Oscillations in the Olsen Model

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Outline

The Olsen Model - A 4D System in Nonstandard Form
(joint work with **Peter Szmolyan**, Vienna):

(1) Some Background on Multiple Time Scales
(2) Olsen Model - Transformation
(3) The Main Two Subsystems
(4) The Blow-Up Method
(5) Non-Classical Relaxation Oscillations

Reference: preprint arXiv:1403.5658

Fast-Slow Systems - Standard Form

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, time scale separation $0 < \epsilon \ll 1$.

$$\left\{ \begin{array}{rcl} x' & = & f(x,y) \\ y' & = & \epsilon g(x,y) \end{array} \right. \quad \overset{\epsilon t = s}{\longleftrightarrow} \quad \left\{ \begin{array}{rcl} \epsilon \dot{x} & = & f(x,y) \\ \dot{y} & = & g(x,y) \end{array} \right.$$

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fast subsystem — slow subsystem

- $C := \{f = 0\}$ = critical manifold = equil. of fast subsystem.
- C is normally hyperbolic if $D_x f$ has no zero-real-part eigenvalues.
- **Fenichel's Theorem:** Normal hyperbolicity $\Rightarrow$ "nice" perturbation.

A 4D System in Nonstandard Form

Olsen Model for peroxidase-oxidase reaction (Olsen 1979)

$$
\begin{aligned}
\frac{dA}{dT} &= -k_3ABY + k_7 - k_{-7}A, \\
\frac{dB}{dT} &= -k_3ABY - k_1BX + k_8, \\
\frac{dX}{dT} &= k_1BX - 2k_2X^2 + 3k_3ABY - k_4X + k_6, \\
\frac{dY}{dT} &= -k_3ABY + 2k_2X^2 - k_5Y.
\end{aligned}
$$

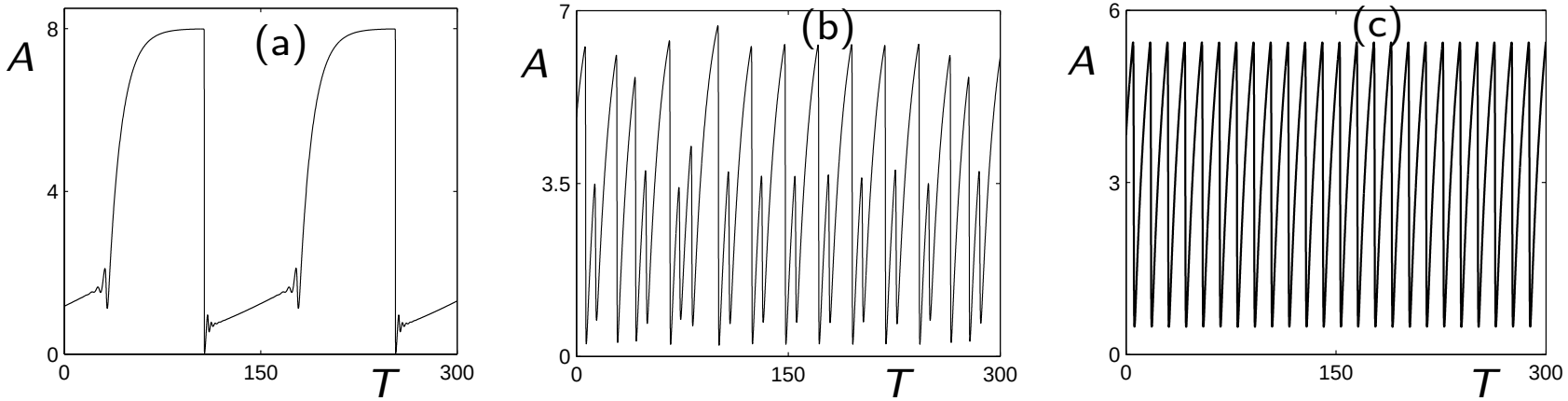

Figure : (a) MMOs for $k_1 = 0.16$, (b) chaotic/aperiodic oscillations for $k_1 = 0.35$ and (c) periodic oscillations for $k_1 = 0.41$.

