Weierstrass Institute for
Applied Analysis and Stochastics

Halbleiter-Nanophotonik
SFB 787

Leibniz Association

Robust homocline in a predator-prey system with hysteresis.

Alexander Pimenov, Dmitrii Rachinskii.

Outline

- Motivation: epidemiology, ecology
- Equilibria: partial stability and homoclinic orbits
- Construction of homoclinic orbits
- Predator-prey model with hysteresis
- Numerical examples of homoclinic orbits
- Conclusion and outlook

- SI-model in epidemiology. Population behaviour: *safe* and *risky*

$$\begin{aligned}
\dot{S} &= b - (\beta_R(1-P_S) + \beta_S P_S)\, SI - \mu S, \\
\dot{I} &= (\beta_R(1-P_S) + \beta_S P_S)\, SI - (\gamma + \mu + \sigma) I, \\
\dot{P}_S(t) &= k\big(A(t) - y(t)\big), \qquad P_S(t) = \big(\mathcal{P}[\eta_0]y\big)(t).
\end{aligned}$$

Pimenov, Korobeinikov, Rachinskii, Pokrovskii et al, *Memory Effects in Population Dynamics: Spread of Infectious Disease as a Case Study*, Math. Modelling of Natural Phenomena, 2012.

- Predator-prey model with *safe* and *risky* patches.

$$\begin{gathered}
\dot{u} = (b_S P_S(t) + b_R(1-P_S(t)))u - (c_S P_S(t) + c_R(1-P_S(t)))u^2 \\
-(a_S P_S(t) + a_R(1-P_S(t)))vu, \\
\dot{v} = -dv + e(a_S P_S(t) + a_R(1-P_S(t)))vu, \\
P_S(t) = (\mathcal{P}[\eta_0]y)(t), \qquad y(t) = \kappa v(t).
\end{gathered}$$

Pimenov, Korobeinikov, Rachinskii et al, *in preparation*.

Motivation: near-equilibrium dynamics in planar systems

- Connected sets of neutrally stable equilibria

$$\begin{aligned} \dot{u} &= f(u,v,x), \\ \dot{v} &= g(u,v,x), \\ x &= \mathcal{P}[\eta_0]v. \end{aligned}$$

- Partially stable equilibria.

$$\begin{aligned} \dot{x} &= f(u,v), \\ \dot{v} &= g(u,v), \\ x &= \mathcal{P}[\eta_0]v. \end{aligned}$$

  S. McCarthy and D. Rachinskii: Dynamics of systems with Preisach memory near equilibria. Mathematica Bohemica, *accepted*.

- Homoclinic orbits?

Motivation: excitable behaviour in a predator-prey system

Predator v, prey u_R in the *risky* patch, prey u_S in the *safe* patch,

$$\begin{aligned}\dot{u}_R &= a_R(u_R) - f_R(u_R)g(v) + h_R(t)u_S - h_S(t)u_R,\\ \dot{u}_S &= a_S(u_S) - f_S(u_S)g(v) - h_R(t)u_S + h_S(t)u_R,\\ \dot{v} &= \sigma(f_R(u_R) + f_S(u_S))g(v) - c(v),\end{aligned}$$

where the flows h_R, h_S are hysteretic

$$h_S(t) = \left(k_{S0} + k_S \frac{d}{dt}(\mathcal{P}[\eta_0]v)(t)\right)^+, \; h_R(t) = \left(k_{R0} - k_R \frac{d}{dt}(\mathcal{P}[\eta_0]v)(t)\right)^+,$$

Pimenov, Rachinskii, *Homoclinic orbits in a two-patch predator-prey model with Preisach hysteresis operator*, WIAS preprint 1849, Mathematica Bohemica EQUADIFF 2013, accepted.
$k_R \equiv k_{R0} \equiv f_S \equiv 0 \to$ a planar system.

