Modeling jumps in rate-independent systems using balanced-viscosity solutions

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Joint work with Riccarda Rossi, Giuseppe Savaré, Sergey Zelik

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Overview

1. Introduction
2. An ODE example
3. A PDE example
4. Energy-dissipation balance for gradient systems
5. Parametrized Balanced-Viscosity solutions
6. Non-parametrized Bal. Visc. solutions

1. Introduction

We study **multi-rate evolutionary system** with
rate-independent friction,
which have the form of **generalized gradient systems**.

Prototype equation

$$
\text{(CS)} \quad \begin{cases} 0 &= \varepsilon^\alpha \mathbb{A}\dot{u} &+ \mathrm{D}_u\mathcal{E}(t,u,z) \\ 0 &= \rho\,\mathsf{Sign}(\dot{z}) + \varepsilon\mathbb{V}\dot{z} &+ \mathrm{D}_z\mathcal{E}(t,u,z) \end{cases}
$$

$u(t) \in \boldsymbol{U}$ "elastic" variable with viscosity $\varepsilon^\alpha \mathbb{A}\dot{u}$

$z(t) \in \boldsymbol{Z}$ internal variable (describing hysteresis)
with dry friction $\rho\,\mathsf{Sign}(\dot{z})$ and viscosity $\varepsilon\mathbb{V}\dot{z}$

Longterm goal:
Understand the rate-independent limit obtained for $\varepsilon \to 0$.

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Three cases

(A) Only elastic part (no z component) $\quad 0 = \varepsilon^\alpha \mathbb{A}\dot{u}(t) + \mathrm{D}_u\mathcal{E}(t,u(t))$

motion through local minima $u_j(t)$ & jump when u_j becomes unstable

- Zanini'07 via dynamical systems theory
- Agostiniani–Rossi–Savaré'13,'14 (using genericity and BV solutions)

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(B) Only internal part (no u comp.) $\quad 0 = \rho\,\mathsf{Sign}(\dot{z}) + \varepsilon\mathbb{V}\dot{z} + \mathrm{D}_z\mathcal{E}(t,z)$

main topic today (for simplicity/experiment)

Efendiev–M.'06, M.–Zelik'14, M.–Rossi-Savaré'09,10,12,12,13,13,...

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(C) coupled system (CS): see M.–Rossi–Savaré in Proc. of MURPHYS-HSFS

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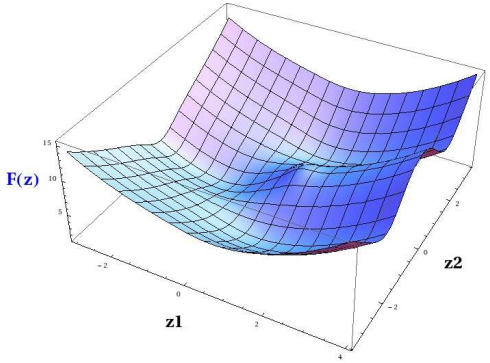
$z = (z_1, z_2) \in \boldsymbol{Z} = \mathbb{R}^2, \quad \mathcal{E}(t, z) = \mathcal{F}(z_1, z_2) - t z_1$

intial condition $z(0) = (-1, \delta)$ with $0 < \delta \ll 1$



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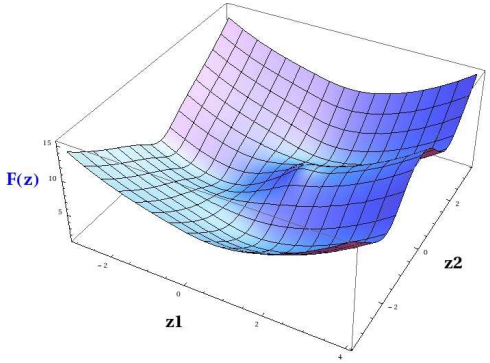

$$\boxed{0 \in \mathrm{Sign}(\dot{z}) + \varepsilon \dot{z} + \mathrm{D}_z \mathcal{E}(t, z)}$$

Dissipation potential $\Psi(\dot{z}) = |\dot{z}| + \frac{\varepsilon}{2}|\dot{z}|^2$

subdifferential $\partial\Psi(\dot{z}) = \mathrm{Sign}(\dot{z}) + \varepsilon \dot{z} \subset \mathbb{R}^2$ is set-valued

$$\mathrm{Sign}(\dot{z}) \stackrel{\text{def}}{=} \begin{cases} \{\, \xi \in \mathbb{R}^2 \mid |\xi| \leq 1 \,\} & \text{for } \dot{z} = 0, \\ \{\frac{\dot{z}}{|\dot{z}|}\} & \text{for } \dot{z} \neq 0. \end{cases}$$

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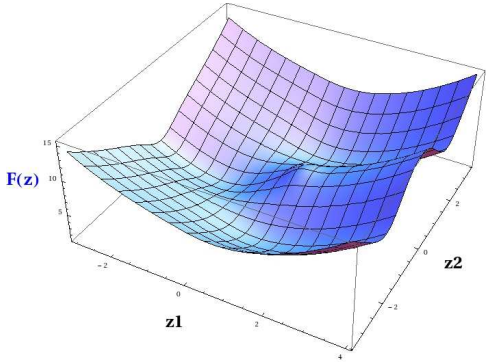
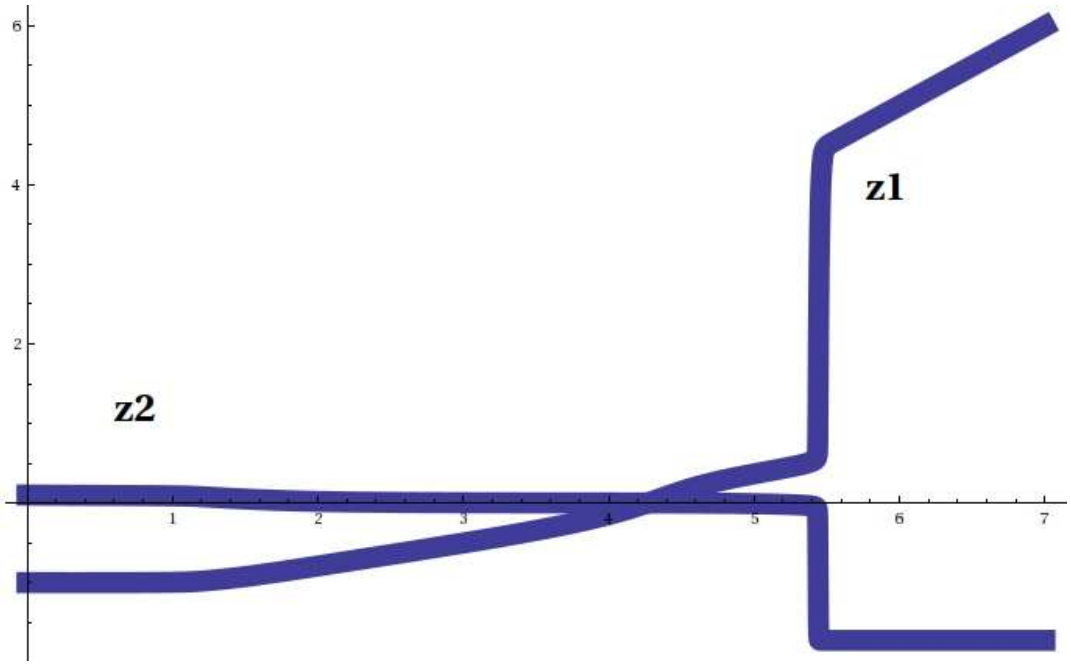


