Controlling the Controller: Hysteresis-Delay Differential Equations

Eyal Ron

April 10, 2014

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t - T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

Hysteresis-Delay Differential equations

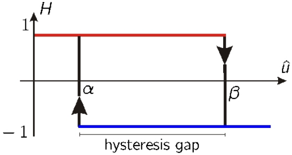
$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

where Initial Conditions for $(\varphi, v) \in (\mathbb{L}^2(-T,0))^N \times \mathbb{R}^N$

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t - T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

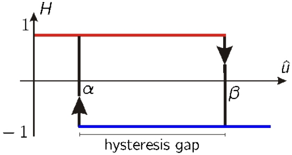
where Initial Conditions for $(\varphi, v) \in (\mathbb{L}^2(-T, 0))^N \times \mathbb{R}^N$
Hysteresis: nonideal relay



Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

where Initial Conditions for $(\varphi, v) \in (\mathbb{L}^2(-T, 0))^N \times \mathbb{R}^N$
Hysteresis: nonideal relay

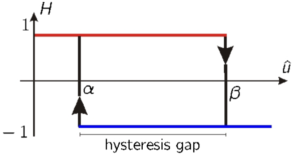


- $u(t) \in \mathbb{R}^n$.

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

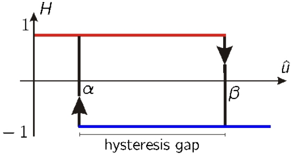
where Initial Conditions for $(\varphi, v) \in (\mathbb{L}^2(-T, 0))^N \times \mathbb{R}^N$
Hysteresis: nonideal relay



- $u(t) \in \mathbb{R}^n$.
- $u \in \mathbb{L}^2(-T, \infty))^N \cap \mathbb{H}^1(0, \infty))^N$.

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

where Initial Conditions for $(\varphi, v) \in (\mathbb{L}^2(-T, 0))^N \times \mathbb{R}^N$
Hysteresis: nonideal relay



- $u(t) \in \mathbb{R}^n$.
- $u \in \mathbb{L}^2(-T, \infty))^N \cap \mathbb{H}^1(0, \infty))^N$.
- Solution becomes piecewise smoother with time.

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

- We study orbital stability of a given periodic solution of period T in the $(\mathbb{L}^2(-T,0))^N \times \mathbb{R}^N$ space.

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

- We study orbital stability of a given periodic solution of period T in the $(\mathbb{L}^2(-T,0))^N \times \mathbb{R}^N$ space.
- Infinite dimensional problem due to the delay.

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

- We study orbital stability of a given periodic solution of period T in the $(\mathbb{L}^2(-T,0))^N \times \mathbb{R}^N$ space.
- Infinite dimensional problem due to the delay.

Main result: finite dimensional reduction

Reduce the question of stability of periodic solutions to a finite dimensional eigenvalue problem.

Hysteresis-Delay Differential equations

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

- We study orbital stability of a given periodic solution of period T in the $(\mathbb{L}^2(-T,0))^N \times \mathbb{R}^N$ space.
- Infinite dimensional problem due to the delay.

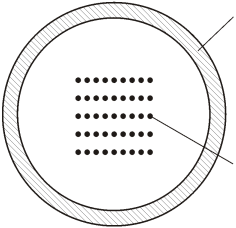
Main result: finite dimensional reduction

Reduce the question of stability of periodic solutions to a finite dimensional eigenvalue problem.

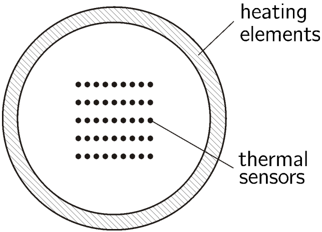
Stabilization 3D systems

Sufficient conditions under which an unstable 3D system can be stabilized.