We consider a coupled system of differential equations

$$u' = f(u,v) + x' h(u,v), \qquad v' = g(u,v) \tag{1}$$

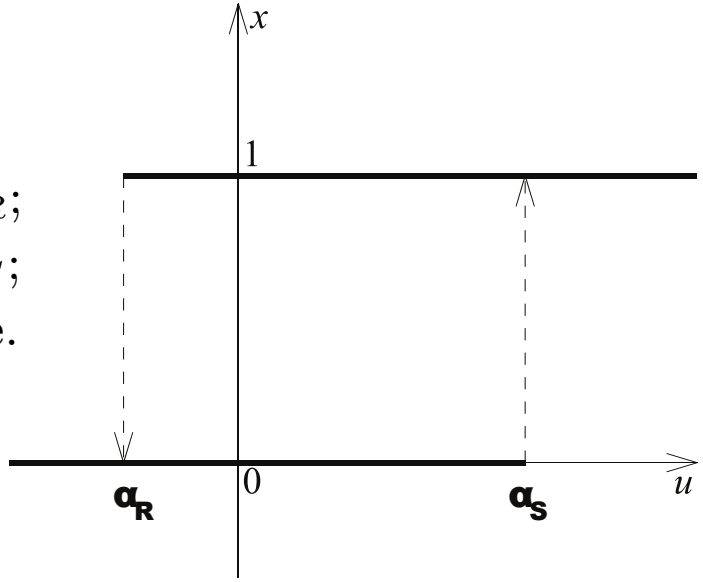
and an operator equation

$$x(t) = (\mathcal{P}[\eta_0]v)(t), \tag{2}$$

$$(\mathcal{P}[\eta_0]v)(t) = \int_0^\infty \int_0^{\alpha_S} \mu(\alpha_R, \alpha_S)(R_{\alpha_R,\alpha_S}[\eta_0(\alpha_R,\alpha_S)]v)(t)\, d\alpha_R d\alpha_S, \tag{3}$$

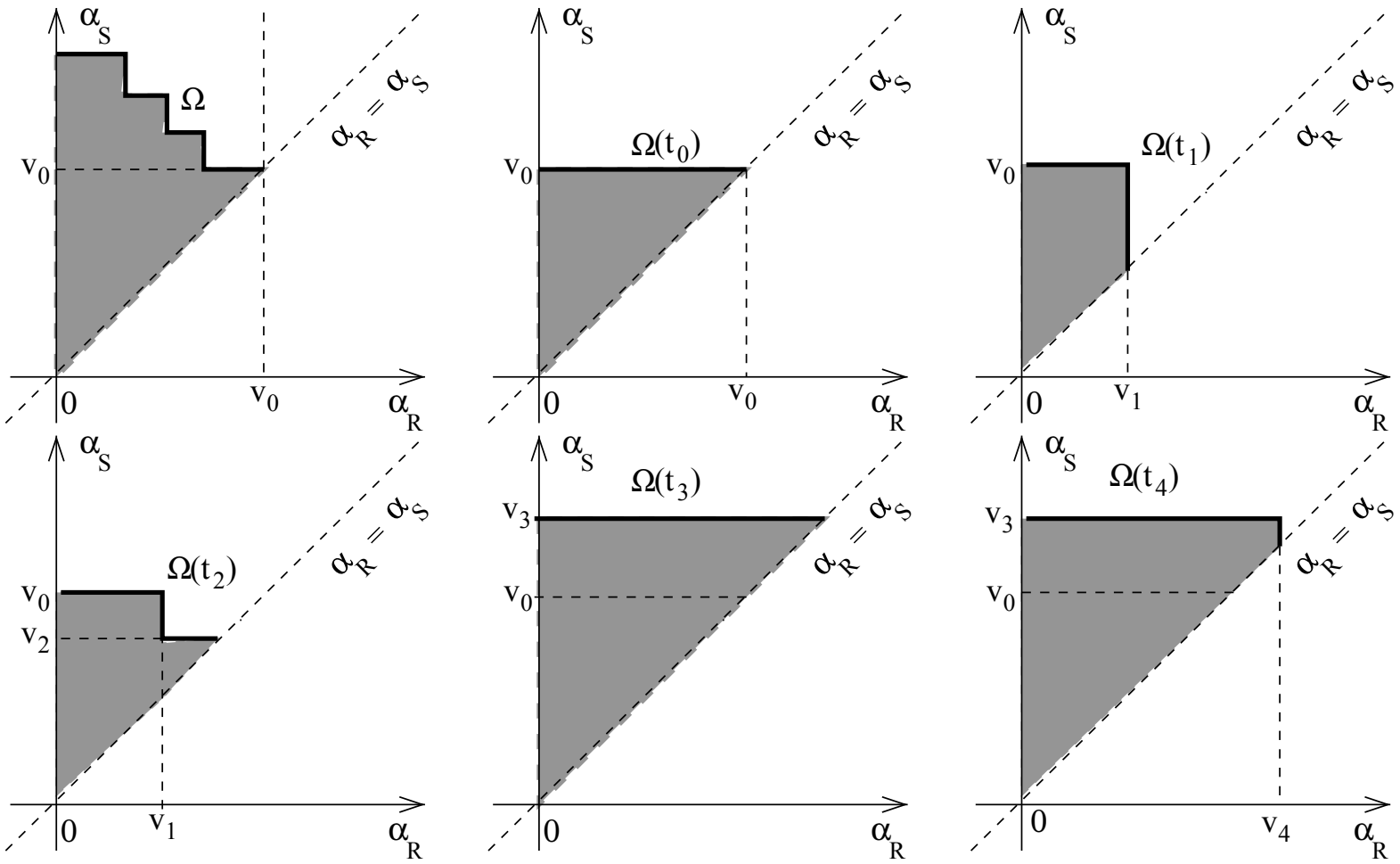
where R_{α_R,α_S} is as a **non-ideal relay**: for $v(t)$ monotone on $t \in [t_0, t]$

$$y(t) = (R_{\alpha_R,\alpha_S}[\eta(t_0)]v)(t) = \begin{cases} 0 & v(t) < \alpha_R; \\ 1 & v(t) \geq \alpha_S; \\ \eta(t_0) & \text{otherwise.} \end{cases}$$

