# About closed-loop control and observability of max-plus linear systems: Application to manufacturing systems

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# Outline

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- Motivation
- Dioid theory in a few words
- Timed event graph modeling
- Optimal closed-loop control
- Observer

# Dioid theory in a few words

## Dioid (or idempotent semiring)

A dioid is a set endowed with two operations $\oplus$ and $\otimes$ such that

- $\oplus$: associative, commutative, zero element denoted $\varepsilon$
- $\otimes$: associative, unit element denoted $e$
- $\otimes$ distributes over the sum: $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$ and $c \otimes (a \oplus b) = c \otimes a \oplus c \otimes b$
- Zero element $\varepsilon$ is absorbing: $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$
- $\oplus$ is idempotent: $a \oplus a = a$
- A dioid admits an *order relation* $\preceq$ defined by $b \preceq a \Leftrightarrow a \oplus b = a \Leftrightarrow a \wedge b = b$

## Example: (max,+)-algebra $\overline{\mathbb{Z}}_{max}$

More

$\mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with max as $\oplus$ and $+$ as $\otimes$. For example, $1 \oplus 1 = 1 = max(1,1)$ and $2 \otimes 1 = 3 = 2 + 1$.

# Diod theory in a few words

## Inequality $a \otimes x \preceq b$ and $x \otimes a \preceq b$

In a complete dioid, inequality $a \otimes x \preceq b$ (resp. $x \otimes a \preceq b$) admits a greatest solution, denoted $x = a \circ\!\!\!\backslash b$ (resp. $x = b \circ\!\!\!/ a$).

## Example

In $\overline{\mathbb{Z}}_{max}$, inequality $5 \otimes x \preceq 3$ admits a greatest solution $5 \circ\!\!\!\backslash 3 = 3 - 5 = -2$.

## Fixed-point equation $x = ax \oplus b$

Theorem: In a complete dioid, the least solution of $x = ax \oplus b$ is $x = a^* b$ with $a^* = \bigoplus_{i \geq 0} a^i = e \oplus a \oplus a^2 \oplus \ldots$ (*i.e.*, $a^*$ is the Kleene star of $a$).

# Diod theory in a few words

## Extension to matrix case

More

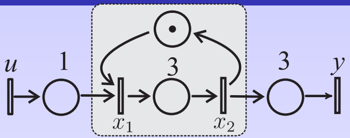
Let $A, B$ two matrices in $\overline{\mathbb{Z}}_{\max}^{n\times n}$

- $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$
- $(A \wedge B)_{ij} = A_{ij} \wedge B_{ij}$
- $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$
- $(A \backslash\!\!\circ B)_{ij} = \bigwedge_{k=1}^{n} A_{ki} \backslash\!\!\circ B_{kj}$, where $A \backslash\!\!\circ B$ is the greatest solution of $AX \preceq B$
- $(B \,/\!\!\circ A)_{ij} = \bigwedge_{k=1}^{n} B_{ik} \,/\!\!\circ A_{jk}$, where $B \,/\!\!\circ A$ is the greatest solution of $XA \preceq B$

## Other extensions

- Dioid of formal power series
- Quotient dioid

# Modeling in $\overline{\mathbb{Z}}_{\max}$




## Dater [Cohen et al., 85]

Dater: $t : \mathbb{Z} \to \overline{\mathbb{Z}}_{\max}$ such that $t(k)$ is the date of firing $k$ of transition $t$

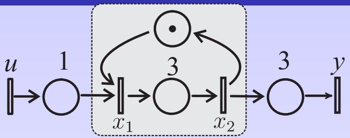
## Equations of the system

$$x_1(k) = \max(1 + u(k), x_2(k-1)) = 1u(k) \oplus x_2(k-1)$$
$$x_2(k) = 3 + x_1(k) = 3x_1(k) = 4u(k) \oplus 3x_2(k-1)$$
$$y(k) = 3 + x_2(k) = 3x_2(k)$$

## Matrix equations of the system

$$\begin{cases} x(k) = \begin{pmatrix} \varepsilon & e \\ \varepsilon & 3 \end{pmatrix} x(k-1) \oplus \begin{pmatrix} 1 \\ 4 \end{pmatrix} u(k) \\ y(k) = \begin{pmatrix} \varepsilon & 3 \end{pmatrix} x(k) \end{cases}$$