Application: thermal control

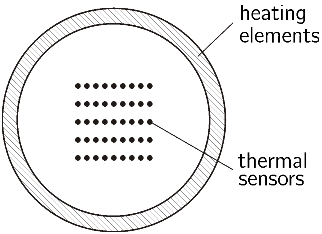


Application: thermal control



Reactor - $Q \subset \mathbb{R}^m$

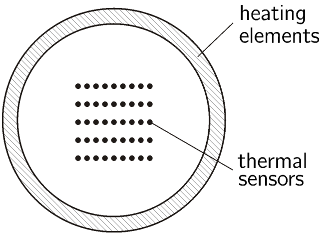
Application: thermal control



Reactor - $Q \subset \mathbb{R}^m$

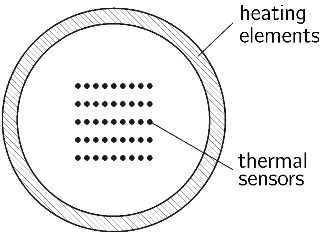
Temperature - $u(x, t)$

Application: thermal control



Reactor - $Q \subset \mathbb{R}^m$
Temperature - $u(x,t)$
Density of sensors - $m(x)$

Application: thermal control



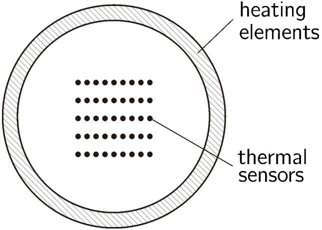
Reactor - $Q \subset \mathbb{R}^m$

Temperature - $u(x,t)$

Density of sensors - $m(x)$

Density of heating elements - $K(x)$

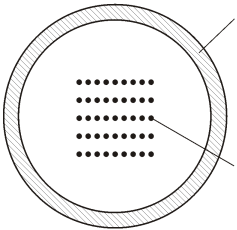
Application: thermal control



Reactor - $Q \subset \mathbb{R}^m$
Temperature - $u(x, t)$
Density of sensors - $m(x)$
Density of heating elements - $K(x)$

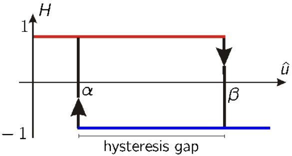
$u_t = \Delta u$, $x \in Q \subset \mathbb{R}^m$ bounded
$u(x, 0) = \phi(x)$

Application: thermal control



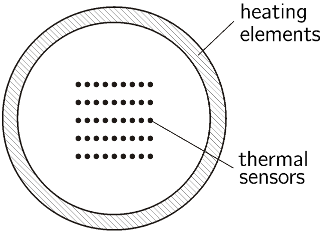
Reactor - $Q \subset \mathbb{R}^m$
Temperature - $u(x,t)$
Density of sensors - $m(x)$
Density of heating elements - $K(x)$

$u_t = \Delta u$, $x \in Q \subset \mathbb{R}^m$ bounded
$u(x,0) = \phi(x)$
$\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t)$ on ∂Q,



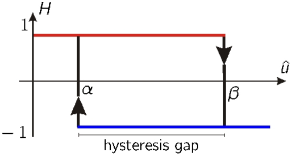
where $\hat{u}(t) = \int_Q m(x)u(x,t)dx$ is the mean temperature.

Application: thermal control



Reactor - $Q \subset \mathbb{R}^m$
Temperature - $u(x,t)$
Density of sensors - $m(x)$
Density of heating elements - $K(x)$

$u_t = \Delta u$, $x \in Q \subset \mathbb{R}^m$ bounded
$u(x,0) = \phi(x)$
$\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t)$ on ∂Q,



where $\hat{u}(t) = \int_Q m(x)u(x,t)dx$ is the mean temperature.

<u>Goal:</u> Study stability of periodic solutions.

History Survey

- Problem suggested by Glashoff and Sprekels [1981, 1982].

History Survey

- Problem suggested by Glashoff and Sprekels [1981, 1982].
- Pruess [1985], Friedman, L.-S. Jiang [1988]. Dimension 1: periodicty.

History Survey

- Problem suggested by Glashoff and Sprekels [1981, 1982].
- Pruess [1985], Friedman, L.-S. Jiang [1988]. Dimension 1: periodicty.
- Kopfová, Kopf [2002]. Compared thermal control with hysteresis to ODE with delay in the hysteresis: $\mathcal{H}(\hat{u})(t - T)$.

History Survey

- Problem suggested by Glashoff and Sprekels [1981, 1982].
- Pruess [1985], Friedman, L.-S. Jiang [1988]. Dimension 1: periodicty.
- Kopfová, Kopf [2002]. Compared thermal control with hysteresis to ODE with delay in the hysteresis: $\mathcal{H}(\hat{u})(t-T)$.
- Gurevich [2011] and Gurevich and Tikhomirov [2012]: periodicity at higher dimensions. Existence of stable and unstable periodic solutions.