$$\boxed{0 \in \mathrm{Sign}(\dot{z}) + \varepsilon \dot{z} + \mathrm{D}_z \mathcal{E}(t, z)}$$

Numerical simulation
for $\varepsilon = 0.01$



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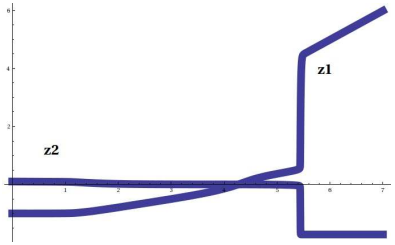
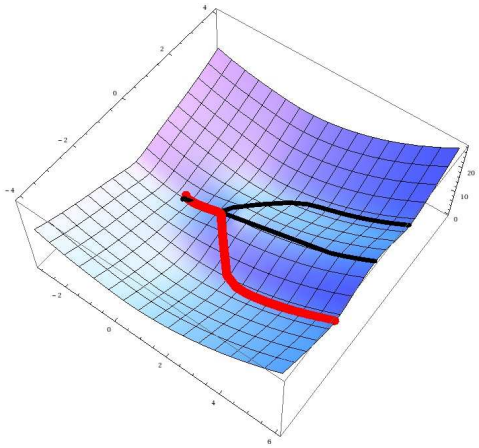


$$0 \in \mathrm{Sign}(\dot{z}) + \varepsilon \dot{z} + \mathrm{D}_z \mathcal{E}(t, z)$$

- Nice regular solution with $|\dot{z}| = O(\varepsilon^0)$ for $t \in [0, 5.4] \cup [5.6, 7]$
- **Jump** at $t \approx 5.5$ with velocities of order $1/\varepsilon$

  The jump start is at $z_{\text{start}} \approx (1, 0)$
  but there are several possible final points of the jump:
  - one symmetric solution $z_{\text{final}} \approx (4.4, 0)$
  - two unsymmetric slns $z_{\text{final}} \approx (4.5, \pm 1.8)$



2. An ODE example

Idea: two different regimes

- **Slow regime:** $|\dot{z}| \approx 1 = \varepsilon^0$ $\qquad 0 \in \mathsf{Sign}(\dot{z}(t)) + \cancel{\varepsilon\ddot{z}} + \mathrm{D}_z\mathcal{E}(t, z(t))$

Consequence: $\boxed{|\mathrm{D}_z\mathcal{E}(t, z(t))| \leq 1}$

since $-\mathrm{D}_z\mathcal{E}(t, z(t)) \in \mathsf{Sign}(\dot{z}(t)) \subset B_1(0) := \{\, \xi \in \mathbb{R}^2 \mid |\xi| \leq 1 \,\}$

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- **Fast regime = jump regime** $|\dot z| \approx 1/\varepsilon$

Fast time scale $t = t_{\mathsf{jump}} + \varepsilon\tau$

$$0 \in \mathsf{Sign}(z'(\tau)) + 1 z'(\tau) + \mathrm{D}_z\mathcal{E}(t_{\mathsf{jump}}, z(\tau))$$

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Aim: Give a rigorous convergence proof

- nonuniqueness/symmetry breaking must be allowed
- problem of delayed loss of stability
  (Tikhonov'52,Pontryagin'60,, Neĭshtadt'87/88, exponentially small terms $\mathrm{e}^{-1/\varepsilon}$)

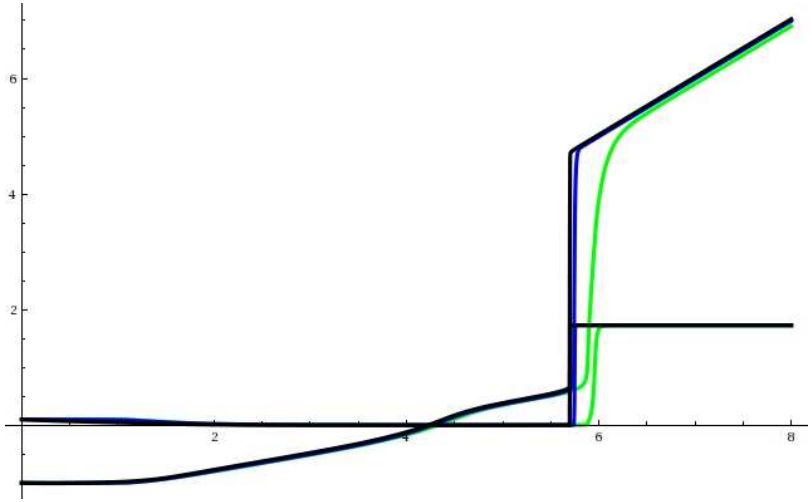
2. An ODE example

Delayed loss of stability for $0 = \frac{\dot z}{|\dot z|} + \varepsilon \dot z + \mathrm{D}\mathcal{F}(z) - \binom{t}{0}$