# Modeling in $\overline{\mathbb{Z}}_{\max}$




## Dater [Cohen et al., 85]

Dater: $t : \mathbb{Z} \rightarrow \overline{\mathbb{Z}}_{\max}$ such that $t(k)$ is the date of firing $k$ of transition $t$

## State-space representation

$$\begin{cases} x(k) = Ax(k-1) \oplus Bu(k) \\ y(k) = Cx(k) \end{cases}$$

## Drawback:

The previous input-output relation is not very handy.

# Modeling in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

## Operators [Cohen et al., 89]

- $\gamma$-operator: $(\gamma t)(k) = t(k-1)$
- $\delta$-operator: $(\delta t)(k) = 1t(k)$

The previous operators (and their linear combinations) correspond to elements in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.

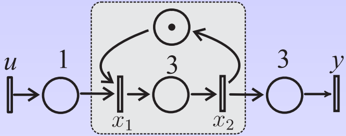
## State-space representation

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases}$$

## Transfer function matrix $H$

$$y = CA^*Bu = Hu$$

# Modeling in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$




## Equations of the system in $\overline{\mathbb{Z}}_{\max}$

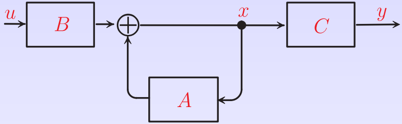
$$x_1(k) = 1u(k) \oplus x_2(k-1) = (\delta u)(k) \oplus (\gamma x_2)(k)$$
$$x_2(k) = 3x_1(k) = (\delta^3 x_1)(k)$$
$$y(k) = 3x_2(k) = (\delta^3 x_2)(k)$$

## State-space representation and transfer function matrix

▸ More

$$\begin{cases} x = \begin{pmatrix} \varepsilon & \gamma \\ \delta^3 & \varepsilon \end{pmatrix} x \oplus \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix} u \\ y = \begin{pmatrix} \varepsilon & \delta^3 \end{pmatrix} x \end{cases} \Rightarrow y = \delta^7 (\gamma\delta^3)^* u$$

# Optimal closed-loop control




**State-space representation**

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases}$$

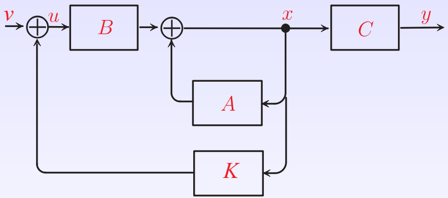
**Transfer function matrix**

$$H = CA^*B$$

# Optimal closed-loop control

## State feedback

$$u = Kx \oplus v$$




## State-space representation

$$\begin{cases} x = (A \oplus BK)x \oplus Bv \\ y = Cx \end{cases}$$

## Transfer function matrix

$$H_{cl} = C(A \oplus BK)^* B$$

## Objective

Compute the greatest controller $K$ such that
$$C(A \oplus BK)^* B \preceq G$$

# Optimal closed-loop control

## Synthesis

▸ More

$$C(A \oplus BK)^* B \preceq G \Leftrightarrow H(KA^*B)^* \preceq G$$
$$\Leftrightarrow (KA^*B)^* \preceq H \circ\backslash G$$

## A particular class of model reference

Proposition: If there exists $M$ (with entries in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$) such that $G = M^* H$, then $C(A \oplus BK)^* B \preceq G$ admits a greatest solution, denoted $\hat{K}$, and given by

$$\hat{K} = H \circ\backslash G \circ/ (A^* B)$$

# Optimal closed-loop control

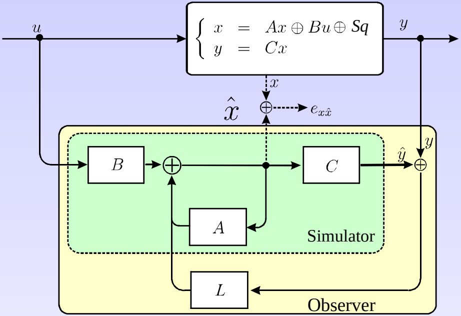
## The neutral state feedback

If $G = H$, *i.e.*, $M = Id$, the greatest state feedback is

$$\hat{K} = H \circ\!\!\!\!\backslash\, H \circ\!\!\!\!/\, (A^* B)$$

This controller delays as much as possible the input, whitout modifying the input/output behavior.