Delayed Feedback Control (Pyragas Control)

- $\dot{u}(t) = f(u(t))$

Delayed Feedback Control (Pyragas Control)

- $\dot{u}(t) = f(u(t))$
- $u_p(t)$ is a solution with minimal period $T > 0$:
  $u_p(t + T) = u_p(t)$.

Delayed Feedback Control (Pyragas Control)

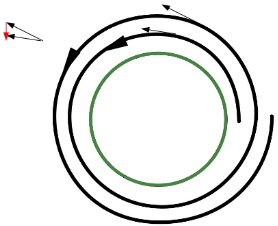
- $\dot{u}(t) = f(u(t))$
- $u_p(t)$ is a solution with minimal period $T > 0$:
  $u_p(t + T) = u_p(t)$.
- Pyragas [1992] Delayed feedback control:
  $\dot{u}(t) = f(u(t)) + b(u(t - T) - u(t))$.

Delayed Feedback Control (Pyragas Control)

- $\dot{u}(t) = f(u(t))$
- $u_p(t)$ is a solution with minimal period $T > 0$:
  $u_p(t + T) = u_p(t)$.
- Pyragas [1992] Delayed feedback control:
  $\dot{u}(t) = f(u(t)) + b(u(t - T) - u(t))$.
- $u_p(t) - u_p(t - T) = 0$

- $\dot{u}(t) = f(u(t))$
- $u_p(t)$ is a solution with minimal period $T > 0$: $u_p(t + T) = u_p(t)$.
- Pyragas [1992] Delayed feedback control: $\dot{u}(t) = f(u(t)) + b(u(t - T) - u(t))$.
- $u_p(t) - u_p(t - T) = 0$

Why does it work? Geometrically:



Thermal control with Pyragas Control

Thermal control problem

$$
\begin{aligned}
&u_t = \Delta u \quad x \in Q \\
&u(x,0) = \phi(x) \\
&\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t) \quad x \in \partial Q.
\end{aligned}
$$

Is turned into

$$
\begin{aligned}
&u_t = \Delta u \quad x \in Q \\
&u(x,0) = \phi(x) \\
&\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t) + b(x)(\hat{u}(t) - \hat{u}(t-T)) \quad x \in \partial Q,
\end{aligned}
$$

where $\hat{u}(t) = \int_Q m(x)u(x,t)dx$

Thermal control with Pyragas Control

Thermal control problem

$$
\begin{aligned}
&u_t = \Delta u \quad x \in Q \\
&u(x,0) = \phi(x) \\
&\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t) \quad x \in \partial Q.
\end{aligned}
$$

Is turned into

$$
\begin{aligned}
&u_t = \Delta u \quad x \in Q \\
&u(x,0) = \phi(x) \\
&\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t) + b(x)(\hat{u}(t) - \hat{u}(t-T)) \quad x \in \partial Q,
\end{aligned}
$$

where $\hat{u}(t) = \int_Q m(x)u(x,t)dx$

- Fourier decomposition: reduction to infinite dimensional ODE.

Thermal control with Pyragas Control

Thermal control problem

$$
\begin{aligned}
&u_t = \Delta u \quad x \in Q \\
&u(x,0) = \phi(x) \\
&\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t) \quad x \in \partial Q.
\end{aligned}
$$

Is turned into

$$
\begin{aligned}
&u_t = \Delta u \quad x \in Q \\
&u(x,0) = \phi(x) \\
&\frac{\partial u}{\partial \nu} = K(x)\mathbb{H}(\hat{u})(t) + b(x)(\hat{u}(t) - \hat{u}(t-T)) \quad x \in \partial Q,
\end{aligned}
$$

where $\hat{u}(t) = \int_Q m(x)u(x,t)dx$

- Fourier decomposition: reduction to infinite dimensional ODE.
- If the sensor density $m(x)$ has a finite number of nonzero modes, then reduction to a finite dimensional ODE.