Symmetry breaking occurs later because

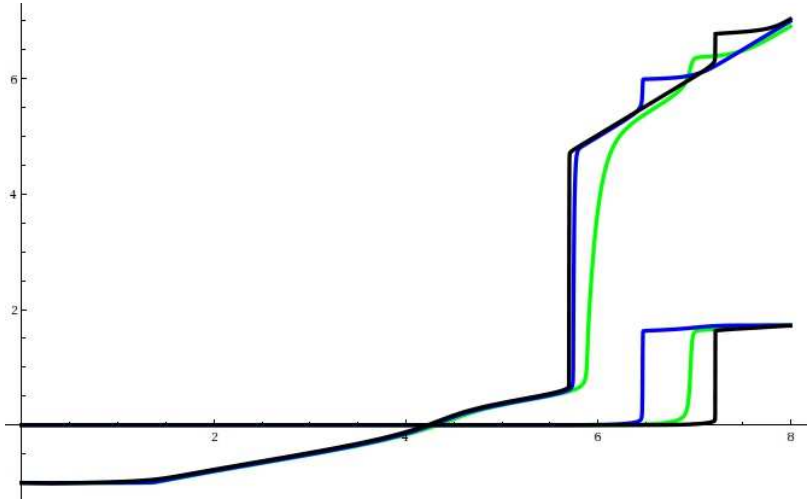
- symmtrization before the instability is very large $\mathrm{e}^{-1/\varepsilon}$
- instability needs time to become effective

Simulation with $\varepsilon = 0.1$, $\varepsilon = 0.01$, $\varepsilon = 0.001$



$z_1(t)$

$z_2(t)$



$z(0) = (-1, 0.1)$

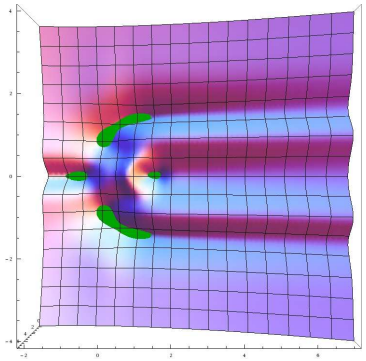
$z(0) = (-0.9, 10^{-6})$

2. An ODE example

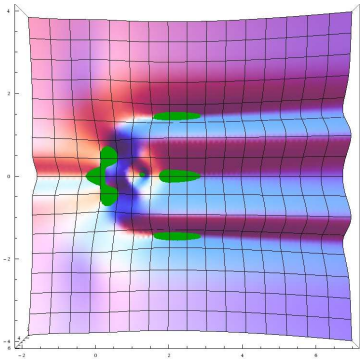
Slow regime: $|\dot z| \approx 1$ $0 \in \mathrm{Sign}(\dot z(t)) + \cancel{\dot z} + \mathrm{D}_z\mathcal{E}(t, z(t)) \implies \boxed{|\mathrm{D}_z\mathcal{E}(t, z(t))| \le 1}$

Necessity of jumps: slow domains with $|\mathrm{D}_z\mathcal{E}(t, z)| \le 1$ disappear.

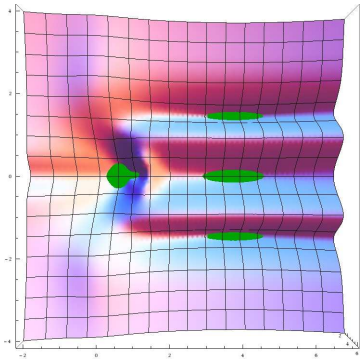
Plot of $\max\{1, |\mathrm{D}_z\mathcal{E}(t_j, \cdot)|\}$



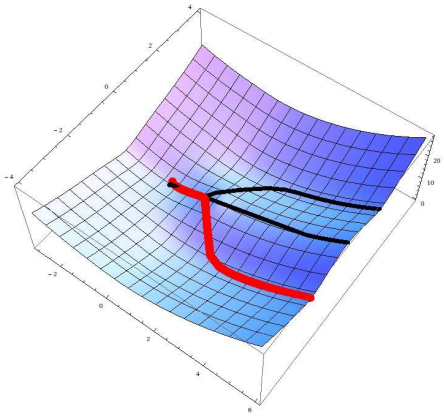
$t = 0.0$

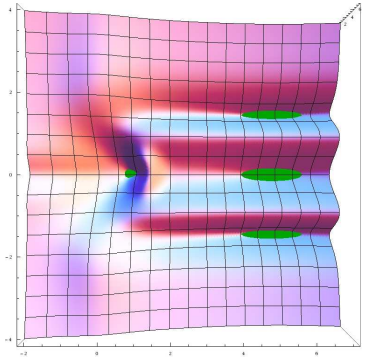


$t = 2.0$

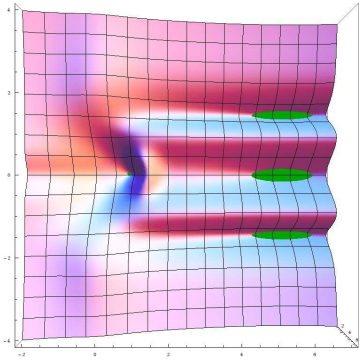


$t = 4.0$

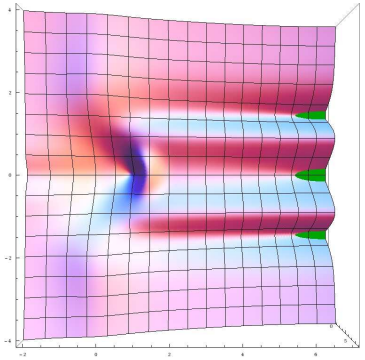




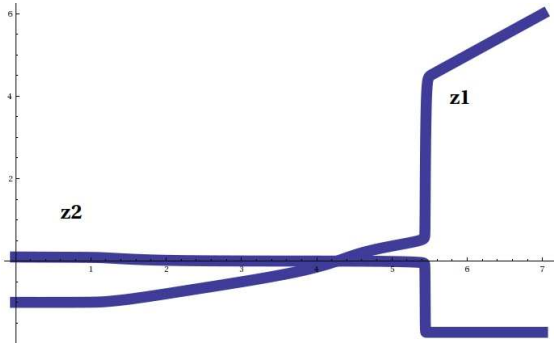
$t = 5.2$



$t = 5.6$



$t = 7.0$



2. An ODE example

- **Two types of hysteresis emerge for $\varepsilon \to 0$**
  - continuous hysteresis (play op.) via dry friction "Sign"
  - discontinuous hysteresis (relay) via jumps

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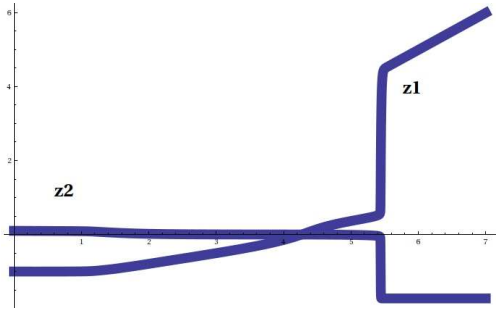
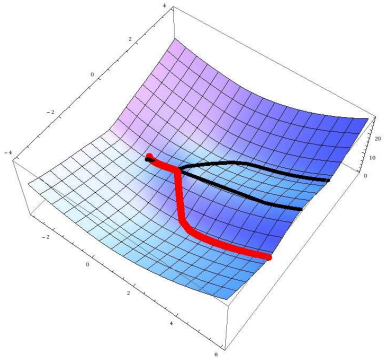
  - continuous hysteresis (play op.) via dry friction "Sign"
  - discontinuous hysteresis (relay) via jumps

