Variational methods for steady-state Darcy/Fick flow in swelling-exhibitting or poro-elastic solids.

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Some elastic media allow for diffusion of a solvent in the atomic grid or the macro-molecular chains causing a swelling (as some metals or polymers) or for a flow of a fluid in pores (as poro-elastic rocks). This rises a nontrivially mechanically-coupled system.

More specifically, confining on the small strains, the momentum equilibrium equation is to be then coupled with the continuity equation for the flow, i.e.:

(1a) $\operatorname{div}\sigma + f = 0$ with $\sigma = \partial_e \varphi(e(u), c)$,

(1b) $\operatorname{div}(\mathbb{M}(c)\nabla\mu) = 0$ with $\mu = \partial_c \varphi(e(u), c)$,

where u is the displacement, $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$ the small-strain tensor, c a concentration of a solvent in interatomic grid or the fluid in pores, $\mathbb{M} = \mathbb{M}(c)$ a mobility matrix, σ stress tensor, μ a chemical potential. The free energy $\varphi = \varphi(e, c)$ is to be prescribed at particular cases, and may lead to the Darcy's or the Fick's flows (i.e. the flux proportional to the gradient of the pressure or the concentration, respectively) or their combination. The system (1) is to be accompanied by suitable boundary conditions.

One should distinguish between the general steady-state situations and purely static situation. The former one means that all fields including the specific dissipation rate $\mathbb{M}(c)\nabla\mu\cdot\nabla\mu$ do not depend on time, while the latter means in addition that that the dissipation rate is zero. In the static case, the system enjoys the variational structure: (u, c) is the unique solution of the constrained minimization problem

(2) $\begin{cases} \text{Minimize} & (u, c) \mapsto \displaystyle\int_\Omega \varphi(e(u), c) - f{\cdot}u \, \mathrm{d}x \\ \text{subject to} & \displaystyle\int_\Omega c \, \mathrm{d}x = C_{\text{total}}, \quad u \in H^1(\Omega; \mathbb{R}^d), \quad c \in L^2(\Omega), \end{cases}$

where $C_{\text{total}} > 0$ is a constant describing the total solvent content in the body which is now considered isolated. The chemical potential μ is constant (assuming the body Ω is connected) and represents the Lagrange multiplier to the affine constraint in (2).

In the general steady-state case, μ is no constant and the situation is more complicated. The variational structure of a certain subproblem, namely

(3) $\begin{cases} \text{Minimize} & (u, c) \mapsto \displaystyle\int_\Omega \varphi(e(u), c) - f{\cdot}u - \mu c \, \mathrm{d}x \\ \text{subject to} & u \in H^1(\Omega; \mathbb{R}^d), \quad c \in L^2(\Omega), \end{cases}$

for μ given can then be combined with the Schauder fixed-point theorem for μ solving a boundary-value problem for (1b) with (u, c) resulting from (3). In fact, a suitable regularization is to be used - the options are to put a gradient on $e(u)$ (i.e. the so-called 2nd-grade nonsimple materials) or on c (i.e. a capillarity) or to consider a gradient phase-field theory.

In the non-static steady-state situation, the flux generates also a heat. If not transferred away fast, it may lead to a substantial variation of temperature. Then the above outlined fixed-point argument is to be augmented by the heat-transfer problem.

Some of these problems bear a generalization for large strains [1]

References

[1] M. Kružík, T. Roubíček: *Mathematical Methods in Continuum Mechanics of Solids.* In preparation.