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“Vector and metric hysteresis evolution processes”

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Sweeping processes

\mathcal{H} is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$,

$$\mathcal{C}_{\mathcal{H}} := \{\mathcal{K} \subseteq \mathcal{H} : \mathcal{K} \neq \emptyset, \text{ closed, convex}\}$$

$$d_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) := \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A}) \right\}, \quad \mathcal{A}, \mathcal{B} \in \mathcal{C}_{\mathcal{H}}.$$

Theorem 1 (*J.J. Moreau, Proc. CIME, 1973*)

$$\forall \mathcal{C} \in Lip([0, T]; \mathcal{C}_{\mathcal{H}}), \quad y_0 \in \mathcal{H} \quad \exists! y \in Lip([0, T]; \mathcal{H})$$

$$\begin{cases} -y'(t) \in N_{\mathcal{C}(t)}(y(t)) & \text{a.e. } t \in [0, T] \\ y(0) = \text{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

Operator solution of sweeping processes

We can define the solution operator

$$\begin{aligned} S : \quad Lip([0, T] ; \mathcal{C}_{\mathcal{H}}) &\longrightarrow \quad Lip([0, T] ; \mathcal{H}) \\ \mathcal{C} &\longmapsto \qquad \qquad \qquad y \end{aligned}$$

S is rate independent:

$$S(\mathcal{C} \circ \phi) = S(\mathcal{C}) \circ \phi$$

for any $\phi \in Lip([0, T] ; \mathbb{R})$ increasing, $\phi([0, T]) = [0, T]$.

Sweeping processes with driving functions u and r

\mathcal{X} Hilbert space, \mathcal{Y} reflexive Banach space, $\mathcal{R} \in \mathcal{C}_{\mathcal{Y}}$, $\overset{\circ}{\mathcal{R}} \neq \emptyset$

Theorem 2 Assume $\mathcal{Z} \in Lip(\mathcal{R}; \mathcal{C}_{\mathcal{X}})$, $z_0 \in \mathcal{H}$.

$\forall u \in Lip([0, T] ; \mathcal{X}), r \in Lip([0, T] ; \mathcal{Y}) \exists! y \in Lip([0, T] ; \mathcal{X})$

$$\begin{cases} \langle u(t) - y(t) - z, y'(t) \rangle \leq 0 & \text{a.e. } t \in [0, T], \ z \in \mathcal{Z}(r(t)) \\ u(0) - y(0) = \text{Proj}_{\mathcal{Z}(r(0))}(z_0) \end{cases}$$

$(\mathcal{C}(t) := u(t) - \mathcal{Z}(r(t)), \quad y_0 := u(0) - z_0)$

(M. Brokate, P. Krejci, H. Schnabel, *J. Convex An.*, 2004)

Sweeping processes with driving functions u and r

\mathcal{X} Hilbert space, \mathcal{Y} reflexive Banach space, $\mathcal{R} \in \mathcal{C}_{\mathcal{Y}}$, $\overset{\circ}{\mathcal{R}} \neq \emptyset$

Theorem 3 Assume $\mathcal{Z} \in Lip(\mathcal{R}; \mathcal{C}_{\mathcal{X}})$, $z_0 \in \mathcal{H}$.

$$\forall u \in Lip([0, T] ; \mathcal{X}), r \in Lip([0, T] ; \mathcal{Y}) \exists! y \in Lip([0, T] ; \mathcal{X})$$

$$\begin{cases} \langle u(t) - y(t) - z, y'(t) \rangle \leq 0 & \text{a.e. } t \in [0, T], z \in \mathcal{Z}(r(t)) \\ u(0) - y(0) = \text{Proj}_{\mathcal{Z}(r(0))}(z_0) \end{cases}$$

$$Q : Lip([0, T] ; \mathcal{X}) \times Lip([0, T] ; \mathcal{Y}) \longrightarrow Lip([0, T] ; \mathcal{X})$$

$$(u, r) \longmapsto y$$

Special case: the play operator

Theorem 4 Assume $\mathcal{Z} \in \mathcal{C}_{\mathcal{H}}$, $0, z_0 \in \mathcal{Z}$.