- **Derivation of rate-independent limit model**
  - using the energy methods for generalized gradient systems
    $0 \in \partial\Psi(\dot{z}) + \varepsilon\dot{z} + \mathrm{D}_z\mathcal{E}(t,z)$
    energy-dissipation balance, De Giorgi's (Ψ, Ψ^*)-formulation
  - using arc-length parametrized solution curves to control jump path



jump behavior disappears in the limit $\varepsilon \to 0$

jump curves show structure that reflects the energy landscape along jump path



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$$0 \in \rho\,\mathrm{Sign}(\dot z) + \varepsilon \dot z - \Delta z - \lambda z + z^3 - g(t,x) \text{ in } \Omega,\ \ z|_{\partial\Omega} = 0$$

$\rho > 0$ nontrivial dry friction and $\varepsilon \to 0$ vanishing viscosity

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Generalized, time-dependent gradient system $(\boldsymbol{X}, \mathcal{E}(t,\cdot), \Psi_\varepsilon)$

Doubly nonlinear equation: $0 \in \partial\Psi_\varepsilon(\dot z(t)) + \mathrm{D}_z\mathcal{E}(t, z(t)) \ \subset \boldsymbol{X}^*$

Energy functional

$$\mathcal{E}(t,z) = \int_\Omega \tfrac{1}{2}|\nabla z(x)|^2 \underbrace{-\tfrac{\lambda}{2} z(x)^2 + \tfrac{1}{4} z(x)^4}_{\text{nonconvex}} - g(t,x) z(x)\,\mathrm{d}x$$

Dissipation potential $\Psi_\varepsilon(\dot z) = \int_\Omega \rho|\dot z(x)| + \frac{\varepsilon}{2}\dot z(x)^2\,\mathrm{d}x$

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Energy balance

$$\mathcal{E}(T, z_\varepsilon(T)) + \int_0^T\!\!\int_\Omega \underbrace{\rho|\dot z_\varepsilon|}_{\text{hyst. loss}} + \underbrace{\varepsilon|\dot z_\varepsilon|^2}_{\text{visc. loss}}\,\mathrm{d}x\,\mathrm{d}t = \mathcal{E}(t, z_\varepsilon(0)) + \int_0^T \underbrace{\partial_t \mathcal{E}(s, z_\varepsilon(s))}_{\text{power ext. forces}}\,\mathrm{d}s$$

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Convergence analysis for $\varepsilon \to 0$ is suitable function spaces

- dry-friction space $\boldsymbol{X} = \mathrm{L}^1(\Omega)$ $\qquad \|\dot z_\varepsilon\|_{\mathrm{L}^1(0,T;\boldsymbol{X})} \le C$
- viscosity space $\boldsymbol{V} = \mathrm{L}^2(\Omega)$ $\qquad \|\dot z_\varepsilon\|_{\mathrm{L}^2(0,T;\boldsymbol{V})} \le C/\varepsilon$
- energy space $\boldsymbol{Z} = \mathrm{H}^1_0(\Omega)$ $\qquad \|\dot z_\varepsilon\|_{\mathrm{L}^\infty(0,T;\boldsymbol{Z})} \le C$

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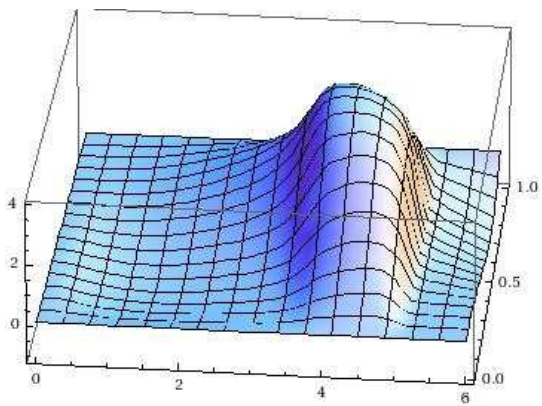
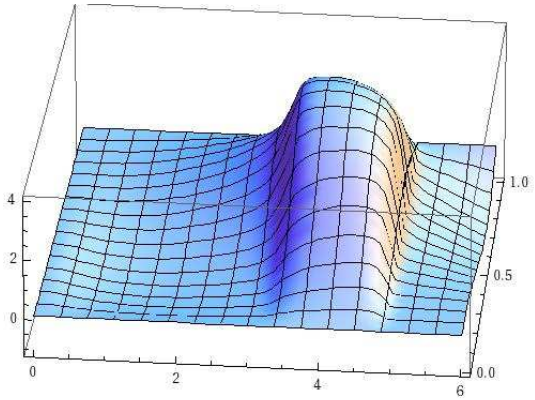
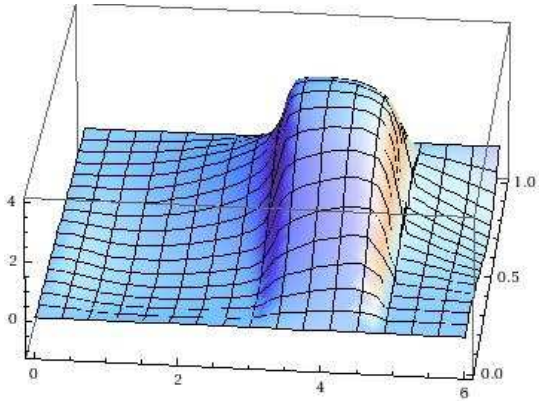
Numerical example $0 \in \mathrm{Sign}\,\dot u + \varepsilon \dot u + \partial_x^2 u - 6u(u-2)(u-4) - t(5-t)h(x)$



$\varepsilon = 2.4$



$\varepsilon = 1.2$



$\varepsilon = 0.6$

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Generalized gradient flow have general dissipation potentials:

$\Psi : \boldsymbol{X} \to [0, \infty]$ convex and lsc

$\Psi(0) = 0, \quad \partial\Psi(v) \subset \boldsymbol{X}^*$ convex subdifferential

