



Weierstrass Institute for
Applied Analysis and Stochastics



Rate-independent partial damage in viscoelastic materials with thermal effects

joint work with G. Lazzaroni (SISSA Trieste), R. Rossi (Brescia) & R. Toader (Udine)

Marita Thomas

Motivation

Damage in viscoelastic materials with thermal effects



small pieces of "minor importance",
exposed to
mech. forces & temperature changes
– but if they fail...

Modeling of damage in solids

Modeling concept:

Generalized Standard Materials [Halphen/Nguyen75]:

↔ Continuum damage mechanics [Kachanov58]

Introduce internal variable z (=damage variable) that

- models the changes of elastic behavior of the material due to evolving damage:

 stored elastic energy density $W(z, e(u))$

- is governed by rate-independent evolution

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For all $t \in [0, T]$ and all $x \in \Omega \subset \mathbb{R}^d$ we define:

local damage variable $z(t, x) := \frac{\mathcal{L}^d((\Omega \cap B_r(x)) \setminus (\text{holes at time } t))}{\mathcal{L}^d(\Omega \cap B_r(x))}$ $(r \text{ fixed}).$

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$z(t, x) = 0 :$ complete damage,

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In addition:

- damage z & temperature θ influence viscosity of material
- evolution of θ via heat equation

1. The model
2. Energetic formulation & Existence result
3. Analytical challenges

1. The model

body $\Omega \subset \mathbb{R}^3$: bdd., Lipschitz domain

state variables: displacement field $u : \Omega \rightarrow \mathbb{R}^3$, small strain tensor $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$
damage variable $z : \Omega \rightarrow [0, 1]$,
temperature $\theta : \Omega \rightarrow \mathbb{R}$, $\underbrace{\theta > 0}_{\text{to show}}$

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Densities (defined in Ω) of

free energy: $F(e(u), z, \nabla z, \theta) := \frac{1}{2}e(u) : \mathbb{C}(z) : e(u) + G(z, \nabla z) + \varphi(\theta) - \theta \mathbb{B} : e(u)$

$\mathbb{C} : \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3 \times 3 \times 3}$ cont., symm. & $\forall z \in \mathbb{R} \forall e \in \mathbb{R}^{3 \times 3} : C_1|e|^2 \leq e : \mathbb{C}(z) : e \leq C_2|e|^2$

$G(z, \nabla z) := I_{[0,1]}(z) + \text{Regularization}(z, \nabla z)$,

e.g. $\text{Regularization}(\nabla z) := |\nabla z|^r \quad r \in (1, \infty)$

or $\text{Regularization}_\varepsilon(z, \nabla z) := \varepsilon z^2(1-z)^2 + \varepsilon^{-1}|\nabla z|^2$ Modica-Mortola-type

$\varphi(\theta) := \theta(\log(\theta) - 1)$

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$\mathbb{D} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3 \times 3 \times 3}$ cont., symm. & $\forall (z, \theta) \in \mathbb{R}^2 \forall e \in \mathbb{R}^{3 \times 3} : C_1|e|^2 \leq e : \mathbb{D}(z, \theta) : e \leq C_2|e|^2$

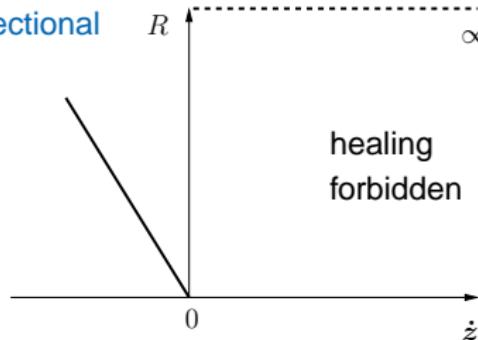
R_1 : pos. 1-homogeneous (\leftrightarrow rate-independent), unidirectional

$$R_1(\dot{z}) := \begin{cases} \rho |\dot{z}| & \text{if } \dot{z} \leq 0, \\ \infty & \text{otw.} \end{cases}$$