Change of variables (PhD thesis of A. Milik)

$$A = \frac{k_1 k_5}{k_3\sqrt{2k_2k_8}} a_2, \;\; B = \frac{\sqrt{2k_2k_8}}{k_1} b_2, \;\; X = \frac{k_8}{2k_2} x_2,$$
$$Y = \frac{k_8}{k_5} y_2, \;\; T = \frac{k_1 k_5}{k_3 k_8 \sqrt{2k_2k_8}} s,$$

transforms the Olsen model into

$$\text{(Ols2)} \begin{cases} \frac{da_2}{ds} &= \mu - \alpha a_2 - a_2 b_2 y_2, \\ \frac{db_2}{ds} &= \epsilon_b (1 - b_2 x_2 - a_2 b_2 y_2), \\ \epsilon^2 \frac{dx_2}{ds} &= b_2 x_2 - x_2^2 + 3 a_2 b_2 y_2 - \xi x_2 + \delta, \\ \epsilon^2 \frac{dy_2}{ds} &= \kappa (x_2^2 - y_2 - a_2 b_2 y_2). \end{cases}$$

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	μ	α	ϵ_b	ϵ^2	ξ	δ	κ
$k_1 = 0.16$	0.97	0.15	**0.0095**	**0.033**	0.98	$1.2 \cdot 10^{-5}$	3.93
$k_1 = 0.35$	0.97	0.32	0.045	0.015	0.98	$1.2 \cdot 10^{-5}$	3.93
$k_1 = 0.41$	0.97	0.37	**0.062**	**0.013**	0.98	$1.2 \cdot 10^{-5}$	3.93

Table : Standard parameter values for the Olsen model.

We need one more transformation:

$$a = a_2, \qquad b = b_2, \qquad x = \epsilon x_2, \qquad y = \epsilon^2 y_2, \qquad \tau = \epsilon^{-2} s$$

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- nonstandard form,
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- nonstandard form,
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- for (Ols2), $\epsilon^2 \to 0 \Rightarrow$ two fast and two slow variables,
- for (Ols2), $\epsilon^2 \neq 0$ and $\epsilon_b \to 0 \Rightarrow$ three fast and one slow,
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- for (Ols1), $\epsilon \to 0$ and $\epsilon_b \neq 0 \Rightarrow$ one fast and three slow,
- role of $\delta \to 0$,
- loss of normal hyperbolicity.

The "Fastest-Scale" Model

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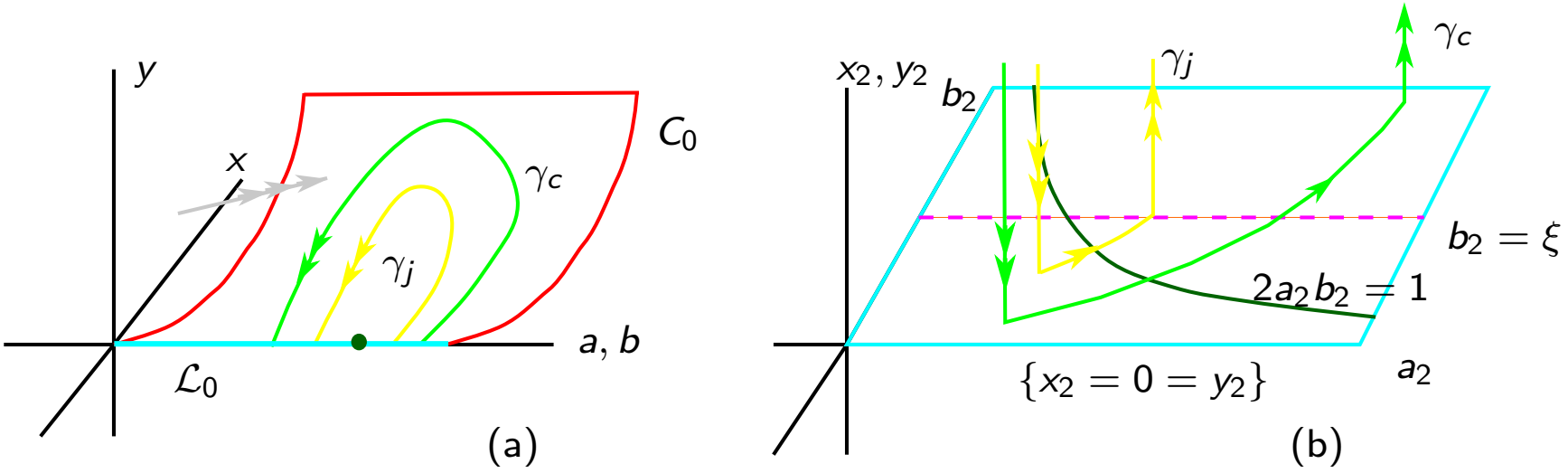