$$\forall u \in Lip([0, T]; \mathcal{H}) \quad \exists! y \in Lip([0, T]; \mathcal{H})$$

$$\begin{cases} \langle u(t) - y(t) - z, y'(t) \rangle \leq 0 & \text{a.e. } t \in [0, T], z \in \mathcal{Z} \\ y(0) - u(0) = z_0 \end{cases}$$

$$(\mathcal{C}(t) := u(t) - \mathcal{Z}, \quad y_0 := u(0) - z_0)$$

$$\mathsf{P} : Lip([0, T]; \mathcal{H}) \longrightarrow Lip([0, T]; \mathcal{H})$$

$$u \qquad \longmapsto \qquad y$$

A general extension theorem

Theorem 5 (*V. Recupero, JDE, 2011*)

Assume \mathcal{M}, \mathcal{N} are complete metric spaces and

$$R : Lip([0, T] ; \mathcal{M}) \longrightarrow Lip([0, T] ; \mathcal{N})$$

- (i) is rate independent,
- (ii) is continuous, where

$Lip([0, T] ; \mathcal{M})$ is endowed with the strict metric

$Lip([0, T] ; \mathcal{N})$ is endowed with the $\|\cdot\|_\infty$ metric

Then $\exists! \bar{R} : CBV([0, T] ; \mathcal{M}) \longrightarrow CBV([0, T] ; \mathcal{N})$ continuous

$$\bar{R} = R \quad \text{on } Lip([0, T] ; \mathcal{M}).$$

Applications

The result applies to the sweeping processes:

$$\exists! \bar{S} : CBV([0, T] ; \mathcal{C}_{\mathcal{H}}) \longrightarrow CBV([0, T] ; \mathcal{H})$$

$$\exists! \bar{Q} : CBV([0, T] ; \mathcal{X}) \times CBV([0, T] ; \mathcal{Y}) \longrightarrow CBV([0, T] ; \mathcal{X})$$

$$\exists! \bar{P} : CBV([0, T] ; \mathcal{H}) \longrightarrow CBV([0, T] ; \mathcal{H})$$

All these extensions coincide with the extension in

J.J. Moreau, *JDE*, 1977

The Moreau extension for Q

\mathcal{X} Hilbert space, \mathcal{Y} reflexive Banach space, $\mathcal{R} \in \mathcal{C}_{\mathcal{Y}}$, $\overset{\circ}{\mathcal{R}} \neq \emptyset$

Theorem 6 Assume $\mathcal{Z} \in Lip(\mathcal{R}; \mathcal{C}_{\mathcal{X}})$, $z_0 \in \mathcal{H}$.

$\forall u \in CBV([0, T] ; \mathcal{X}), r \in CBV([0, T] ; \mathcal{Y}) \exists! y \in CBV([0, T] ; \mathcal{X})$

$$\begin{cases} \int_0^T \langle u(t+) - y(t+) - z(t), dy(t) \rangle \leq 0 & z(t) \in \mathcal{Z}(r(t)) \\ u(0) - y(0) = \text{Proj}_{\mathcal{Z}(r(0))}(z_0) \end{cases}$$

P. Krejčí, M. Liero, *Appl. Math.*, 2009

P. Krejčí, T. Roche, *DCDS-B*, 2011

A general extension theorem 2

Theorem 7 (*V. Recupero, 2014*) Assume that \mathcal{M} admits a unique (or some canonical) geodesic connecting two points. If

$$R : Lip([0, T] ; \mathcal{M}) \longrightarrow Lip([0, T] ; \mathcal{N})$$

- (i) is rate independent,
- (ii) is continuous, where

$Lip([0, T] ; \mathcal{M})$ is endowed with the strict metric

$Lip([0, T] ; \mathcal{N})$ is endowed with the L^1 -metric

Then $\exists! \bar{R} : BV([0, T] ; \mathcal{M}) \longrightarrow BV([0, T] ; \mathcal{N})$ continuous