# Observer: Synthesis

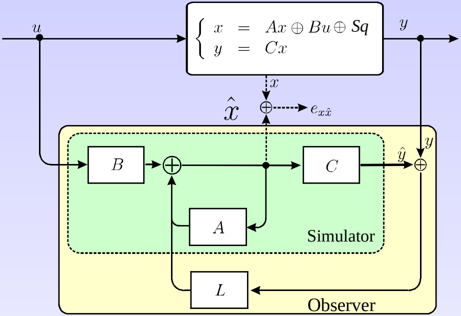



## Objective

Compute the greatest observer $L$ such that

$$\hat{x} \preceq x$$

# Observer: Synthesis




## System equations

▸ Matrix $S$

$$\begin{aligned} x &= Ax \oplus Bu \oplus Sq = A^*Bu \oplus A^*Sq \\ y &= Cx = CA^*Bu \oplus CA^*Sq \end{aligned}$$

## Observer equations

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \\ \hat{y} &= C\hat{x}. \end{aligned}$$

# Observer: Synthesis

## Objective

Compute the greatest observer $L$ such that

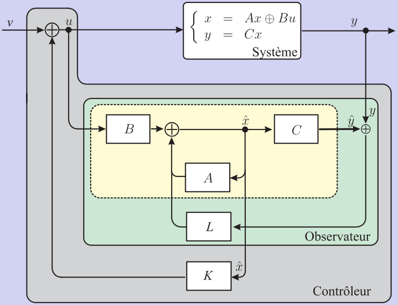
$$\begin{array}{rcl}(A \oplus LC)^* B & \preceq & A^* B \\ (A \oplus LC)^* LCA^* S & \preceq & A^* S\end{array}$$

## Optimal Observer

$$L_{opt} = ((A^* B) \phi H) \wedge ((A^* S) \phi (CA^* S))$$

# Observer: Control

## Principle




## Transfer function matrix

$$H_{cl} = H(K(A \oplus LC)^* B)^*$$

# Observer: Control

## Objective

Compute the greatest controller $K$ such that

$$H(K(A \oplus LC)^* B)^* \preceq G$$

## Controller $\hat{K}$

If $G = M^* H$, then the optimal controller exists and is given by

$$\hat{K} = H \backslash\!\!\!\circ\, G \,\circ\!\!\!/\, ((A \oplus LC)^* B)$$

# Conclusion

## Related works

- Application to High-Throughput Screening Systems
- Control the system in order to keep the state in a semi-module, *e.g.*, ensuring that $Dx = Ex$
- and more .......

## Software tools

- http://www.istia.univ-angers.fr/~hardouin
- http://www.scilab.org/contrib/

# Conclusion

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# References


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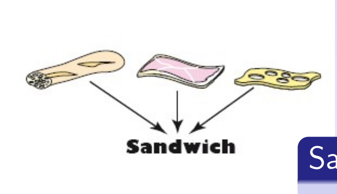
- Cottenceau B., Hardouin L., Boimond J.L. Model Reference Control for Timed Event Graphs in Dioids. Automatica, 37, 1451-1458, August 2001.
- L. Hardouin, C.A. Maia, B. Cottenceau , M. Lhommeau, E. Le Corronc Observer Design for (max,+) linear systems , IEEE Trans. on Automatic Control, vol.55-2, 2010, pp 538-543.

# Dioid theory in a few words

Back




**Sandwiches Algebra [Cohen et al. ]**

1 piece of Bread + 1 slice of ham + 1 slice of cheese is equal to 1 sandwich. Another way of counting !