Generalized gradient system $(\boldsymbol{X}, \mathcal{E}, \Psi)$**:**

$$\text{gradient flow} \quad 0 \in \underbrace{\partial\Psi(\dot{z}(t))}_{\text{friction forces}} + \underbrace{\mathrm{D}_z\mathcal{E}(t, z(t))}_{\text{potential forces}} \quad \subset \boldsymbol{X}^*$$

Existence theory if $\Psi(v) \geq \varepsilon \|v\|_{\boldsymbol{V}}^2$ for

doubly nontinear equations: Colli,Visintin'91,'92

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Legendre-Fenchel theory

Lengendre-Fenchel transform of Ψ: $\quad \Psi^*(\xi) \stackrel{\text{def}}{=} \sup_{v \in \boldsymbol{X}} \langle \xi, v \rangle - \Psi(v)$

Fenchel equivalence (1940):

$$\xi \in \partial\Psi(v) \iff v \in \partial_\xi \Psi^*(\xi) \iff \Psi(v) + \Psi^*(\xi) = \langle \xi, v \rangle$$

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Generalized gradient flow have general dissipation potentials:

$\Psi : \boldsymbol{X} \to [0, \infty]$ convex and lsc

$\Psi(0) = 0, \quad \partial\Psi(v) \subset \boldsymbol{X}^*$ convex subdifferential

Generalized gradient system $(\boldsymbol{X}, \mathcal{E}, \Psi)$**:**

$$\text{gradient flow} \quad 0 \in \underbrace{\partial\Psi(\dot{z}(t))}_{\text{friction forces}} + \underbrace{\mathrm{D}_z\mathcal{E}(t, z(t))}_{\text{potential forces}} \quad \subset \boldsymbol{X}^*$$

Three equivalent gradient-flow formulations:

Force balance	$0 \in \partial\Psi(\dot{z}(t)) + \mathrm{D}\mathcal{E}(t, z(t)) \subset \boldsymbol{X}^*;$
Rate equation	$\dot{z}(t) \in \partial\Psi^*(-\mathrm{D}\mathcal{E}(t, z(t))) \subset \boldsymbol{X};$
Power balance	$\Psi(\dot{z}) + \Psi^*(-\mathrm{D}\mathcal{E}(t, z)) = \langle -\mathrm{D}\mathcal{E}(t, z), \dot{z} \rangle \in \mathbb{R}.$

4. Energy-dissipation balance for gradient systems

Force balance $\quad 0 \in \partial\Psi(\dot z(t)) + \mathrm{D}\mathcal{E}(t, z(t)) \subset X^*;$

Energy balance $\quad \Psi(\dot z) + \Psi^*(-\mathrm{D}\mathcal{E}(t,z)) = \langle -\mathrm{D}\mathcal{E}(t,z), \dot z\rangle \in \mathbb{R}.$

General inequality $\Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$

Chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, z(t)) - \langle \mathrm{D}\mathcal{E}(t, z(t)), \dot z(t)\rangle = \partial_t \mathcal{E}(t, z(t))$

Proposition (Energy-Dissipation estimate)

z satisfies force balance for a.a. $t \in [0, T]$

$$\Longleftrightarrow$$

the following upper energy inequality holds:

$$\textbf{(EDe)} \quad \underbrace{\mathcal{E}(T, z(T))}_{\text{final energy}} + \underbrace{\int_0^T \Psi(\dot z) + \Psi^*(-\mathrm{D}\mathcal{E}(t,z))\,\mathrm{d}t}_{\text{dissipated energy}}$$

$$\leq \underbrace{\mathcal{E}(0, z(0))}_{\text{initial energy}} + \underbrace{\int_0^T \partial_t \mathcal{E}(t, z(t))\,\mathrm{d}t}_{\text{work of external forces}}$$

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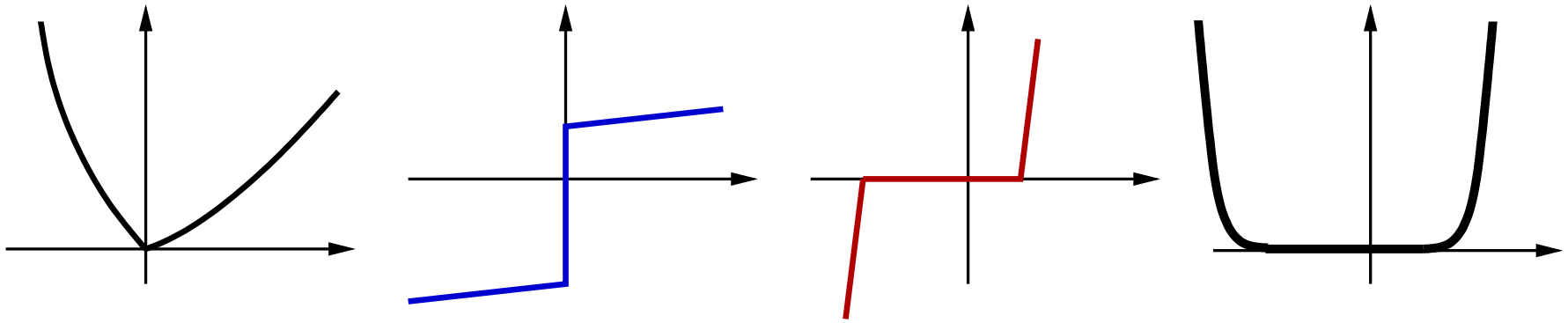
5. Parametrized Balanced-Viscosity solutions

Now consider the case $\Psi_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2$ with

- rate-independent Ψ, i.e. $\Psi(\gamma v) = \gamma^1 \Psi(v)$
- viscous space $\boldsymbol{V}$, i.e. $\|v\|_{\boldsymbol{V}}^2 = \langle \mathbb{V} v, v\rangle$

With $\boldsymbol{K} := \partial\Psi(0) \subset \boldsymbol{X}^*$ we have $\boxed{\Psi_\varepsilon^*(\xi) = \frac{1}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2}$



$\Psi(v)$ $\partial\Psi(v)$ $\partial\Psi^*(\xi)$ $\Psi^*(\xi)$

5. Parametrized Balanced-Viscosity solutions

$\Psi_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2 \qquad \Psi_\varepsilon^*(\xi) = \frac{1}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2$

Reformulation of the generalized gradient flow via the **(EDe)**

$$\mathcal{E}(T, z_\varepsilon(T)) + \int_0^T \mathrm{M}_\varepsilon(\dot{z}_\varepsilon, -\mathrm{D}_z\mathcal{E}(t, z_\varepsilon))\,\mathrm{d}t = \mathcal{E}(0, z_\varepsilon(0)) + \int_0^T \underbrace{\partial_t\mathcal{E}(s, z_\varepsilon(s))}_{-\langle \dot{\ell}(s), z_\varepsilon(s)\rangle}\,\mathrm{d}s$$

with $\boxed{\mathrm{M}_\varepsilon(v, \xi) = \Psi(v) + \frac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2 + \frac{1}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2}$

5. Parametrized Balanced-Viscosity solutions

$\Psi_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2 \qquad \Psi_\varepsilon^*(\xi) = \frac{1}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2$

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For limit passage $\varepsilon \to 0$ use a reparametrization to slow down jumps

(Efendiev–M.'06, M.–Zelik'09 $\overset{\text{publ}}{\rightsquigarrow}$ 14; M,–Rossi–Savaré'09,10,12,12,13,13,...)