Theorem (K., Szmolyan 2012/2013)

- **Canard case** $0 < \epsilon \ll 1$ $\delta = \mathcal{O}(\epsilon^2 e^{-K_1/\epsilon^2})$ *then nonclassical relaxation oscillation has a canard segment.*
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- Way-in, way-out function for canards + exchange lemma.
- Global return map analysis, explicit estimates.

The Blow-Up Method - An Example

$$\text{vector field } X \begin{cases} \frac{dx}{dt} &= -x^2 + y, \\ \frac{dy}{dt} &= \epsilon, \\ \frac{d\epsilon}{dt} &= 0. \end{cases}$$

Blow-up map $\Phi : S^2 \times [0, r_0] \to \mathbb{R}^2 \Rightarrow \phi_*(\bar{X}) = X.$

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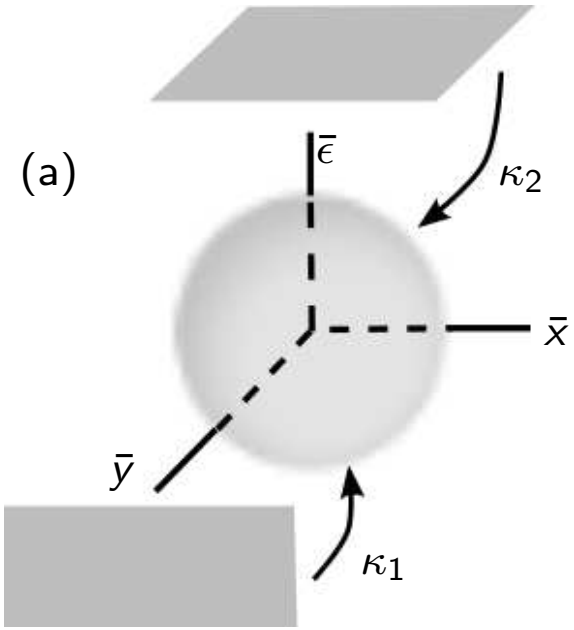

