Diffusive interface dynamics in lattices

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Microscopic regularizations of ill-posed forward-backward diffusion equations have many applications and provide dynamical models for diffusive interfaces. Typical examples are the Cahn-Hilliard equation, the viscous approximations studied in [1, 4], and spatially discrete gradient systems such as

$$\dot{u}_j = \Delta p_j\,, \qquad p_j = \Phi'(u_j)\,, \qquad j \in \mathbb{Z}\,,$$

where Δ denotes the discrete Laplacian and Φ' is the bistable derivative of a double-well potential. While the Cahn-Hilliard case is well understood, very little is known about the mathematical analysis of viscous or lattice regularizations.

In this talk we choose Φ' as a trilinear function and demonstrate that the dynamics of both moving and standing phase interface is intimately related to a family of entropy inequalities and a hysteretic Stefan condition, see the figure and [2] for a simpler case.

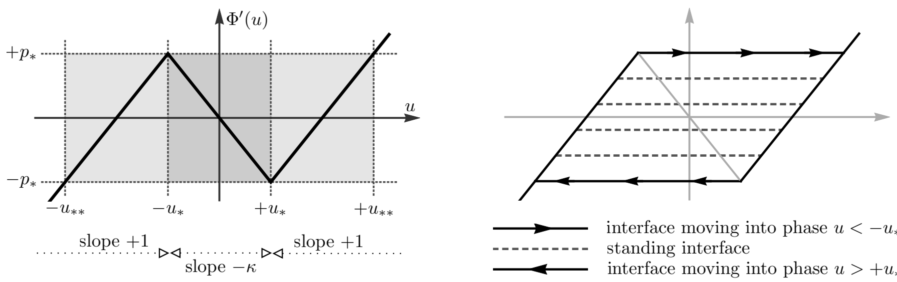

Figure. Cartoon of the nonlinearity Φ' and the macroscopic hysteresis loop.

Theorem (macroscopic evolution in the parabolic scaling limit)**.** *Identifying*

$$u_j(t) = U(\tau,\, \xi)\,, \qquad p_j(t) = P(\tau,\, \xi)\,, \qquad \tau = \varepsilon^2 t\,, \qquad \xi = \varepsilon j\,,$$

the limit $\varepsilon \to 0$ for well-prepared single-interface initial data is governed by

$$\partial_\tau U = \partial_\xi^2 P\,, \qquad U = P + \mathcal{M}[P]\,,$$

where the relay operator $\mathcal{M}$ acts pointwise in ξ on temporal functions. In particular, there exists a unique interface curve $\xi = \xi_*(\tau)$ which separates space-time regions with either $U \le -u_*$ or $U \ge +u_*$.

The key arguments in the proof are derived by a careful analysis of the lattice ODE and can informally be summarized as follows (see [3] for the details):

1. (1) Macroscopic interfaces exist according to a *persistence lemma.*
2. (2) The *waiting lemma* provides an upper bound for the interface speed.
3. (3) Microscopic phase transitions can be characterized by a *slow-fast splitting.*
4. (4) Several *fluctuation estimates* control the impact of the mesoscopic oscillations.

References

[1] L.C. Evans and M. Portilheiro, Irreversibility and hysteresis for a forward-backward diffusion equation, *Math. Models Methods Appl. Sci.*, **14-11**, (2004), 1599–1620.

[2] M. Helmers and M. Herrmann, Interface dynamics in discrete forward-backward diffusion equations, *SIAM Multiscale Model. Simul.*, **11-4**, (2013), 1261–1297.

[3] ——, *Macroscopic dynamics of diffusive phase interfaces in lattices*, *in preparation*, (2016).

[4] P.I. Plotnikov, Passing to the limit with respect to viscosity in an equation with variable parabolicity direction, *Differential Eqns.*, **30-4**, (1994), 614–622.