$$\bar{R} = R \quad \text{on } Lip([0, T] ; \mathcal{M}),$$

In both cases

$$\bar{\mathsf{R}}(u) := \mathsf{R}(\tilde{u}) \circ \ell_u, \quad u \in BV([0, T] ; \mathcal{M}),$$

where $\ell_u : [0, T] \longrightarrow [0, T]$ by

$$\ell_u(t) := \begin{cases} \frac{T}{\mathbf{V}(u, [0, T])} \mathbf{V}(u, [0, t]) & \text{if } \mathbf{V}(u, [0, T]) \neq 0 \\ 0 & \text{if } \mathbf{V}(u, [0, T]) = 0 \end{cases}$$

and \tilde{u} is the reparametrization by the arclength ℓ_u :

$$\tilde{u} \in Lip([0, T] ; \mathcal{M})$$

$$u = \tilde{u} \circ \ell_u,$$

\tilde{u} is a geodesic on jumps $[\ell_u(t-), \ell_u(t+)]$

$$\tilde{u}'(t) = \mathbf{V}(u, [0, T])/T \quad \text{for a.e. } t$$

Applications

The result applies to the sweeping processes:

$$\bar{S} : BV([0, T] ; \mathcal{C}_{\mathcal{H}}) \longrightarrow BV([0, T] ; \mathcal{H})$$

$$\bar{Q} : BV([0, T] ; \mathcal{X}) \times BV([0, T] ; \mathcal{Y}) \longrightarrow BV([0, T] ; \mathcal{X})$$

$$\bar{P} : BV([0, T] ; \mathcal{H}) \longrightarrow BV([0, T] ; \mathcal{H})$$

All these extensions differ from the Moreau extensions:

$\overline{Q}(u, r)$ is not the Moreau extension

\mathcal{X} Hilbert space, \mathcal{Y} reflexive Banach space, $\mathcal{R} \in \mathcal{C}_{\mathcal{Y}}$, $\overset{\circ}{\mathcal{R}} \neq \emptyset$

Theorem 8 Assume $\mathcal{Z} \in Lip(\mathcal{R}; \mathcal{C}_{\mathcal{X}})$, $z_0 \in \mathcal{H}$.

$\forall u \in BV([0, T] ; \mathcal{X}), r \in BV([0, T] ; \mathcal{Y}) \exists! y \in BV([0, T] ; \mathcal{X})$

$$\begin{cases} \int_0^T \langle u(t) - y(t) - z, y'(t) \rangle \leq 0 & z \in \mathcal{Z}(r(t)) \\ u(0) - y(0) = \text{Proj}_{\mathcal{Z}(r(0))}(z_0) \end{cases}$$

$$\overline{Q}(u, r) \neq y$$

V. Recupero, *Ann. SNS*, 2011;

P. Krejčí, V. Recupero, *J. Convex. An.*, 2014

A continuity property

Theorem 9 (*V. Recupero, Ann. SNS, 2011*) Assume that

$$R : Lip([0, T] ; \mathcal{H}) \longrightarrow Lip([0, T] ; \mathcal{H})$$

- (i) *is rate independent,*
- (ii) *is continuous w.r.t. the $W^{1,1}$ -topology*

Then

$$\bar{R} : CBV([0, T] ; \mathcal{H}) \longrightarrow CBV([0, T] ; \mathcal{H})$$

is continuous w.r.t. the strict metric

Corollary 1 $P : CBV([0, T] ; \mathcal{H}) \longrightarrow CBV([0, T] ; \mathcal{H})$ *is continuous w.r.t. the strict metric.*

A continuity property: the scalar case

Theorem 10 (*V. Recupero, M²AS, 2009*) Assume that

$$R : Lip([0, T] ; \mathbb{R}) \longrightarrow Lip([0, T] ; \mathbb{R})$$

- (i) *is locally isotone,*
- (ii) *is rate independent,*
- (iii) *is continuous w.r.t. the $W^{1,1}$ -topology*