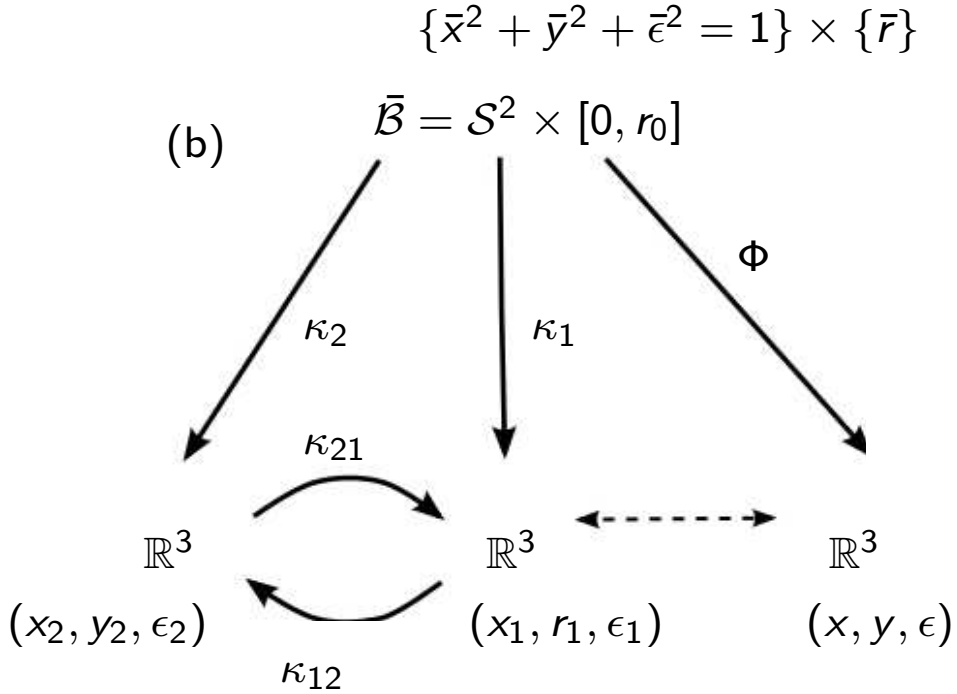

Main Olsen Blow-Up

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$$

Define the manifold

$$
\bar{\mathcal{D}} := [a^*, \infty) \times [b^*, \infty) \times (\mathcal{S}^2)_0^+ \times [0, r_0].
$$

Blow-up transformation $\Phi : \bar{\mathcal{D}} \to \mathcal{D}$ defined via

$$
a = \bar{a}, \qquad b = \bar{b}, \qquad x = \bar{r}\bar{x}, \qquad y = \bar{r}^2\bar{y}, \qquad \epsilon = \bar{r}\bar{\epsilon}
$$

Dynamics in the Classical / Rescaling Chart

System (Ols2) - nonclassical relaxation orbit, canard case.