$t = \mathsf{t}_\varepsilon(s)$ and $\mathsf{z}_\varepsilon(s) := z_\varepsilon(\mathsf{t}_\varepsilon(s))$

such that $\int_0^S \mathsf{t}_\varepsilon'(s)^2 + \|\mathsf{z}_\varepsilon'(s)\|_{\boldsymbol{X}}^2\,\mathrm{d}s \le C$

Example: arclength $s = \widehat{s}(t) = t + \int_0^t \|\dot z(\tau)\|_{\boldsymbol{V}}\,\mathrm{d}\tau$

5. Parametrized Balanced-Viscosity solutions

$$\Psi_\varepsilon(v) = \Psi(v) + \tfrac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2 \qquad \Psi_\varepsilon^*(\xi) = \tfrac{1}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2$$

Reformulation of the generalized gradient flow via the **(EDe)**

$$\mathcal{E}(T, z_\varepsilon(T)) + \int_0^T \mathrm{M}_\varepsilon(\dot z_\varepsilon, -\mathrm{D}_z\mathcal{E}(t, z_\varepsilon))\,\mathrm{d}t = \mathcal{E}(0, z_\varepsilon(0)) + \int_0^T \underbrace{\partial_t\mathcal{E}(s, z_\varepsilon(s))}_{-\langle \dot\ell(s), z_\varepsilon(s)\rangle}\,\mathrm{d}s$$

with $\mathrm{M}_\varepsilon(v,\xi) = \Psi(v) + \frac{\varepsilon}{2}\|v\|_{\boldsymbol{V}}^2 + \frac{1}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2$

Reparametrized **(EDe)**:

$$\mathcal{E}(\mathsf{t}_\varepsilon(S), \mathsf{z}_\varepsilon(S)) + \int_0^T \mathcal{M}_\varepsilon(\mathsf{t}_\varepsilon'(s), \mathsf{z}_\varepsilon'(s), -\mathrm{D}_z\mathcal{E}(\mathsf{t}_\varepsilon, \mathsf{z}_\varepsilon))\,\mathrm{d}s$$
$$= \mathcal{E}(0, \mathsf{z}_\varepsilon(0)) - \int_0^S \langle \dot\ell(\mathsf{t}(s)), \mathsf{z}_\varepsilon(s)\rangle \mathsf{t}_\varepsilon'(s)\,\mathrm{d}s$$

with $\boxed{\mathcal{M}_\varepsilon(\alpha, v, \xi) = \alpha \mathrm{M}_\varepsilon(\tfrac{1}{\alpha}v, \xi) = \Psi(v) + \tfrac{\varepsilon}{2\alpha}\|v\|_{\boldsymbol{V}}^2 + \tfrac{\alpha}{2\varepsilon}\mathrm{dist}_{\boldsymbol{V}}(\xi, \boldsymbol{K})^2}$

5. Parametrized Balanced-Viscosity solutions

$$\mathcal{E}(\mathsf{t}_\varepsilon(S),\mathsf{z}_\varepsilon(S)) + \int_0^T \mathcal{M}_\varepsilon(\mathsf{t}_\varepsilon',\mathsf{z}_\varepsilon',-\mathrm{D}_z\mathcal{E}(\mathsf{t}_\varepsilon,\mathsf{z}_\varepsilon))\,\mathrm{d}s = \mathcal{E}(0,\mathsf{z}_\varepsilon(0)) - \int_0^S \langle \dot\ell(\mathsf{t}_\varepsilon),\mathsf{z}_\varepsilon\rangle \mathsf{t}_\varepsilon'\,\mathrm{d}s$$

with $\mathcal{M}_\varepsilon(\alpha,v,\xi) = \Psi(v) + \frac{\varepsilon}{2\alpha}\|v\|_{\boldsymbol{V}}^2 + \frac{\alpha}{2\varepsilon}\mathsf{dist}_{\boldsymbol{V}}(\xi,\boldsymbol{K})^2$

For the family $\mathcal{M}_\varepsilon : \mathbb{R}_\geq \times \boldsymbol{X} \times \boldsymbol{X}^* \to [0,\infty]$

we have weak Γ-convergence as $\varepsilon \to 0$ as follows

$$\mathcal{M}_\varepsilon \xrightarrow{\Gamma} \mathcal{M}_0 : (\alpha,v,\xi) \mapsto \begin{cases} \Psi(v) + \chi_{\boldsymbol{K}}(\xi) & \text{for } \alpha > 0, \\ \Psi(v) + \|v\|_{\boldsymbol{V}}\mathsf{dist}_{\boldsymbol{V}}(\xi,\boldsymbol{K}) & \text{for } \alpha = 0. \end{cases}$$

5. Parametrized Balanced-Viscosity solutions

$$\mathcal{E}(\mathsf{t}_\varepsilon(S),\mathsf{z}_\varepsilon(S)) + \int_0^T \mathcal{M}_\varepsilon(\mathsf{t}'_\varepsilon,\mathsf{z}'_\varepsilon,-\mathrm{D}_z\mathcal{E}(\mathsf{t}_\varepsilon,\mathsf{z}_\varepsilon))\,\mathrm{d}s = \mathcal{E}(0,\mathsf{z}_\varepsilon(0)) - \int_0^S \langle \dot{\ell}(\mathsf{t}_\varepsilon),\mathsf{z}_\varepsilon\rangle \mathsf{t}'_\varepsilon\,\mathrm{d}s$$

with $\mathcal{M}_\varepsilon(\alpha,v,\xi) = \Psi(v) + \frac{\varepsilon}{2\alpha}\|v\|_{\boldsymbol{V}}^2 + \frac{\alpha}{2\varepsilon}\mathsf{dist}_{\boldsymbol{V}}(\xi,\boldsymbol{K})^2$

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The two different regimes emerge automatically:

- $\mathsf{t}'(s) = \alpha > 0$ classical rate-independent friction $\dot{z}(t) \approx O(1)$
- $\mathsf{t}'(s) = \alpha = 0$ fast jump: $\underbrace{\Psi(v)}_{\text{hyst. loss}} + \underbrace{\|v\|_{\boldsymbol{V}}\mathsf{dist}_{\boldsymbol{V}}(\xi,\boldsymbol{K})}_{\text{viscous loss}}$

The limiting viscosity at the jumps is retained in a "balanced way".

