \leq 0$.

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- We know that $D(H) = \mathcal{H}^m_A(\mathbb{R}^d)$ if $m > 0$ and $D(H) = L^2(\mathbb{R}^d)$ if $m \leq 0$.
- Due to the above remarks $f(x) :=< x >^p$ satisfies Hypothesis W with respect to H.

Proof of Theorem A - Polynomial Decay

Point 2:

- $\mathbf{m > 0}$: For any $n \geq 1$ we have $D(H^n) = \mathcal{H}_A^{nm}(\mathbb{R}^d)$.
  Thus: $u \in \mathrm{Ker}\big(H(h, A) - \lambda\big)$ implies
  $< x >^p u \in \mathcal{H}_A^{\infty}(\mathbb{R}^d) = \bigcap_{k \in \mathbb{N}} \mathcal{H}_A^k(\mathbb{R}^d), \ \forall p \in \mathbb{N}$.
  And for $A_j \in C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$ we conclude that
  $< x >^p \partial^\alpha u \in L^2(\mathbb{R}^d)$ for any $p \in \mathbb{N}$ and any $\alpha \in \mathbb{N}^d$

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  And for $A_j \in C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$ we conclude that
  $< x >^p \partial^{\alpha} u \in L^2(\mathbb{R}^d)$ for any $p \in \mathbb{N}$ and any $\alpha \in \mathbb{N}^d$
- $\mathbf{m < 0}$ and $\lambda \neq 0$: H_ϵ is a magnetic pseudodifferential operator of order m.
  $\forall \epsilon \in (0, \epsilon_0]$, $\exists n_\epsilon \in \mathbb{N}^*$ such that $\forall u \in \mathrm{Ker}(H - \lambda)$ we can find $v \in L^2(\mathbb{R}^d)$ such that $(H_\epsilon - \lambda)^{n_\epsilon} v = 0$ and $u = f_\epsilon^{-1} v$.
  $\lambda \neq 0$ implies that $v = Q_\epsilon v$ with Q_ϵ a magnetic pseudodifferential operator of order $m < 0$ and thus $v \in \mathcal{H}_A^{\infty}(\mathbb{R}^d)$ and the proof can continue as in the case $m > 0$.

Proof of Theorem B - Exponential Decay

Let us consider the function $f : \mathbb{R}^d \to [1, \infty)$; $f(x) = e^{<x>}$.

- $b_{\epsilon,j}(x, y) := \epsilon(x_j + y_j)\big(< \epsilon x > + < \epsilon y > \big)^{-1}$,
- $c_\epsilon : \mathbb{R}^{2d} \times \mathbb{R}^d \to \mathbb{C}$; $c_\epsilon(x, y, \eta) := \widetilde{h}\left(\frac{x+y}{2}, \eta + i\epsilon b_\epsilon(x, y)\right)$.

Proposition B.1

Under the assumptions of Theorem B, we denote by
$\epsilon_0 := \min\{1, \delta/4\}$ and for any $\epsilon \in (0, \epsilon_0]$,
we have $c_\epsilon \in S^m(\mathbb{R}^{2d} \times \mathbb{R}^d)$ uniformly with respect to $\epsilon \in (0, \epsilon_0]$
and the following equality is true

$$f_\epsilon \mathfrak{Op}^A(h) f_\epsilon^{-1} = \mathfrak{E}(c_\epsilon), \quad \epsilon \in (0, \epsilon_0].$$

Proof of Theorem B - Exponential Decay

We can write:

$$c_\epsilon(x,y,\eta) \;=\; h\left(\frac{x+y}{2},\eta\right) + \epsilon d_\epsilon(x,y,\eta), \quad \forall(x,y,\eta)\in\mathbb{R}^{3d}, \forall\epsilon\in(0,\epsilon_0]$$

with

$$d_\epsilon(x,y,\eta) := i\int_0^1 \left\langle b_\epsilon(x,y), (\nabla_\eta \widetilde{h})\left(\frac{x+y}{2}, \eta + it\epsilon b_\epsilon(x,y)\right)\right\rangle dt$$

a family of symbols of class $S^{m-1}(\mathbb{R}^{2d}\times\mathbb{R}^d)$ uniformly with respect to $\epsilon\in(0,\epsilon_0]$.

Proposition B.2

Under the assumptions of Theorem B there exists a bounded family of symbols $\{r_\epsilon\}_{\epsilon\in(0,\epsilon_0]}\subset S^{m-1}(\mathbb{R}^d)$ such that

$$f_\epsilon \mathfrak{Op}^A(h) f_\epsilon^{-1} u \;=\; \mathfrak{Op}^A(h) \,+\, \epsilon\mathfrak{Op}^A(r_\epsilon), \quad \forall\epsilon\in(0,\epsilon_0].$$