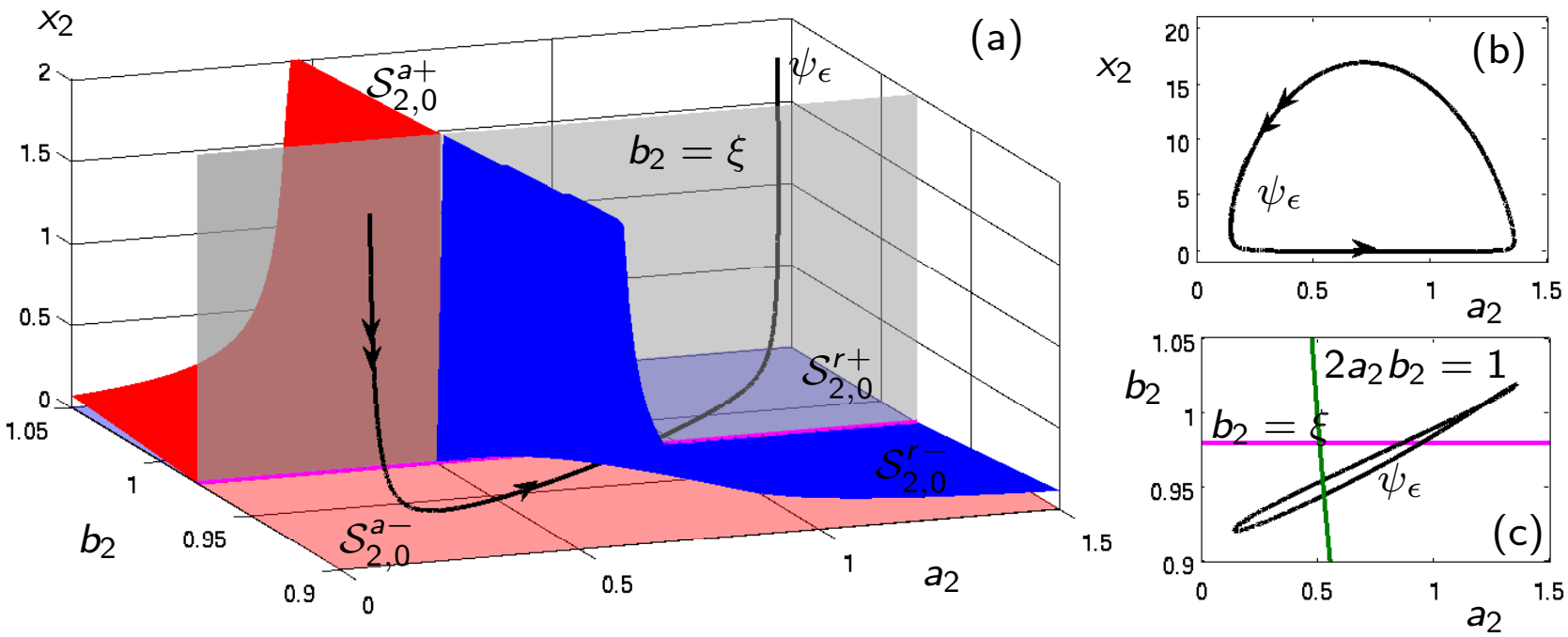

Figure : (a) Projection into (a_2, b_2, x_2)-space. $\delta = 0$. Critical manifold in blow-up (blue=repelling, red=attracting), transcritical (magenta) and hyperplane $\{b_2 = \xi\}$. (b) Projection of the full periodic solution into (a_2, x_2)-space. (c) Important curves in the (a_2, b_2)-plane.

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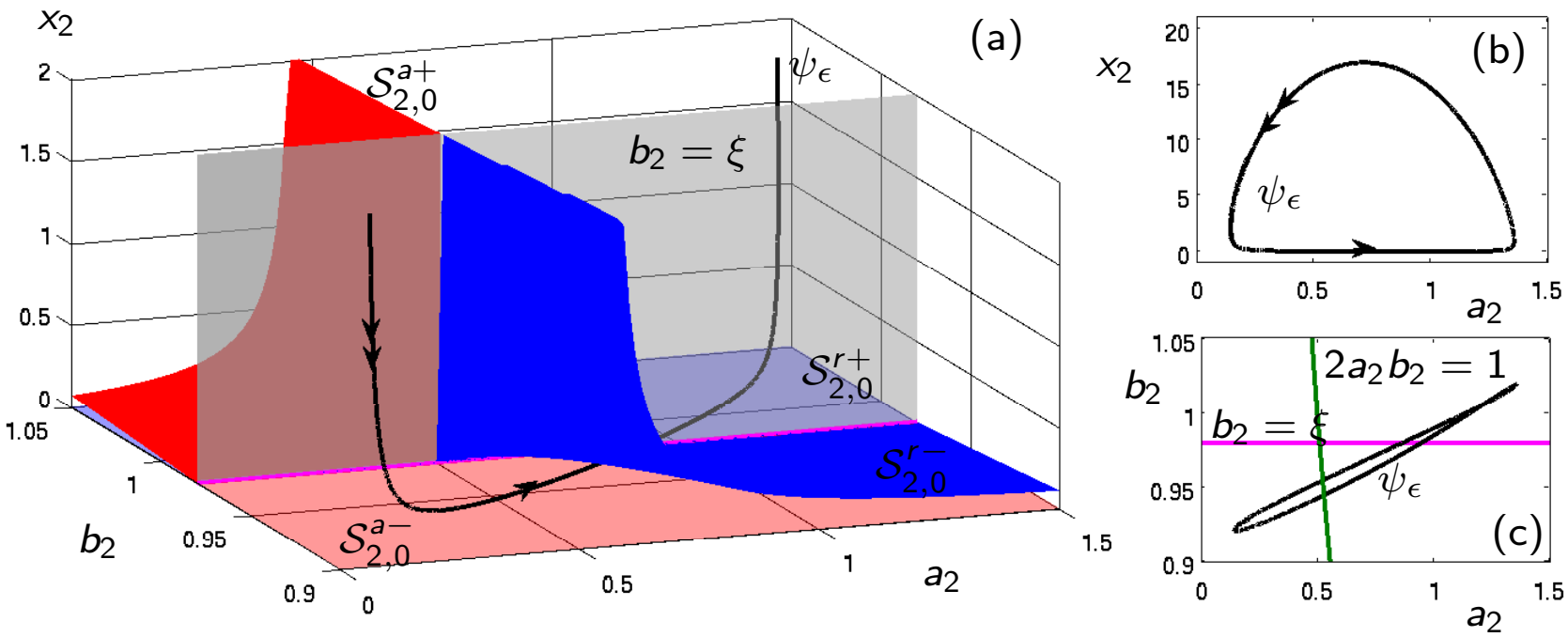