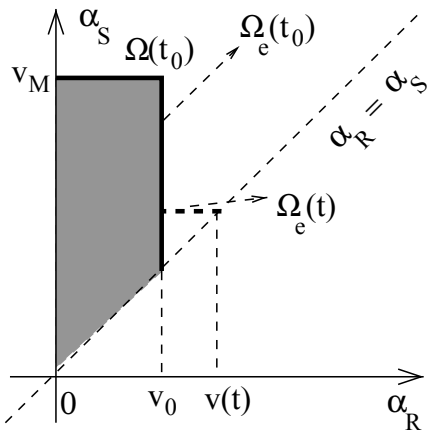


Here $\eta(t_0), y(t)$ are boolean values

Memory evolution



Denote by $(v_m, v), (v, v_M)$ the end points of the segment Ω_e, where $v_m = v$ if Ω_e is a vertical segment and $v_M = v$ if Ω_e is horizontal.



If $v = v(t)$ increases, then the time derivative of the output of the Preisach operator satisfies

$$\frac{d(\mathcal{P}[\eta_0]v)}{dt} = \dot{v}H(v, v_m) \quad \text{with} \quad H(v, v_m) = \int_{v_m}^{v} \mu(\alpha_R, v)\, d\alpha_R. \tag{4}$$

If v decreases, then

$$\frac{d(\mathcal{P}[\eta_0]v)}{dt} = \dot{v}V(v, v_M) \quad \text{with} \quad V(v, v_M) = \int_{v}^{v_M} \mu(v, \alpha_S)\, d\alpha_S. \tag{5}$$

Hence at an interval of monotonicity solution of (1)-(2) satisfies a system of ODEs, and we solve (1)-(2),(4)/(5), where $\dot{v}$ in (4)/(5) is replaced by (2).

Partially stable equilibrium

Without loss of generality we consider system (1)-(2) near to the zero equilibrium.

Definition

The zero equilibrium is partially asymptotically stable if

- for any $\varepsilon > 0$ there is an open ball in the product of the (u, v) phase plane and the set Σ of state space of the Preisach operator such that any solution of system (1)-(2) with initial data from this ball satisfies $|u(t)| + |v(t)| < \varepsilon$ for all $t > t_0$ and $u(t), v(t) \to 0$ as $t \to \infty$;and,at the same time,
- there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is an open ball in the product of the disk $|u| + |v| < \delta$ and the set Σ of states of the Preisach operator such that any solution of system (1)-(2) with initial data from this ball satisfies $|u(t)| + |v(t)| \geq \varepsilon$ at some moment $t > t_0$.

Partially stable equilibria are related to the change of stability of zero equilibrium in the different systems of ODEs on different intervals of monotonicity.

We study homoclinic orbits attached to the partially stable equilibria.

Definition

We call the tripple $\{u(t), v(t), \eta(t)\}$ the solution of (1)-(2) for all $-\infty < t < \infty$ if for each t_0 the tripple $\{u(t), v(t), \eta(t)\}$ is the solution of (1)-(2) with the initial values $u_0 = u(t_0), v = v(t_0), \eta_0 = \eta(t_0)$ for all $t \geq t_0$.

Definition

We call the solution $\{u(t), v(t), \eta(t)\}$ of (1)-(2) for all $-\infty < t < \infty$ *homoclinic* if $u(t) \to 0, v(t) \to 0$ for $t \to \pm\infty$, and $u(t) \not\equiv 0$ or $v(t) \not\equiv 0$.