\Downarrow

Dissipation distance (density):

$$D(z_1, z_2) = R_1(z_2 - z_1)$$



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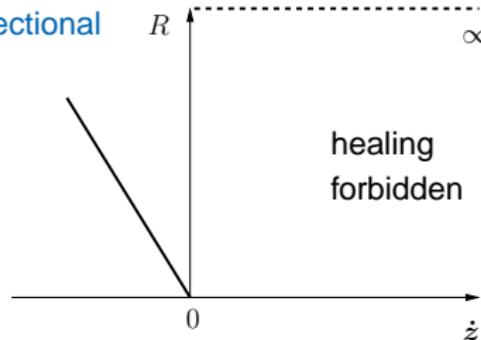
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$\mathbb{K} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3}$ cont., symm.,

$\exists \kappa \in (1, 5/3) \forall (z, \theta) \in \mathbb{R}^2 \forall \xi \in \mathbb{R}^3 :$

$$c_1(|\theta|^\kappa + 1)|\xi|^2 \leq \xi \cdot \mathbb{K}(z, \theta)\xi \leq c_2(|\theta|^\kappa + 1)|\xi|^2$$



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stress σ involves Kelvin-Voigt rheology & thermal expansion

$$\sigma := \partial_e F + \partial_{\dot{e}} R = \mathbb{C}(z) e(u) + \mathbb{D}(z, \theta) e(\dot{u}) - \theta \mathbb{B}$$

System of PDEs in $(0, T) \times \Omega$:

balance of momentum: $\rho \ddot{u} - \operatorname{div} \sigma = f$

flow rule for z : $0 \in \partial_z F - \operatorname{div} D_{\nabla z} F + \partial_z R_1$

heat equation: $\dot{\theta} - \operatorname{div} (\mathbb{K}(z, \theta) \nabla \theta) = |\dot{z}| + 2R_2(e(\dot{u})) - \theta \mathbb{B} : e(\dot{u}) + h$
+ BCs + ICs

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Challenges in the existence analysis:

- mixed character of the system:
rate-independent vs. rate-dependent
- nonlinear coupling of the variables
- nonsmooth terms
- dissipation rates act as heat sources

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Challenges in the existence analysis:

Suitable weak formulation
needed!



- mixed character of the system:
rate-independent vs. **rate-dependent**
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2. Energetic formulation [Roubíček10]

in terms of 4 conditions:

1. Weak formulation of the momentum balance for u
2. Semistability for z
3. mechanical energy balance
4. weak formulation of the heat equation for θ

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Subdifferential flow rule $0 \in \partial_z F - \operatorname{div} \partial_{\nabla z} F + \partial_z R_1$
replaced by semistability & energy balance
(formulation in terms of functionals instead of derivatives)

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Notation: State space $\Omega = \mathcal{U} \times \mathcal{Z} \times \mathcal{W}$ (Banach space)

\mathcal{U} for displacements

\mathcal{Z} for damage variable

\mathcal{W} for temperature

Mechanical energy: $\mathcal{E}(t, u, z) := \int_{\Omega} \frac{\rho}{2} |\dot{u}|^2 + \frac{1}{2} e(u) : \mathbb{C}(z) : e(u) \, dx - \langle f(t), u \rangle$

2. Energetic formulation of the process

Definition (Energetic formulation [Roubíček10]):

A triple $q = (u, z, \theta) : [0, T] \rightarrow \Omega$ is called an energetic solution to the IBVP iff $q(0) = q_0, \dot{u}(0) = u_1$ for sufficiently regular initial data $q_0 = (u_0, z_0, \theta_0), u_1$, and:

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$$\int_{\Omega} \rho \dot{u}(T) \cdot v(T) \, dx + \int_0^T \int_{\Omega} ((e(u) : \mathbb{C}(z) + e(\dot{u}) : \mathbb{D}(z, \theta) - \theta \mathbb{B}) : e(v) - \dot{u} \cdot \dot{v}) \, dx \, dt \\ = \int_{\Omega} u_0 \cdot v(0) \, dx + \int_0^T \langle f, v \rangle \, dt$$

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4. weak formulation of the heat equation:

For all $t \in [0, T]$ & all test fct.s $\eta \in H^1(0, T; L^2(\Omega)) \cap C^0(0, T; W^{2,3+\delta}(\Omega))$:

$$\begin{aligned} \langle \theta(t), \eta(t) \rangle + \int_0^t \int_{\Omega} (\mathbb{K}(\theta, z) \nabla \theta \cdot \nabla \eta - \theta \dot{\eta}) dx dt \\ = \int_{\Omega} \theta_0 \eta(0) dx + \int_0^t \int_{\Omega} (2\mathcal{R}_2(e(\dot{u}), z, \theta) + \mathcal{R}_1(\dot{z}) - \theta \mathbb{B} : e(\dot{u})) \eta dx dt \end{aligned}$$

2. Energetic formulation – Existence result

Theorem [Lazzaroni/Rossi/Th/Toader14] : For all f, h, q_0, u_1 suff. regular there exists an energetic solution $q = (u, z, \theta) : [0, T] \rightarrow \Omega$ and $\theta > 0$ provided $\theta_0 > \theta_* > 0$.