Dimension Reduction: Poincaré Map

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

Dimension Reduction: Poincaré Map

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

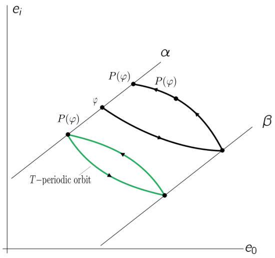


Figure : Poincaré without delay

$$\dot{u}(t) = Au + k\mathcal{H}(u)(t) + B[u(t-T) - u(t)] \text{ a.e for } t \geq 0$$
$$u(t) = \varphi(t) \text{ a.e for } t \in [-T, 0]$$
$$u(0+) = v$$

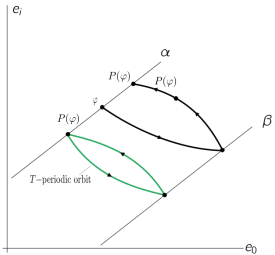


Figure : Poincaré without delay

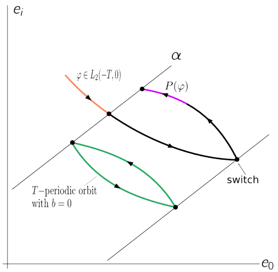


Figure : Poincaré with delay

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V}$$

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V}$$

- **F** is a finite rank operator, i.e., Range(F) is finite dimensional.

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V}$$

- **F** is a finite rank operator, i.e., Range(F) is finite dimensional.
- **V** is a volterra operator (in N-dimensions!)

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V}$$

- **F** is a finite rank operator, i.e., Range(F) is finite dimensional.
- **V** is a volterra operator (in N-dimensions!)
- **V** has only the eiganvalue 0 at its spectrum.

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V}$$

- **F** is a finite rank operator, i.e., Range(F) is finite dimensional.
- **V** is a volterra operator (in N-dimensions!)
- **V** has only the eiganvalue 0 at its spectrum.
- $D\mathbf{P}$ is a compact operator, whose spectrum has only eigenvalues.

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V},$$

Dimension Reduction

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V},$$

Dimension Reduction

- $\lambda \neq 0$ is an eigenvalue if and only if there is a ξ such that $(\lambda I - V - F)\xi = 0$.

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V},$$

Dimension Reduction

- $\lambda \neq 0$ is an eigenvalue if and only if there is a ξ such that $(\lambda I - V - F)\xi = 0$.
- Let $\rho = (\lambda I - V)\xi \Rightarrow \boxed{\xi = (\lambda I - V)^{-1}\rho.}$

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V},$$

Dimension Reduction

- $\lambda \neq 0$ is an eigenvalue if and only if there is a ξ such that $(\lambda I - V - F)\xi = 0$.
- Let $\rho = (\lambda I - V)\xi \Rightarrow \boxed{\xi = (\lambda I - V)^{-1}\rho.}$
- Then

$$0 = (\lambda I - V - F)(\lambda I - V)^{-1}\rho = (I - F(\lambda I - V)^{-1})\rho$$

Dimension Reduction: Analyzing Poincaré Map

Derivative of Poincaré map has the form:

$$D\mathbf{P} = \mathbf{F} + \mathbf{V},$$

Dimension Reduction

- $\lambda \neq 0$ is an eigenvalue if and only if there is a ξ such that $(\lambda I - V - F)\xi = 0$.
- Let $\rho = (\lambda I - V)\xi \Rightarrow \boxed{\xi = (\lambda I - V)^{-1}\rho.}$
- Then

$$0 = (\lambda I - V - F)(\lambda I - V)^{-1}\rho = (I - F(\lambda I - V)^{-1})\rho$$

nonzero λ is an eigenvalue of $D\mathbf{P}$ if and only if the finite rank operator $(I - F(\lambda I - V)^{-1})\rho$ has a non-trivial kernel.

Stabilization: 3D Example

Stabilizing a 3D example from Gurevich and Tikhomirov.

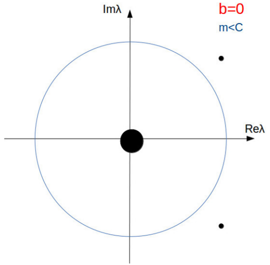
Stabilization: 3D Example

Stabilizing a 3D example from Gurevich and Tikhomirov.

- m: control parameter of the original system

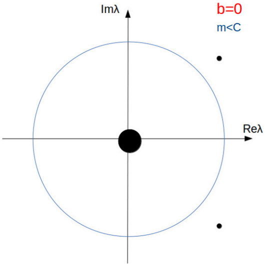
Stabilizing a 3D example from Gurevich and Tikhomirov.

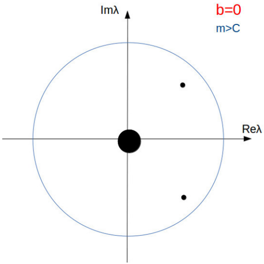
- m: control parameter of the original system



Stabilizing a 3D example from Gurevich and Tikhomirov.