5. Parametrized Balanced-Viscosity solutions

$$\mathcal{E}(\mathsf{t}_\varepsilon(S),\mathsf{z}_\varepsilon(S)) + \int_0^T \mathcal{M}_\varepsilon(\mathsf{t}_\varepsilon',\mathsf{z}_\varepsilon',-\mathrm{D}_z\mathcal{E}(\mathsf{t}_\varepsilon,\mathsf{z}_\varepsilon))\,\mathrm{d}s = \mathcal{E}(0,\mathsf{z}_\varepsilon(0)) - \int_0^S \langle \dot\ell(\mathsf{t}_\varepsilon),\mathsf{z}_\varepsilon\rangle \mathsf{t}_\varepsilon'\,\mathrm{d}s$$

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Theorem [MRossiSav'13]: (some techn. assumpt.)
If $(\mathsf{t}_\varepsilon,\mathsf{z}_\varepsilon)$ solve **(EDe)**$_\varepsilon$, then there exists a subsequence such that $(\mathsf{t}_\varepsilon(s),\mathsf{z}_\varepsilon(s)) \to (\mathsf{t}(s),\mathsf{z}(s))$ and the pair satisfies the rate-independent limit model **(EDe)**$_0$, viz.

$$\mathcal{E}(\mathsf{t}(S),\mathsf{z}(S)) + \int_0^T \mathcal{M}_0(\mathsf{t}'(s),\mathsf{z}'(s),-\mathrm{D}_z\mathcal{E}(\mathsf{t}(s),\mathsf{z}(s)))\,\mathrm{d}s = \mathcal{E}(0,\mathsf{z}(0)) - \int_0^S \langle \dot\ell(\mathsf{t}(s)),\mathsf{z}(s)\rangle \mathsf{t}'(s)\,\mathrm{d}s$$

5. Parametrized Balanced-Viscosity solutions

$$\mathcal{E}(\mathsf{t}(S),\mathsf{z}(S)) + \int_0^T \mathcal{M}_0(\mathsf{t}',\mathsf{z}',-\mathrm{D}_z\mathcal{E}(\mathsf{t},\mathsf{z}))\,\mathrm{d}s = \mathcal{E}(0,\mathsf{z}(0)) - \int_0^S \langle \dot{\ell}(\mathsf{t}),\mathsf{z}\rangle \mathsf{t}'\,\mathrm{d}s$$

$$\text{with } \mathcal{M}_0(\alpha,v,\xi) = \begin{cases} \Psi(v) + \chi_{\boldsymbol{K}}(\xi) & \text{if } \alpha > 0, \\ \Psi(v) + \|v\|\mathsf{dist}(\xi,\boldsymbol{K}) & \text{if } \alpha = 0. \end{cases}$$

Analyzing the contact set $\{\,(\alpha,v,\xi) \mid \mathcal{M}_0(\alpha,v,\xi) = \langle \xi, v\rangle\,\}$ we find:

Rate-independent limit model:

$$0 \in \partial\Psi(\mathsf{z}'(s)) + \lambda(s)\mathsf{z}'(s) + \mathrm{D}_z\mathcal{E}(\mathsf{t}'(s),\mathsf{z}(s)) \;\subset \boldsymbol{X}^*$$

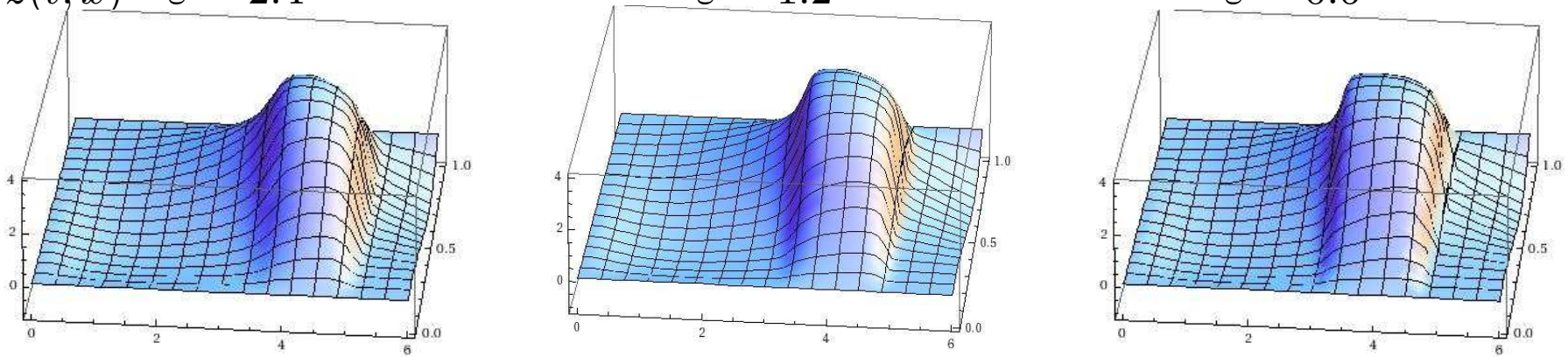
with switching conditions $\mathsf{t}'(s) \geq 0, \quad \lambda(s) \geq 0, \quad \mathsf{t}'(s)\lambda(s) \equiv 0$

- $\mathsf{t}'(s) > 0 \implies \lambda(s) = 0$: classical dry-friction evolution
- $\lambda(s) > 0 \implies \mathsf{t}'(s) = 0$: jump with $\mathsf{t}(s) = \text{const}$ for $s \in [s_1, s_2]$

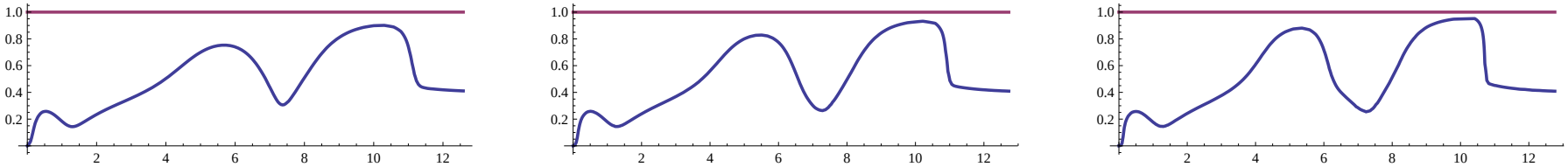
5. Parametrized Balanced-Viscosity solutions