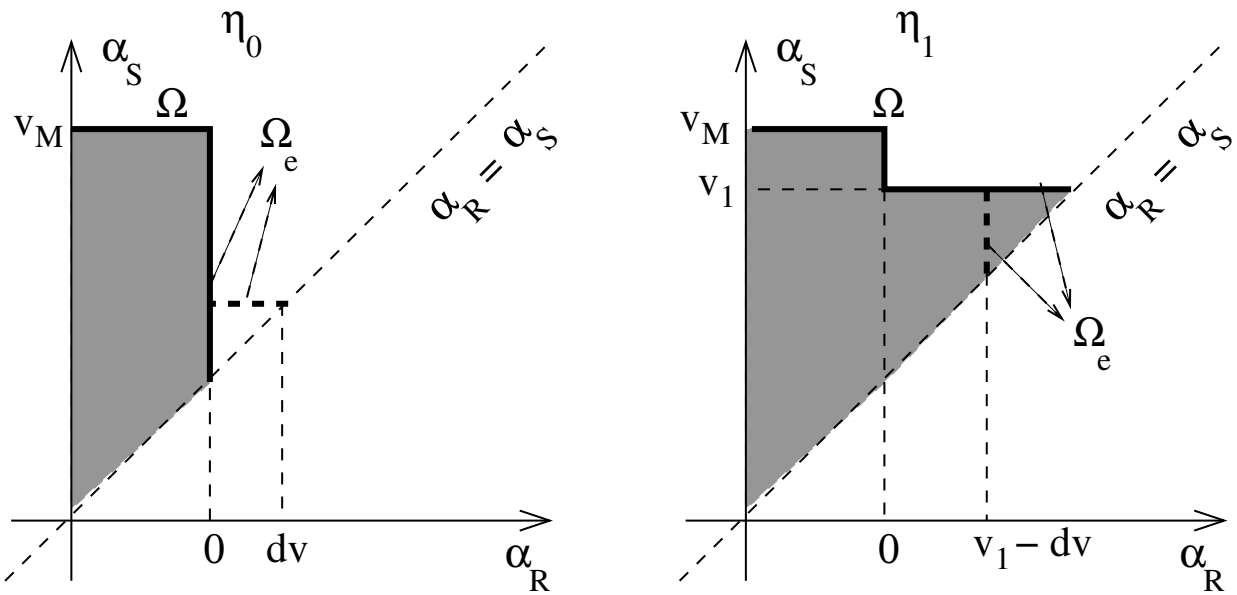
We are interested in the homoclinic orbits with the following property: there exist moments $t_1 \leq t_2$ such that $v(t)$ is monotone for all $t < t_1$ and $t > t_2$.

We consider the system (1)-(2) in a small vicinity $|u| + |v| < \varepsilon$ of the zero equilibrium. For example, if $h(u, v) \sim 1/\varepsilon$ inside this ball, the local dynamics of (1) is given by

$$u' = au + bv - \kappa x', \qquad v' = cu + dv, \tag{6}$$

where the measure $\mu(\alpha_R, \alpha_S) \equiv 1$ and $a = \frac{\partial f}{\partial u}(0), b = \frac{\partial f}{\partial v}(0), c = \frac{\partial g}{\partial u}(0), d = \frac{\partial g}{\partial v}(0)$, $\kappa \sim 1/\varepsilon$.

We assume $\Delta = ad - bc > 0, \tau = a + d > 0, D = \tau^2 - 4\Delta > 0$, i.e. the zero equilibrium is an unstable node, and $a > 0, c > 0, d < 0, b < 0$.



Initially, $v(t)$ increases:

$$u' = au + bv - \kappa v'(v - 0), \qquad v' = cu + dv. \tag{7}$$

At $v(t_1) = v_1$ we have $v'(t_1) = 0$, and $v(t)$ decreases for $t > t_1$ (6),(2), (5) with $v_M = v(t_1) = v_1$:

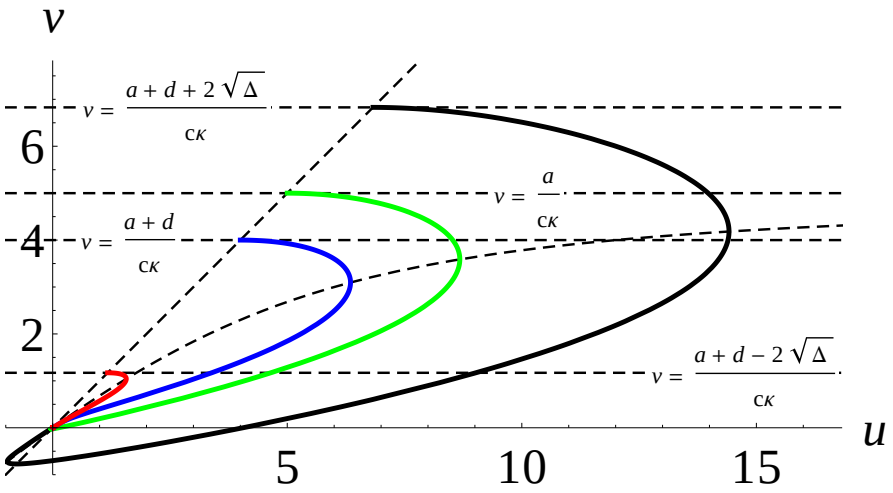
$$u' = au + bv - \kappa v'(v_1 - v), \qquad v' = cu + dv. \tag{8}$$

Eq. (8): $\Delta = ad - bc > 0$

$$\tau = a - cv_1\kappa + d.$$

$v_1 < \frac{a+d-2\sqrt{\Delta}}{c\kappa}$: unstable node,
$v_1 > \frac{a+d}{c\kappa}$: stable,
$v_1 > \frac{a+d+2\sqrt{\Delta}}{c\kappa}$: stable node.
Eq. (7): $u \to \infty, v \to \frac{a}{\kappa c}$.