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Remark: Normal form unfolding for jumps across singularities at infinity **CK** arXiv:1204.0947.

Case 2: Mixed-Mode Oscillations

$\epsilon_b \to 0$, three fast variables and one slow variable

$$\begin{cases} a' &= \epsilon^3(\mu - \alpha a) - \epsilon aby, \\ b' &= \epsilon^2(\epsilon_b \epsilon - \epsilon_b bx) - \epsilon \epsilon_b aby, \\ x' &= -x^2 + \epsilon(b - \xi)x + 3aby + \epsilon^2 \delta, \\ y' &= \epsilon\kappa(x^2 - y - aby). \end{cases}$$

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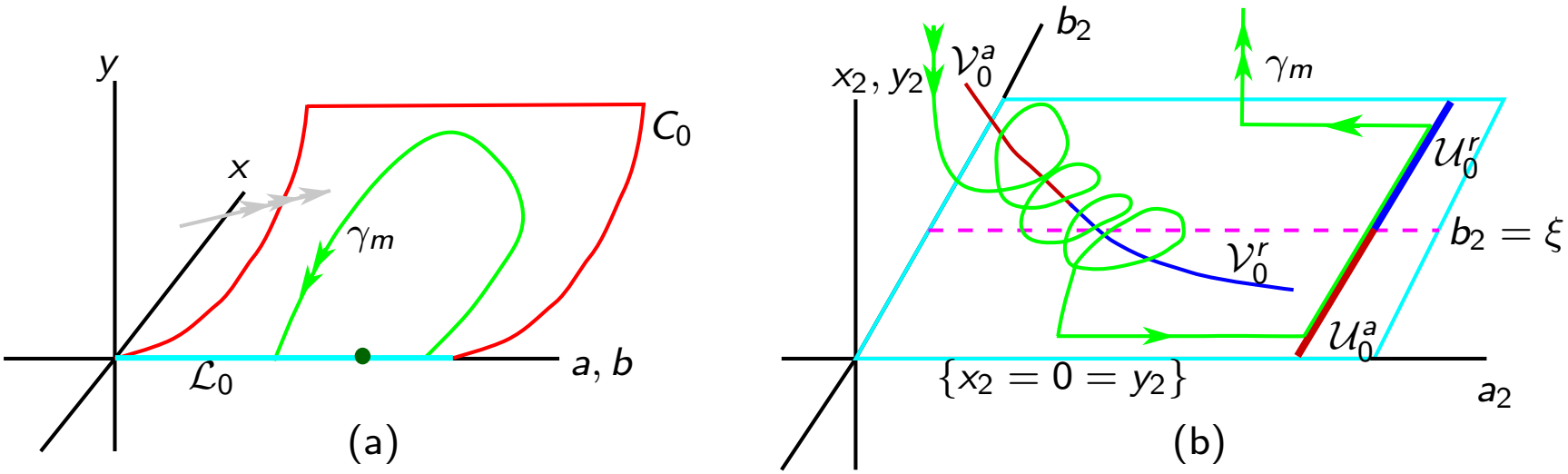
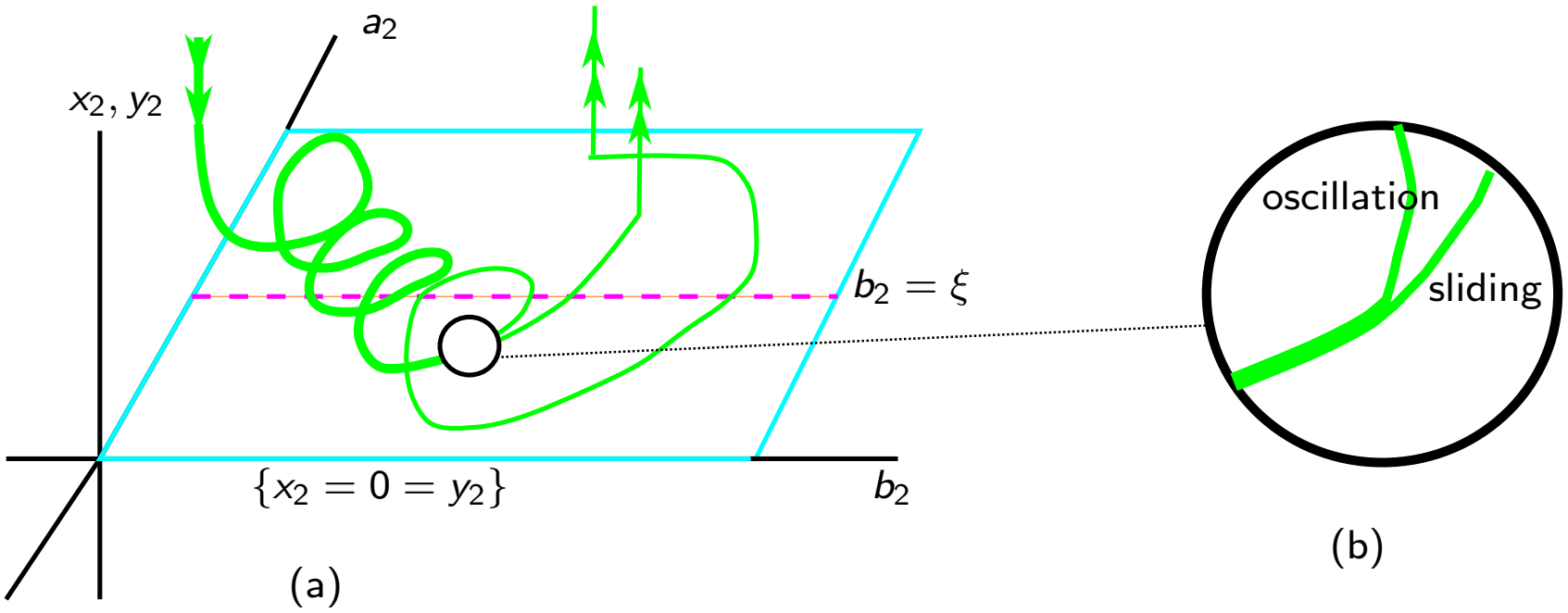


Note: Delayed Hopf bifurcation is involved.

Case 3: A Chaos-Generating Mechanism

"Intermediate" between MMOs and Nonclassical Relaxation



Conjectures:

1. Smale Horseshoe "splitting-streching-folding".
2. Grazing-sliding nonsmooth limit scenario.

Other Topics

Remark: Multiscale Dynamics (almost) everywhere!

1. Critical transitions / tipping points in applications
2. Noise and mixed-mode oscillations in SODEs
3. Large deviation principles in SIDE/SPDEs
4. Singular perturbations of PDE operators
5. Self-organized criticality in adaptive networks
6. Averaging / homogenization problems
7. Numerically challenging ("multiscale methods")
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see also:

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