Strategy of the proof:

- time-discretization: Existence at each time step τ_n by *direct method* for z and *theory of pseudomonotone operators* for coupled system of momentum balance & heat equation
- RHS of heat equation $(2R_2(e(\dot{u}), z, \theta) + R_1(\dot{z}) - \theta \mathbb{B} : e(\dot{u}))$ requires *regularization* in discrete momentum balance by γ -Laplacian $\tau_n \operatorname{div}|e(u_n^k)|^{\gamma-2} e(u_n^k)$ with $\gamma > 4$

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- define piecewise const. $(\bar{q}_n, \underline{q}_n)$ & linear q_n interpolants
- prove discrete version of energetic formulation \rightsquigarrow a priori estimates for $(\bar{q}_n, \underline{q}_n, q_n)_n$
But: first a priori information on $(\bar{\theta}_n)_n$ only wrt. $L^\infty(0, T; L^1(\Omega))$...
- *refined* a priori estimates on $(\bar{\theta}_n)_n$ by arguments from [Feireisl/Pezeljová/Rocca09]: clever test of discrete weak heat equation & interpolation

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- RHS of heat equation $(2R_2(e(\dot{u}), z, \theta) + R_1(\dot{z}) - \theta \mathbb{B} : e(\dot{u}))$ requires *regularization* in discrete momentum balance by γ -Laplacian $\tau_n \operatorname{div}|e(u_n^k)|^{\gamma-2} e(u_n^k)$ with $\gamma > 4$
- define piecewise const. $(\bar{q}_n, \underline{q}_n)$ & linear q_n interpolants
- prove discrete version of energetic formulation \rightsquigarrow a priori estimates for $(\bar{q}_n, \underline{q}_n, q_n)_n$
But: first a priori information on $(\bar{\theta}_n)_n$ only wrt. $L^\infty(0, T; L^1(\Omega))$...
- refined a priori estimates on $(\bar{\theta}_n)_n$ by arguments from [Feireisl/Pezelcová/Rocca09]: clever test of discrete weak heat equation & interpolation
- selection of convergent subsequences by version of Helly's selection principle (for rate-independent contributions) \rightsquigarrow limit triple (u, z, θ)
- limit passage $n \rightarrow \infty$: Show that (u, z, θ) satisfies time-continuous energetic formulation

3. Challenge: Limit passage time-discrete → continuous
in momentum balance regularized by γ -Laplacian
via Mosco-convergence and refined estimates for θ

Message: just another application of abstract results in Riccarda's talk!

3. Limit passage in regularized momentum balance

Time-discrete momentum balance:

For every $(n+1)$ -tuple $\{v_n^k\}_{k=0}^n \subset W_D^{1,\gamma}(\Omega, \mathbb{R}^3)$ for some $\gamma > 4$ fixed

set $\bar{v}_n(t) := v_n^k$ and $v_n(t) := \frac{t-t_n^{k-1}}{\tau_n} v_n^k + \frac{t_n^k-t}{\tau_n} v_n^{k-1}$ for $t \in (t_n^{k-1}, t_n^k]$:

$$\begin{aligned} & \rho \int_{\Omega} (\dot{u}_n(T) \cdot v_n(T) - \dot{u}_0 \cdot v_n(0)) dx - \rho \int_0^T \int_{\Omega} \dot{u}_n(t - \tau_n) \cdot \dot{v}_n(t) dx dt \\ & + \int_0^T \int_{\Omega} (e(\dot{u}_n) : \mathbb{D}(z_n, \underline{\theta}_n) + e(\bar{u}_n) : \mathbb{C}(\bar{z}_n) - \bar{\theta}_n \mathbb{B} + \tau_n |e(\bar{u}_n)|^{\gamma-2} e(\bar{u}_n)) : e(\bar{v}_n) dx dt \\ & = \int_0^T \langle \bar{f}_n, \bar{v}_n \rangle dt \end{aligned}$$