$$0 = \dot{u} = v(b - \kappa dv) + u(a - \kappa cv).$$



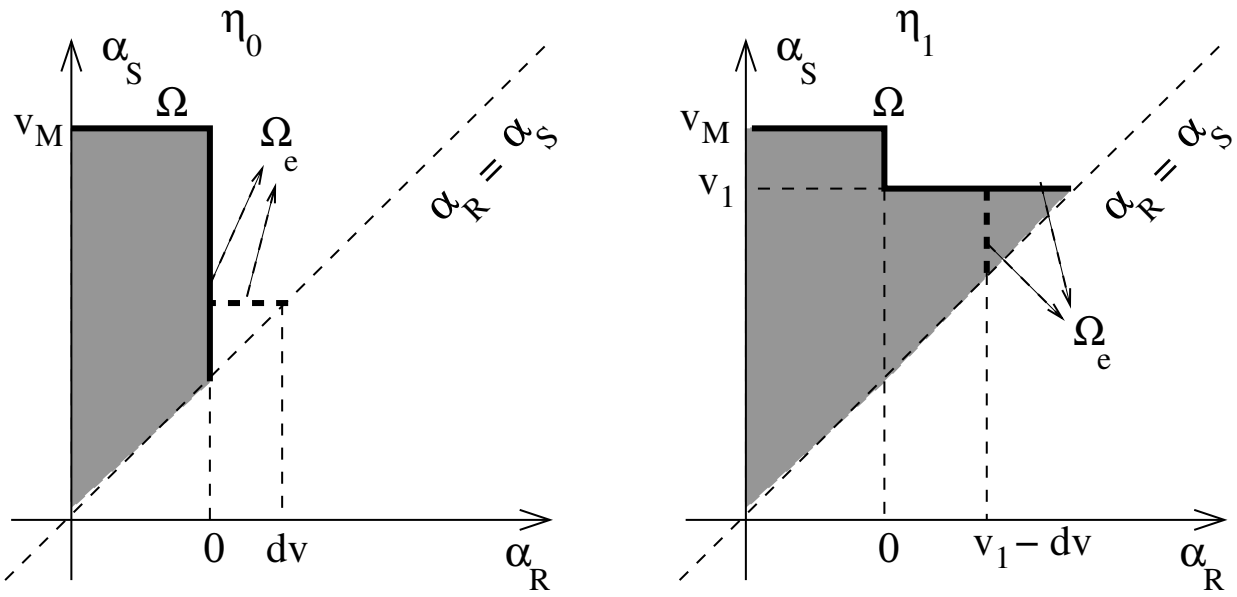
$$u' = au + bv - \kappa(x' - |v'|v), \qquad v' = cu + dv, \tag{9}$$

We pick a point $v_1 > \frac{a+d+2\sqrt{\Delta}}{c\kappa}$ on the line $\dot{v} = 0$. For v, increasing from 0

$$u' = au + bv, \qquad v' = cu + dv, \tag{10}$$

For v, decreasing from $v(t_1) = v_1$

$$u' = (a - \kappa v_1 c)u + (b - \kappa v_1 d)v, \qquad v' = cu + dv. \tag{11}$$



First, we consider the solution of (10) in reverse time $\tilde{t} = -t$ with initial condition $v(0) = v_1, u(0) = -v_1 d/c$

$$v(\tilde{t}) = \frac{v_1}{2\sqrt{D}} \left(e^{-\frac{1}{2}(\tau+\sqrt{D})\tilde{t}} (\sqrt{D} - \tau) + (\sqrt{D} + \tau) e^{-\frac{1}{2}(\tau-\sqrt{D})\tilde{t}} \right), \tag{12}$$

hence for $\tau > \sqrt{D}$: $v(\tilde{t}) \to 0$ with $\tilde{t} \to \infty$ due to $0 \leq \tau^2 - D = 4\Delta$.

$$\dot{v} = \frac{v_1(\tau^2 - D)}{4\sqrt{D}} \left(e^{-\frac{1}{2}(\tau+\sqrt{D})\tilde{t}} - e^{-\frac{1}{2}(\tau-\sqrt{D})\tilde{t}} \right) \neq 0$$

for any $\tilde{t} > 0$, i.e. $t < 0$.
For $t > 0$ the solution has the same form as (12) with $\tilde{\tau} = -(a - \kappa v_1 c + d), \tilde{D} = \tilde{\tau}^2 - 4\Delta$, where $\tilde{\tau} > \sqrt{\tilde{D}}$ due to the choice of $v_1 > \frac{a+d+2\sqrt{\Delta}}{c\kappa}$, and $v(t) \to 0$ with $t \to \infty$.