Numerical example $0 \in \text{Sign}\,\dot{z} + \varepsilon\dot{z} + \partial_x^2 z - 6z(z-2)(z-4) - t(5-t)h(x)$

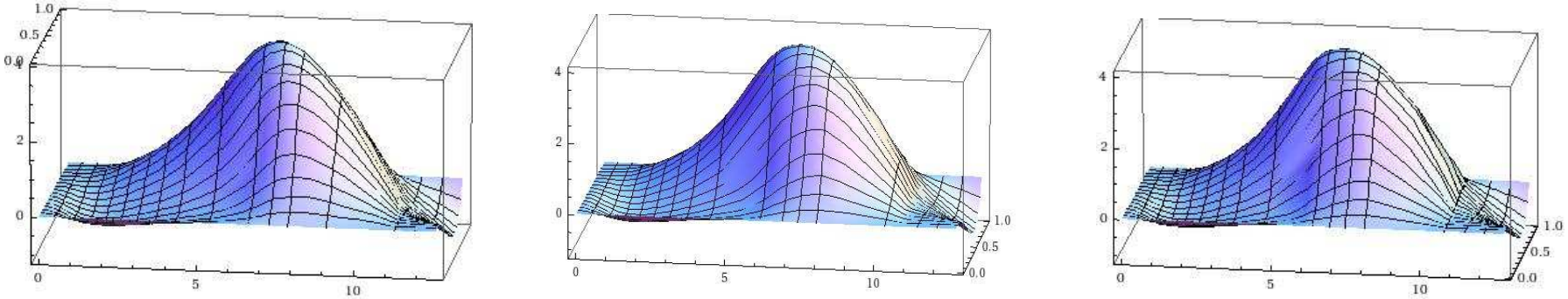
$z(t,x)$ $\varepsilon = 2.4$ $\varepsilon = 1.2$ $\varepsilon = 0.6$



$\|\partial_s \mathsf{z}(s)\|_{\mathrm{L}^2}$



$\mathsf{z}(s,x)$



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We return to non-parametrized solutions $z : [0,T]\boldsymbol{Z}$:

$\Pi(\mathrm{t},\mathrm{z}) \stackrel{\text{def}}{=} \{\, z : [0,T] \to \boldsymbol{Z} \mid \text{Graph}(z) \subset \text{Range}(\mathrm{t},\mathrm{z}) \,\}$

Limits of solutions z_ε of $0 \in \partial\Psi(\dot{z}) + \varepsilon\mathbb{V}\dot{z} + \mathrm{D}_z\mathcal{E}(t,z)$

should have a graph lying in the range $\text{Range}(\mathrm{t},\mathrm{z})$.

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Definition of Balanced Viscosity solutions [M.RossiSav.'11'13]
A function $z \in \mathrm{BV}([0,T];X) \cap \mathrm{L}^\infty(0,T,Z)$ is called Balanced Viscosity solution of the RIS $(Z,\mathcal{E},\Psi,\mathbb{V})$, if

local stability $\quad 0 \in \partial\Psi(0) + \mathcal{E}(t,z(t))$ for all $t \in [0,T]\backslash J(z)$

energy inequality $\quad \mathcal{E}(T,z(T)) + \text{Diss}_{\mathcal{M}_0,\mathcal{E}}(z,[0,T])$
$$\leq \mathcal{E}(0,z(0)) + \int_0^T \partial_t\mathcal{E}(t,z(t))\,\mathrm{d}t$$

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$\text{Diss}_{\mathcal{M}_0,\mathcal{E}}(z,[0,T]) =$ Hysteretic dissip. $\int_0^T \Psi(\dot z(t))\,\mathrm{d}t +$

$+$ Balanced Visc. diss. at jumps $\sum_{t\in J(z)} \mathcal{C}_{\mathcal{M}_0,\mathcal{E}}(z(t_-),z(t_+))$

6. Non-parametrized Bal. Visc. solutions

$$(*)_\varepsilon \qquad 0 \in \partial\Psi(\dot z) + \varepsilon \mathbb{V}\dot z + \mathrm{D}\mathcal{E}(t,z), \quad z(0) = z_0$$

Theorem [MRS'13]
(Vanishing-viscosity solutions are Bal. Visc. solutions)
Under suit. techn. assumtptions, let $z_\varepsilon \in \mathrm{H}^1(0,T;\boldsymbol{V})$ solve $(*)_\varepsilon$.

Then, $\exists\ \varepsilon_k \to 0$ and $z \in \mathrm{BV}([0,T],\boldsymbol{X}) \cap \mathrm{L}^\infty(0,T,\boldsymbol{Z})$ such that

- $z^{\varepsilon_k}(t) \rightharpoonup z(t)$ in $\boldsymbol{Z}$ for all $t \in [0,T]$
- z is a Balanced Viscosity solution to $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{V})$.

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Idea of proof:

Choose subsequence such that parametrized solutions ζ^{ε_k} converge

Show convergence of $z^{\varepsilon_k} \in \Pi(\zeta^{\varepsilon_k})$ in $[0,T]\backslash J(z)$

Treat countable number of jump times

6. Non-parametrized Bal. Visc. solutions

Time discretization: time-step $\tau > 0$ and viscosity $\varepsilon > 0$.

$$\boxed{z_k \in \operatorname*{Argmin}_{z \in \boldsymbol{Z}} \mathcal{E}(k\tau, z) + \Psi(z - z_{k-1}) + \frac{\varepsilon}{2\tau} \|z - z_{k-1}\|_{\boldsymbol{V}}^2}$$

Piecewise affine interpolant $\widetilde{z}^{\tau,\varepsilon} : [0,T] \to \boldsymbol{Z}$.

Theorem (Convergence for $\tau, \varepsilon \to 0$)

If $\varepsilon_k \to 0$, $\tau_k \to 0$, and $\varepsilon_k / \tau_k \to \infty$, then for a subsequence

- $\widetilde{z}^{\tau_l,\varepsilon_l}(t) \rightharpoonup z(t)$ in Z for all $t \in [0,T]$,
- z is a Balanced-Viscosity solution for $(\boldsymbol{Z}, \mathcal{E}, \Psi, \mathbb{V})$.

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Thank You for Your Attention

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