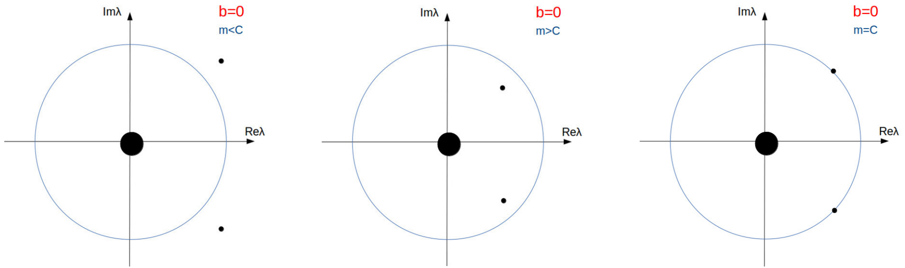
- m: control parameter of the original system





Stabilizing a 3D example from Gurevich and Tikhomirov.

- m: control parameter of the original system



Stabilization: 3D Example

Stabilizing a 3D example from Gurevich and Tikhomirov.

- m: control parameter of the original system
- b: pyragas control parameter.

Stabilization: 3D Example

Stabilizing a 3D example from Gurevich and Tikhomirov.

- m: control parameter of the original system
- b: pyragas control parameter.
- J: a polynomial, achieved from the finite rank operator.

Stabilization: 3D Example

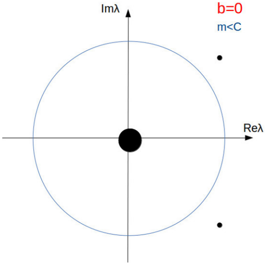
Stabilizing a 3D example from Gurevich and Tikhomirov.

- m: control parameter of the original system
- b: pyragas control parameter.
- J: a polynomial, achieved from the finite rank operator.
- A complex $\lambda = \lambda_1 + i\lambda_2$ is an eigenvalue of the Poincaré map if and only if

$$J(b, m, \lambda_1, \lambda_2) = J_1(b, m, \lambda_1, \lambda_2) + iJ_2(b, m, \lambda_1, \lambda_2) = 0,$$

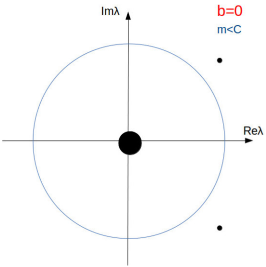
Stabilization: 3D Example

For $b = 0$ (no delay):

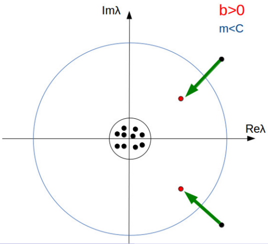


Stabilization: 3D Example

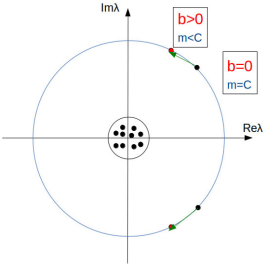
For $b = 0$ (no delay):



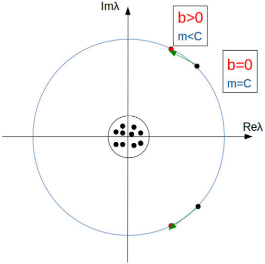
Want ($b \neq 0$):



Stabilization: 3D Example

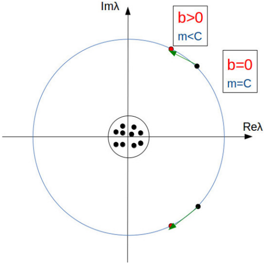


Stabilization: 3D Example



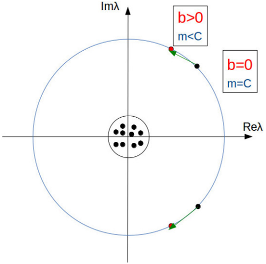
$$\lambda_2 = \sqrt{1 - \lambda_1^2}$$

Stabilization: 3D Example



$$\lambda_2 = \sqrt{1-\lambda_1^2}$$
$$J^*(b,m,\lambda^1) := J(b,m,\lambda_1,\sqrt{1-\lambda_1^2}).$$