Simplest homoclinic orbit

$$\kappa = 1$$

$$a = 5$$

$$b = -7$$

$$c = 1$$

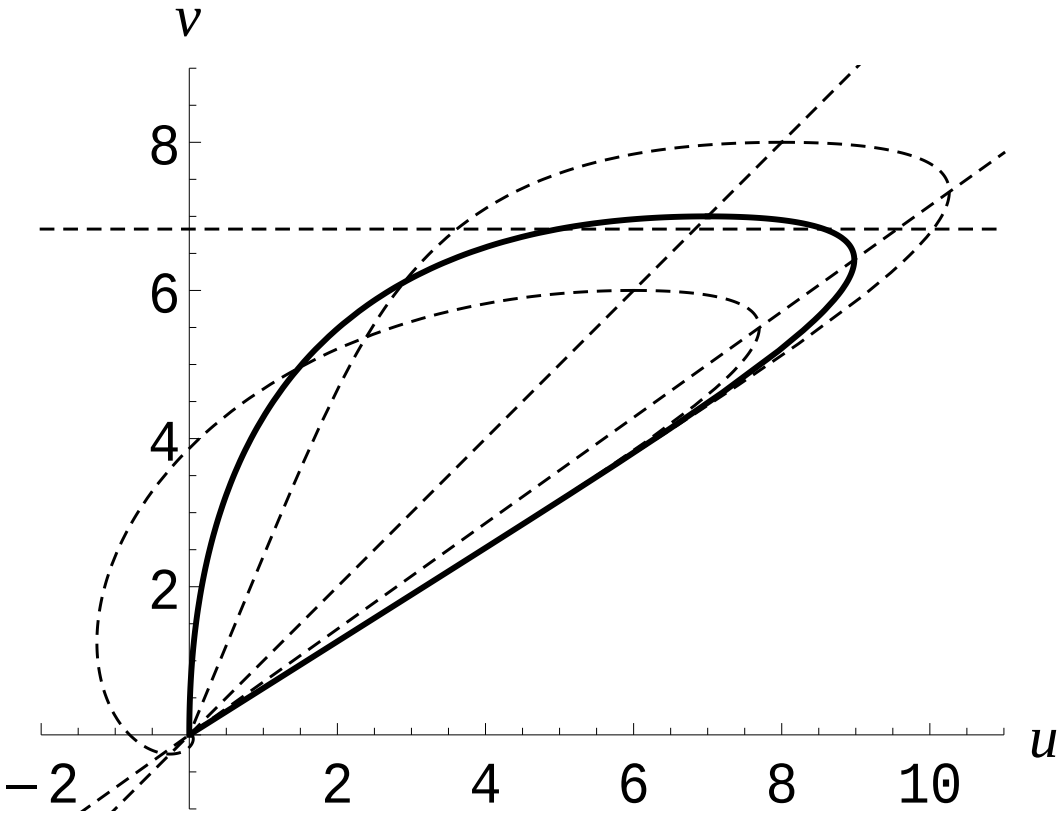
$$d = -1$$

$$\tau = 4$$

$$\Delta = 2$$

$$D = \tau^2 - 4\Delta = 8$$

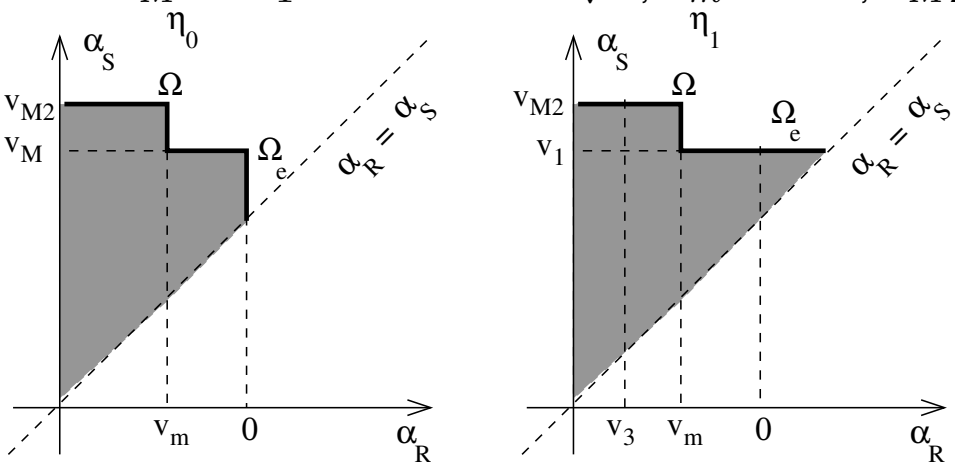
$$v_1 = 7 > 4 + 2\sqrt{2}$$

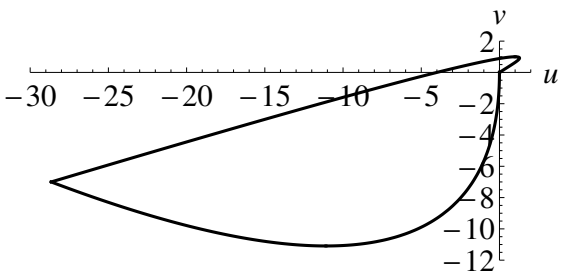


If $v(t)$ hits local minimum v_3 at $t = t_3$ then for $t \geq t_3$ it's the solution of

$$u' = (a + \kappa v_3 c)u + (b + \kappa v_3 d)v, \qquad v' = cu + dv, \tag{13}$$

We take $v_M = v_1 = 1 < 6 - 2\sqrt{5}$, $v_m = -7$, $v_{M2} = 7$: $\tilde{\tau} = 3$, $\tilde{D} = 1$.