Then

$$\bar{R} : BV([0, T] ; \mathbb{R}) \longrightarrow BV([0, T] ; \mathbb{R})$$

is continuous w.r.t. the strict metric

”Proof” for the CBV -vector case

First we prove that

$$u_n \rightarrow u \text{ strictly} \implies \tilde{u}_n \rightarrow \tilde{u} \text{ in } W^{1,1}$$

thus

- $R(u_n) = R(\tilde{u}_n \circ \ell_{u_n}) = R(\tilde{u}_n) \circ \ell_{u_n} \rightarrow R(\tilde{u}) \circ \ell_u = R(u)$
- $V(R(u_n)) = V(R(\tilde{u}_n)) \rightarrow V(R(\tilde{u})) = V(R(u))$

”Proof”

The implication

$$u_n \rightarrow u \text{ strictly} \implies \tilde{u}_n \rightarrow \tilde{u} \text{ in } W^{1,1}$$

needs the uniform convexity of \mathcal{H} : we first prove that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } W^{1,p}, \quad p \in (1, \infty)$$

thus

$$\begin{aligned} \|\tilde{u}'_n\|_{L^p}^p &= \int_0^T \|\tilde{u}'_n(t)\|_{\mathcal{H}}^p dt = \int_0^T \left(\frac{\mathbf{V}(u_n, [0, T])}{T} \right)^p dt \rightarrow \\ &\rightarrow \int_0^T \left(\frac{\mathbf{V}(u, [0, T])}{T} \right)^p dt = \int_0^T \|\tilde{u}'(t)\|_{\mathcal{H}}^p dt = \|\tilde{u}'\|_{L^p}^p \end{aligned}$$

Sweeping processes with driving functions u and r

\mathcal{X} Hilbert space, \mathcal{Y} reflexive Banach space, $\mathcal{R} \in \mathcal{C}_{\mathcal{Y}}$

Theorem 11 Assume $\mathcal{Z} \in Lip(\mathcal{R}; \mathcal{C}_{\mathcal{X}})$, $z_0 \in \mathcal{H}$.

$\forall u \in Lip([0, T] ; \mathcal{X}), r \in Lip([0, T] ; \mathcal{Y}) \exists! y \in Lip([0, T] ; \mathcal{X})$

$$\begin{cases} \langle u(t) - y(t) - z, y'(t) \rangle \leq 0 \text{ for a.e. } t \in [0, T], \forall z \in \mathcal{Z}(r(t)) \\ u(0) - y(0) = \text{Proj}_{\mathcal{Z}(r(0))}(z_0) \end{cases}$$

$$\begin{array}{ccc} Q : & Lip([0, T] ; \mathcal{X}) \times Lip([0, T] ; \mathcal{Y}) & \longrightarrow Lip([0, T] ; \mathcal{X}) \\ & (u, r) & \longmapsto y \end{array}$$

$W^{1,1}$ -continuity of Q

Theorem 12 (*M. Brokate, P. Krejci, H. Schnabel, J. Convex An., 2004*)

Under suitable regularity conditions for $\mathcal{Z}(r)$

$$Q : Lip([0, T] ; \mathcal{X}) \times Lip([0, T] ; \mathcal{Y}) \longrightarrow Lip([0, T] ; \mathcal{X})$$

is continuous w.r.t. the $W^{1,1}$ -topology

Hilbert case

Theorem 13 (*V. Recupero, Ann. SNS. 2011*)

Assume that

$$R : Lip([0, T] ; \mathcal{H}) \longrightarrow Lip([0, T] ; \mathcal{H})$$

- (i) is rate independent,
- (ii) is continuous w.r.t. the $W^{1,1}$ -topology

Then

$$\bar{R} : CBV([0, T] ; \mathcal{H}) \longrightarrow CBV([0, T] ; \mathcal{H})$$

are continuous w.r.t. the strict metric

Banach version?