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whereas:

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For all $v \in L^2(0, T; W_D^{1,2}(\Omega, \mathbb{R}^3)) \cap W^{1,2}(0, T; L^2(\Omega, \mathbb{R}^3))$:

$$\begin{aligned} & \int_{\Omega} \rho \dot{u}(T) \cdot v(T) dx + \int_0^T \int_{\Omega} ((e(u) : \mathbb{C}(z) + e(\dot{u}) : \mathbb{D}(z, \theta) - \theta \mathbb{B}) : e(v) - \dot{u} \cdot \dot{v}) dx dt \\ & = \int_{\Omega} u_0 \cdot v(0) dx + \int_0^T \langle f, v \rangle dt \end{aligned}$$

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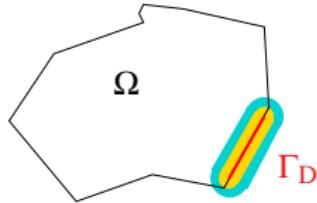
3. Limit passage in regularized momentum balance, ctd.

Construction of the recovery sequence:

Step 1: pointwise wrt. time

1. To preserve homog. Dirichlet BCs and obtain strong $W^{1,2}(\Omega, \mathbb{R}^3)$ -convergence:

Construction from [Mielke/Roubíček/Th.12]:



Given $v \in W_D^{1,2}(\Omega, \mathbb{R}^3)$ $\tilde{v}_n \in W_D^{1,2}(\Omega, \mathbb{R}^3)$ s.th.

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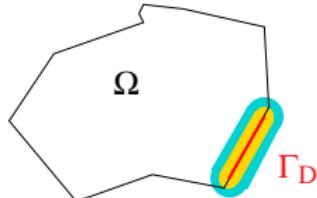
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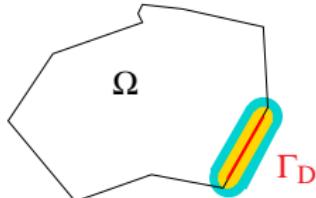
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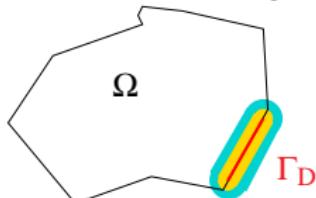
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convergence of all the black terms by construction of $(v_n, \bar{v}_n)_n$ and a priori information:
 $\tau_n^{1/\gamma} \|e(\bar{u}_n)\|_{L^\gamma} \leq C$, $u_n \rightharpoonup u$ in $W^{1,2}(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ & $\forall t: u_n(t) \rightharpoonup u(t)$ in $W^{1,2}(\Omega, \mathbb{R}^3)$

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For orange terms:

Further a priori information: for a.a. $t \in (0, T)$: $\bar{z}_n(t), z_n(t) \rightharpoonup z(t)$ in $W^{1,r}(\Omega)$ (by $G(z, \nabla z)$)

Refined bounds for $(\bar{\theta}_n)_n$ by [Feireisl/Pezeljová/Rocca09]-technique:

$$\|\bar{\theta}_n\|_{L^2(0,T;H^1(\Omega))} + \|\bar{\theta}_n\|_{L^2((0,T) \times \Omega)} + \|\bar{\theta}_n\|_{BV(0,T;W^{2,3+\delta}(\Omega))}$$

Aubin-Lions: $\bar{\theta}_n, \underline{\theta}_n \rightarrow \theta$ in $L^2(0, T; L^2(\Omega))$

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For every $(n+1)$ -tuple $\{v_n^k\}_{k=0}^n \subset W_D^{1,\gamma}(\Omega, \mathbb{R}^3)$ for some $\gamma > 4$ fixed