# Dioid theory in a few words

## Sum of matrices $A \oplus B = C$

Back

$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \oplus \begin{pmatrix} e & 8 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & 7 \end{pmatrix}$$

## Product of matrices $A \otimes B = C$

Back

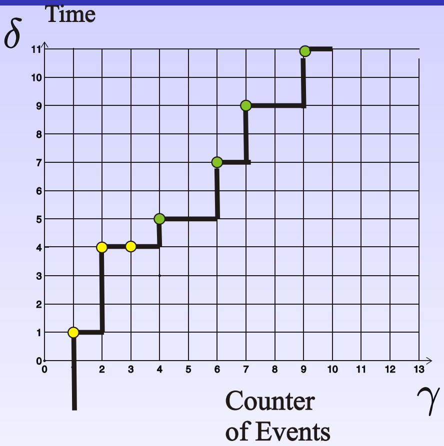
$$\begin{pmatrix} 2 & 5 \\ \varepsilon & 3 \\ 1 & 8 \end{pmatrix} \otimes \begin{pmatrix} e \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \otimes e \oplus 5 \otimes 1 \\ \varepsilon \otimes e \oplus 3 \otimes 1 \\ 1 \otimes e \oplus 8 \otimes 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$$

## Residuation of matrices $A \circ\!\!\!\backslash B$ is the greatest solution of $A \otimes X \preceq B$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \circ\!\!\!\backslash \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} = \begin{pmatrix} (1 \circ\!\!\!\backslash 8) \wedge (3 \circ\!\!\!\backslash 9) \wedge (5 \circ\!\!\!\backslash 10) \\ (2 \circ\!\!\!\backslash 8) \wedge (4 \circ\!\!\!\backslash 9) \wedge (6 \circ\!\!\!\backslash 10) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

# Modeling in $\overline{\mathbb{Z}}_{\max}$

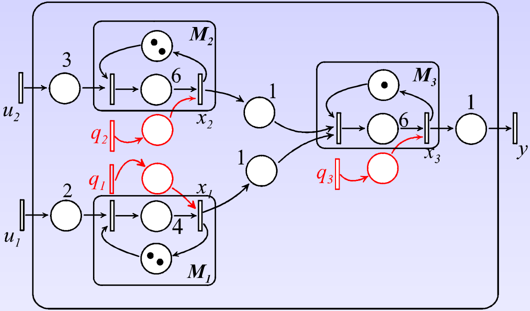
# Modeling in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$




Back

## a series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

$s = \bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k = 1\gamma \oplus 4\gamma^2 \oplus 5\gamma^4 \oplus 7\gamma^6 \oplus \ldots.$

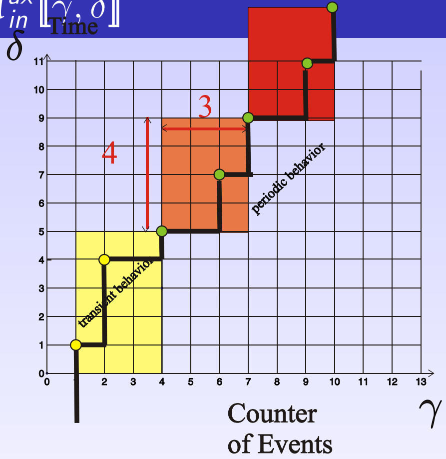



## Matrix $S$ and input $q$:

Back

- vector $q$ represents a vector of exogenous uncontrollable inputs (disturbance) which act on the system through matrix $S$.
- These disturbances lead to disable the transition firing, that is they decrease system performances and delay tokens output.

# Modeling in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$




Back

## a periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

$$s = (1\gamma \oplus 4\gamma^2) \oplus (5\gamma^4 \oplus 7\gamma^6)(4\gamma^3)^*$$

The throughput is denoted by $\sigma_\infty(s) = 3/4$

# Optimal closed-loop control

## Details :

- Recall :

$$(a \oplus b)^* = a^*(ba^*)^*$$
$$a(ba)^* = (ab)^*a$$

- Let

$$C(A \oplus BK)^*B = CA^*(BKA^*)^*B = CA^*B(KA^*B)^*$$

.

- hence :

$$C(A \oplus BK)^*B \preceq G_{ref} \Leftrightarrow (KA^*B)^* \preceq ((CA^*B) \circ\!\!\!\!\backslash\, G_{ref})\ .$$

Back