Theorem 14 Assume that

\mathcal{B} uniformly convex Banach space, \mathcal{E} Banach space,

$$R : Lip([0, T] ; \mathcal{B}) \longrightarrow Lip([0, T] ; \mathcal{E})$$

- (i) is rate independent,
- (ii) is continuous w.r.t. the $W^{1,1}$ -topology

Then

$$\bar{R} : CBV([0, T] ; \mathcal{B}) \longrightarrow CBV([0, T] ; \mathcal{E})$$

are continuous w.r.t. the strict metric

The Banach case does not work

$$\mathcal{B} := \mathcal{X} \times \mathcal{Y},$$

$$\|(x, y)\|_{\mathcal{X} \times \mathcal{Y}} := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$$

$u_n \rightarrow u$ strictly in $BV([0, T] ; \mathcal{X})$, $r_n \rightarrow r$ strictly in $BV([0, T] ; \mathcal{Y})$

$$\Downarrow$$

$(u_n, r_n) \rightarrow (u, r)$ strictly in $BV([0, T] ; \mathcal{B})$

Another argument is needed

$$u_n \rightarrow u \text{ strictly in } BV([0, T] ; \mathcal{X}), \quad r_n \rightarrow r \text{ strictly in } BV([0, T] ; \mathcal{Y})$$

\Downarrow

$$v_n = (u_n, r_n) \rightarrow v(u, r) \quad \text{strictly in } BV([0, T] ; \mathcal{X} \times \mathcal{Y})$$

\Downarrow

$$\tilde{v}_n = (\bar{u}_n, \bar{r}_n) \rightarrow \tilde{v} = (\bar{u}, \bar{r}) \quad \text{strictly in } BV([0, T] ; \mathcal{X} \times \mathcal{Y})$$

\Downarrow

$$\tilde{v}_n = (\bar{u}_n, \bar{r}_n) \rightarrow \tilde{v} = (\bar{u}, \bar{r}) \quad \text{uniformly in } [0, T]$$

\Downarrow

$$\mathbf{S}(\bar{u}_n, \bar{r}_n) \rightarrow \mathbf{S}(\bar{u}, \bar{r}) \quad \text{strictly in } BV([0, T] ; \mathcal{X})$$

$$\begin{aligned}
\mathsf{S}(u_n, r_n) &= \mathsf{S}(\bar{u}_n \circ \ell_{v_n}, \bar{r}_n \circ \ell_{v_n}) \\
&= \mathsf{S}(\bar{u}_n, \bar{r}_n) \circ \ell_{v_n} \\
&\longrightarrow \mathsf{S}(\bar{u}, \bar{r}) \circ \ell_v \\
&= \mathsf{S}(\bar{u} \circ \ell_v, \bar{r} \circ \ell_v) \\
&= \mathsf{S}(u, r)
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}(\mathsf{S}(u_n, r_n), [0, T]) &= \mathbf{V}(\mathsf{S}(\bar{u}_n, \bar{r}_n) \circ \ell_{v_n}, [0, T]) \\
&= \mathbf{V}(\mathsf{S}(\bar{u}_n, \bar{r}_n), [0, T]) \\
&\longrightarrow \mathbf{V}(\mathsf{S}(\bar{u}, \bar{r}), [0, T]) \\
&= \mathbf{V}(\mathsf{S}(\bar{u}, \bar{r}) \circ \ell_v, [0, T]) \\
&= \mathbf{V}(\mathsf{S}(u, r), [0, T])
\end{aligned}$$

Strict BV continuous dependence

Theorem 15 (*V. Recupero 2014*)

Under suitable regularity conditions for $\mathcal{Z}(r)$

$$\overline{Q} : CBV([0, T] ; \mathcal{X} \times \mathcal{Y}) \longrightarrow CBV([0, T] ; \mathcal{X})$$

is continuous w.r.t. the strict metric

Strict BV continuous dependence

i.e. Problem

$$\begin{cases} \int_0^T \langle u(t) - y(t) - z, dy(t) \rangle \leq 0 & \forall z \in \mathcal{Z}(r(t)) \\ u(0) - y(0) = \text{Proj}_{\mathcal{Z}(r(0))}(z_0) \end{cases}$$

is well posed w.r.t. the strict metric in CBV .

Some references

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