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$$\begin{aligned} & \rho \int_{\Omega} (\dot{u}_n(T) \cdot v_n(T) - \dot{u}_0 \cdot v_n(0)) dx - \rho \int_0^T \int_{\Omega} \dot{u}_n(t - \tau_n) \cdot \dot{v}_n(t) dx dt \\ & + \int_0^T \int_{\Omega} (e(\dot{u}_n) : \mathbb{D}(z_n, \underline{\theta}_n) + e(\bar{u}_n) : \mathbb{C}(\bar{z}_n) - \bar{\theta}_n \mathbb{B}) + \tau_n |e(\bar{u}_n)|^{\gamma-2} e(\bar{u}_n) : e(\bar{v}_n) dx dt \\ & = \int_0^T \langle \bar{f}_n, \bar{v}_n \rangle dt \end{aligned}$$

convergence of all the black terms by construction of $(v_n, \bar{v}_n)_n$ and a priori information:
 $\tau_n^{1/\gamma} \|e(\bar{u}_n)\|_{L^\gamma} \leq C$, $u_n \rightharpoonup u$ in $W^{1,2}(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ & $\forall t: u_n(t) \rightharpoonup u(t)$ in $W^{1,2}(\Omega, \mathbb{R}^3)$

For orange terms:

Further a priori information: for a.a. $t \in (0, T)$: $\bar{z}_n(t), z_n(t) \rightharpoonup z(t)$ in $W^{1,r}(\Omega)$ (by $G(z, \nabla z)$)

Refined bounds for $(\bar{\theta}_n)_n$ by [Feireisl/Pezeljová/Rocca09]-technique:

$$\|\bar{\theta}_n\|_{L^2(0,T;H^1(\Omega))} + \|\bar{\theta}_n\|_{L^2((0,T) \times \Omega)} + \|\bar{\theta}_n\|_{BV(0,T;W^{2,3+\delta}(\Omega))}$$

Aubin-Lions: $\bar{\theta}_n, \underline{\theta}_n \rightarrow \theta$ in $L^2(0, T; L^2(\Omega))$

\Rightarrow pointw. a.e. convergent subsequences $(e(\bar{v}_n), \bar{z}_n, z_n, \bar{\theta}_n, \underline{\theta}_n) \rightarrow (e(v), z, \theta)$

\mathbb{C}, \mathbb{D} continuous & L^∞ -bound. Hence, by dominated convergence:

$$(\mathbb{D}(z_n, \underline{\theta}_n) + \mathbb{C}(\bar{z}_n) - \bar{\theta}_n \mathbb{B}) : e(\bar{v}_n) \rightarrow (\mathbb{D}(z, \theta) + \mathbb{C}(z) - \theta \mathbb{B}) : e(v)$$

3. Limit passage in regularized momentum balance, ctd.

To summarize, we have shown for:

$$\mathcal{L}_n : L^\gamma(0, T; W_D^{1,\gamma}(\Omega, \mathbb{R}^3)) \rightarrow \mathbb{R},$$

$$\mathcal{L}_n(v) := \int_0^T \int_\Omega \left(\frac{1}{2} e(v) : \mathbb{C}(\bar{z}_n) : e(v) - \bar{\theta}_n \mathbb{B} : e(v) + \frac{\tau_n}{\gamma} |e(v)|^\gamma \right) dx dt$$

$$\tilde{\mathcal{R}}_{2n}(v) := \int_0^T \int_\Omega \frac{1}{2} e(v) : (\mathbb{D}(\underline{z}_n, \underline{\theta}_n) : e(v)), dx dt$$

that for all $v \in L^2(0, T; W_D^{1,2}(\Omega, \mathbb{R}^3))$ exists $(\bar{v}_n)_n \subset L^\gamma(0, T; W_D^{1,\gamma}(\Omega, \mathbb{R}^3))$ with

$\bar{v}_n \rightarrow v$ strongly in $L^2(0, T; W_D^{1,2}(\Omega, \mathbb{R}^3))$ and

$$\mathcal{L}_n(\bar{v}) \rightarrow \mathcal{L}(v) := \int_0^T \int_\Omega \left(\frac{1}{2} e(v) : \mathbb{C}(z) : e(v) - \theta \mathbb{B} : e(v) \right) dx dt$$

$$\tilde{\mathcal{R}}_{2n}(\bar{v}_n) \rightarrow \int_0^T \mathcal{R}_2(v) dt$$

Hence: Mosco-convergence

and for sequences of approximating solutions $(\bar{u}_n, \dot{u}_n) \rightharpoonup (u, \dot{u})$ in \mathcal{U} :

$$D_e(\mathcal{L}_n(\bar{u}_n) + 2\mathcal{R}_2(e(\dot{u}_n)) dt) \rightarrow D_e(\mathcal{L}(u) + 2\mathcal{R}_2(e(\dot{u})) dt) \quad \text{in } \mathcal{U}^*$$

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Thank You!