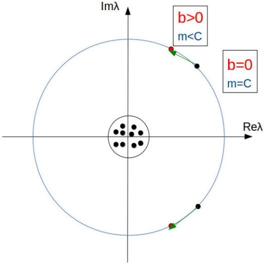
Stabilization: 3D Example

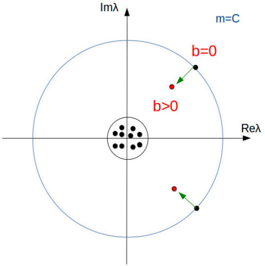


$$\lambda_2 = \sqrt{1-\lambda_1^2}$$
$$J^*(b, m, \lambda^1) := J(b, m, \lambda_1, \sqrt{1-\lambda_1^2}).$$

$$\boxed{\text{Condition 1: } \det D_{(b,\lambda_1)} J^* \neq 0.}$$

Stabilization: 3D Example



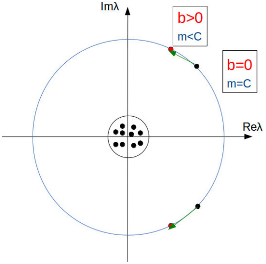


$$\lambda_2 = \sqrt{1-\lambda_1^2}$$

$$J^*(b, m, \lambda^1) := J(b, m, \lambda_1, \sqrt{1-\lambda_1^2}).$$

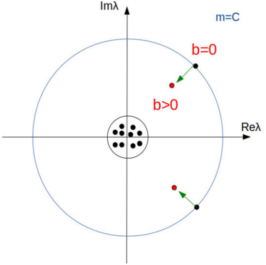
$$\boxed{\text{Condition 1: } \det D_{(b,\lambda_1)} J^* \neq 0.}$$

Stabilization: 3D Example

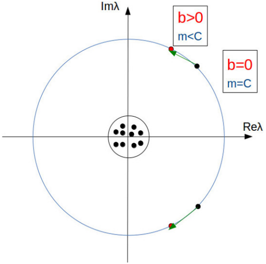


$$\lambda_2 = \sqrt{1-\lambda_1^2}$$
$$J^*(b,m,\lambda^1) := J(b,m,\lambda_1,\sqrt{1-\lambda_1^2}).$$

$$\boxed{\text{Condition 1: } \det D_{(b,\lambda_1)} J^* \neq 0.}$$

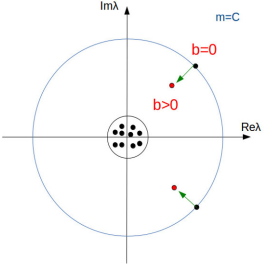


$$r = \lambda_1^2 + \lambda_2^2$$



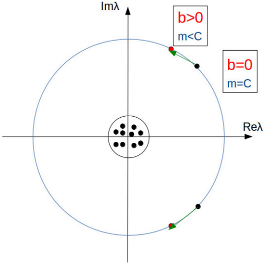
$$\lambda_2 = \sqrt{1-\lambda_1^2}$$
$$J^*(b,m,\lambda^1) := J(b,m,\lambda_1,\sqrt{1-\lambda_1^2}).$$

$\boxed{\text{Condition 1: } \det D_{(b,\lambda_1)} J^* \neq 0.}$



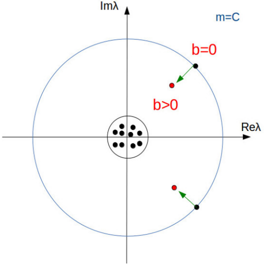
$$r = \lambda_1^2 + \lambda_2^2$$
$$J^{**}(b,r,\lambda_1) := J(b,c,\lambda_1,\sqrt{r-\lambda_1^2}).$$

Stabilization: 3D Example



$$\lambda_2 = \sqrt{1-\lambda_1^2}$$
$$J^*(b, m, \lambda^1) := J(b, m, \lambda_1, \sqrt{1-\lambda_1^2}).$$

$\boxed{\text{Condition 1: } \det D_{(b,\lambda_1)} J^* \neq 0.}$



$$r = \lambda_1^2 + \lambda_2^2$$
$$J^{**}(b, r, \lambda_1) := J(b, c, \lambda_1, \sqrt{r-\lambda_1^2}).$$

$\boxed{\text{Condition 2: } \det D_{(b,\lambda_1)} J^{**} \neq 0.}$

Thank you!

Thank you for your attention!