Here $v(t_2) = v_m$ for $t_2 = t_1 + \log(1 + 2\sqrt{2})$, where $u(t_2) = -23 - 4\sqrt{2}$. The solution $v(t)$ of (11) with $v_1 := v_{M2}$ has the minimum $v_3 = v(t_3) = u(t_3) = -(697 + 64\sqrt{2})/71$ at $t_3 = 2$ arctanh$(10 + \sqrt{2})/49$.

Predator v, prey u

$$\dot{u} = a(u) - f(u)g(v) - h(t)u, \tag{14}$$

$$\dot{v} = \sigma f(u)g(v) - c(v). \tag{15}$$

Population terms

$$a(u) = \rho u - \lambda u^2,$$
$$f(u) = \frac{\omega u}{\phi + u},$$
$$g(v) = \frac{v}{1 + \beta v}, \qquad c(v) = \gamma v,$$

and hysteretic flow

$$h(t) = \left(k_0 + k \frac{d}{dt} (\mathcal{P}[\eta_0]v)(t) - 2k\delta |v'| (v - v^\dagger) \right)^+, \tag{16}$$

where $\delta = \{0, 1\}$, and $k(v^* - v^\dagger) \sim 1$.

$$\dot{v} > 0: \qquad h(t) = \left(k_0 + k\dot{v} H(v, v_m) - 2k\delta v' (v - v^\dagger) \right)^+, \tag{17}$$

$$\dot{v} < 0: \qquad h(t) = \left(k_0 + k\dot{v} V(v, v_M) + 2k\delta v' (v - v^\dagger) \right)^+. \tag{18}$$

Numerical example

We take $\rho = 1.35$, $\phi = 0.1$, $\beta = 1.2$, $\gamma = 0.5$, $\omega = 2$, $k_0 = 0.01$, $\lambda = 0.01$ there are 3 positive equilibria

$$(u^*, v^*) = (0.183649, 0.245752), \tag{19}$$

$$(u^\dagger, v^\dagger) = (0.340215, 0.45473), \tag{20}$$

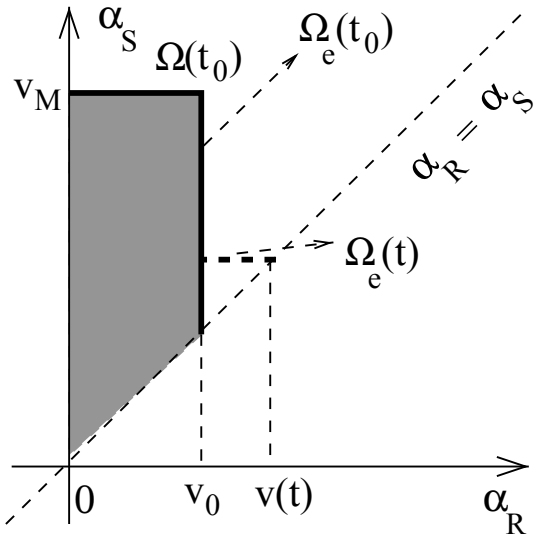
$$(u^\ddagger, v^\ddagger) = (133.376, 0.832085). \tag{21}$$

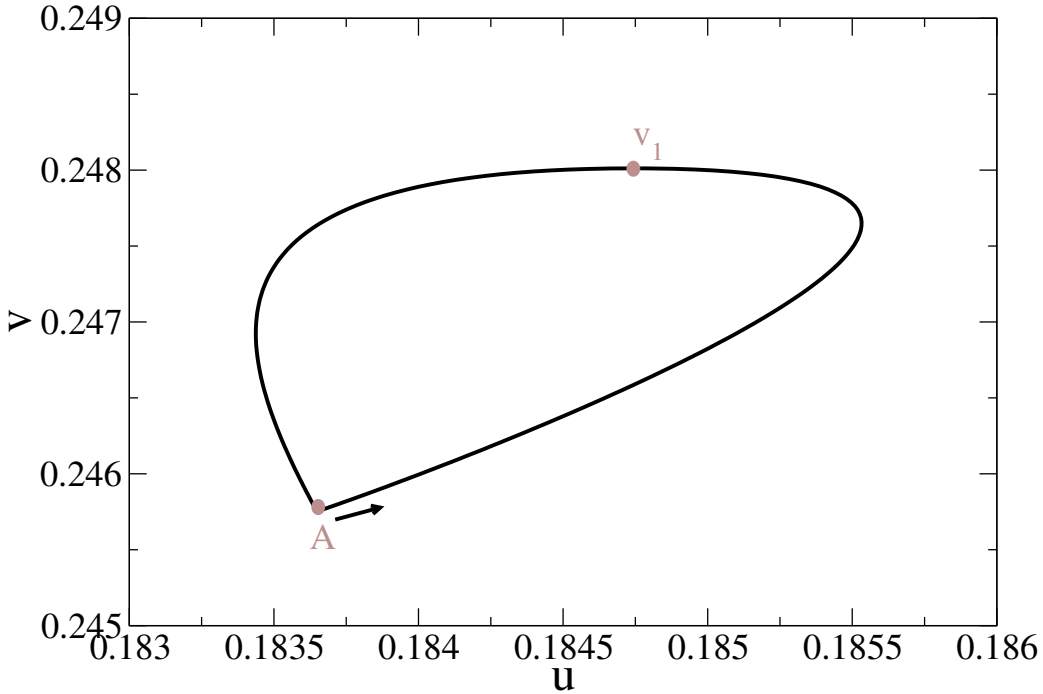
with the eigenvalues for $k \equiv 0$: $(0.136, 0.614)$, $(-0.089, 0.942)$, and $(-1.33, -0.25)$. We define the density function

$$\mu(\alpha_R, \alpha_S) = 2,$$

in the triangle $0 \leq \alpha_R \leq \alpha_S \leq 1$ with $\mu = 0$ outside this triangle.

Numerical example: $\delta = 1$



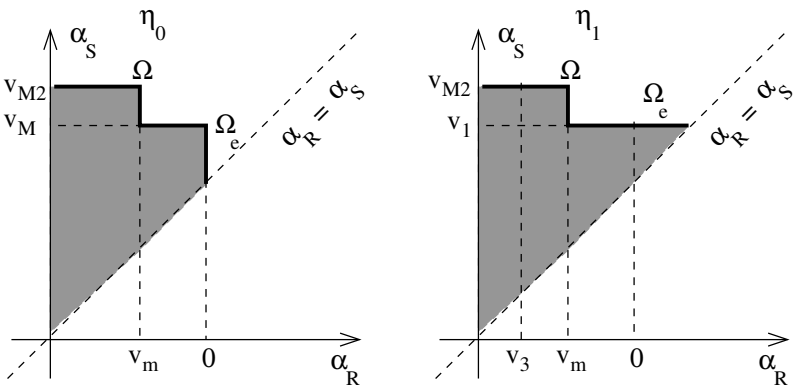


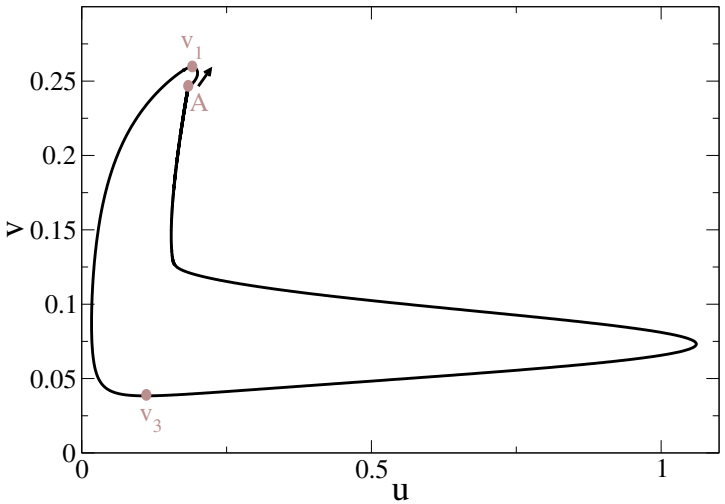
$k = 8000$, $v_M = 1$, $v^\dagger = 0.246$, $v_1 = 0.248$: $(-5.73072, -0.0146083)$.

In another example

$$h(t) = (k_0 + k\dot{v}H(v, v_m))^+ , \quad h(t) = (k_0 + k\dot{v}V(v, v_M))^+ , \tag{22}$$

when $\dot{v} < 0$ we can have $k_0 + k\dot{v}V(v, v_M) < 0$ and there is no flow.
We take $k = 700$, $v_{M2} = v_M = v_1 = 0.2596$, and $v_3 = 0.0384$.





$v_1 : (-0.044584 - 0.285881i, -0.044584 + 0.285881i)$,
$v_3 : (-11.8176, -0.00708397)$.

Conclusion and outlook

- Robust homoclinic orbits in systems with hysteresis: the mechanism.
- Rigorous construction of homoclinic orbits in a planar simplified system.
- We have found numerically that homoclinic orbits can be observed in biological planar systems where
  1. nonlinearity is supressed: artificially or $0 < \det J \ll 1$?
  2. hysteresis is supressed due to natural model limitations.
- Study of the conditions of existence of homoclinic orbits for general systems (1)-(2).
- Periodic orbits near partially stable equilibria.
- Partial stability and excitability in the system with continuous sets of equilibria .