Weierstrass Institute for
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Leibniz Association

String-representation of hysteresis operators acting on vector-valued, left-continuous and piecewise monotaffine and continuous functions

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- *Hysteresis operators* as introduced in Krasnosel'skii-Pokrovskii 1983 can be considered as mathematical operators mapping time dependent functions to time dependent functions being *causal and rate-independent*, see Visintin 1994, Brokate–Sprekels 1996, Krejčí 1996, Mayergoyz 2003.
- Brokate–Sprekels 1996:
  a representation result for hysteresis operators acting on **scalar** inputs
  input function
- Representation result applied in Brokate 1994, Brokate 2000, Ekanayake–Iyer 2008, Gasiński 2004, 2008, Jais-2008, Kaltenbacher–Kaltenbacher-2007, Löschner–Brokate 2008, Löschner-Greenberg 2008, Miettinen–Panagiotopoulos 1998, Tan–Baras–Krishnaprasad 2005, Visone 2008

- K. 2012 a,b, K. 2013:
  extension of this representation result to hysteresis operators acting on **vector-valued** input functions.
- This talk, K. 2013p (=WIAS Preprint No. 1912 (2013)):
  extension of this representation to input functions with a finite number number of discontinuities

- Let $T > 0$ denote some final time.
- Let X be some topological vector space.
- Let Y be some nonempty set, and let $\mathrm{Map}\,([0,T],Y) := \{v : [0,T] \to Y\}$.
- Let $\mathrm{C}\,([0,T];X)$ denote the set of all continuous functions $u : [0,T] \to X$.
- $\alpha : [0,T] \to [0,T]$ is an *admissible time transformation* $:\Longleftrightarrow$ α is continuous and increasing (not necessary strictly increasing), $\alpha(0) = 0$ and $\alpha(T) = T$.

Definition

Let $\mathcal{H} : D(\mathcal{H})(\subseteq \mathrm{Map}\,([0,T],X)) \to \mathrm{Map}\,([0,T],Y)$ with $D(\mathcal{H}) \neq \emptyset$ be given.

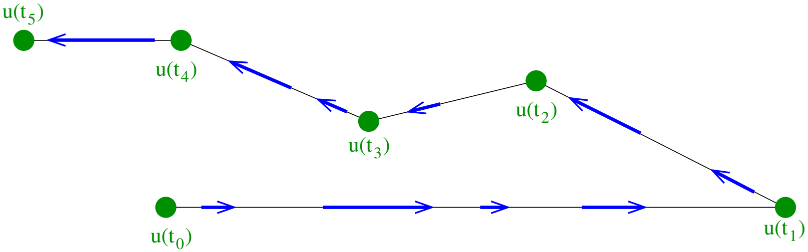
- $\mathcal{H}$ a *hysteresis operator* $:\Longleftrightarrow$ $\mathcal{H}$ is rate-independent and causal.
- $\mathcal{H}$ is *rate-independent* $:\Longleftrightarrow \forall\, v \in D(\mathcal{H}), \forall$ admissible time transformation $\alpha : [0,T] \to [0,T]$ with $v \circ \alpha \in D(\mathcal{H}), \forall\, t \in [0,T]$:

$$\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t)).$$

- $\mathcal{H}$ is *causal* $:\Longleftrightarrow \forall\, v_1, v_2 \in D(\mathcal{H}), \forall\, t \in [0,T]$:
  If $v_1(\tau) = v_2(\tau) \quad \forall\, \tau \in [0,t]$ then $\mathcal{H}[v_1](t) = \mathcal{H}[v_2](t)$.

- **Brokate–Sprekels 1996:** Hysteresis operators act on the set of all continuous and piecewise monotone functions and the set of all piecewise monotone functions from $[0, T]$ to $\mathbb{R}$.
- **K. 2012a, K. 2012b, K. 2013:** Hysteresis operators acting on continuous and piecewise **monotaffine** function from $[0, T]$ to X.
- **Monotaffine function informal:** composition of a **monot**one and an **affine** function, monotone functions being evaluated first
- **Monotaffine function precise:** K. 2012a, b, K. 2013:
  Let some function $u : [0, T] \to X$ be given
  Let some $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ be given.
  u is *monotaffine on* $[t_1, t_2]$:$\Longleftrightarrow$
  $\exists\, \beta : [t_1, t_2] \to [0, 1]$ monotone increasing (not necessary strictly increasing) such that $\beta(t_1) = 0$, $\beta(t_2) = 1$ and

$$u(t) = (1 - \beta(t))u(t_1) + \beta(t)u(t_2), \quad \forall\, t \in [t_1, t_2].$$



- $u : [0, T] \to X$ u is denoted as *piecewise monotaffine* $:\Longleftrightarrow$ there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that for $\forall\, 1 \le i \le n$: u is monotaffine on $[t_{i-1}, t_i]$.
- Let $\mathrm{Map}_{\mathrm{pw.ma.}}([0, T]; X)$ be the set of all **p**iece**w**ise **m**onot**a**ffine functions in $\mathrm{Map}\,([0, T], ()\,[0, T]; X)$.
- $\mathrm{Map}_{\mathrm{pw.ma.}}([0, T]; \mathbb{R})$ is just the set of all piecewise monotone functions
- Let $\mathrm{C}_{\mathrm{pw.ma.}}([0, T]; X) := \mathrm{Map}_{\mathrm{pw.ma.}}([0, T]; X) \cap C([0, T]; X)$.

- The *standard monotonicity partition of $[0,T]$* for piecewise monotone, scalar input functions is introduced in Brokate–Sprekels 1996.
- In K. 2012a, K. 2012b, K. 2013 the *standard monotaffinicity partition of $[0,T]$ for u* with $u \in \mathrm{Map}_{\mathrm{pw.ma.}}([0,T];X)$ being appropriate is defined.
- Now, this definition it extended to:
  Let $u \in \mathrm{Map}_{\mathrm{pw.ma.}}([0,T];X)$ be given.
  The *standard monotaffinicity partition of $[0,T]$ for u* := uniquely defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that for $1 \leq i \leq n$ holds:
  $t_i := \sup \left\{ t \in]t_{i-1}, T] \middle| u \text{ is monotaffine on } [t_{i-1}, t] \right\}$

K. 2012a,b; K. 2013:

- $(v_a, v_b, v_c) \in X^3$ is denoted as *convexity triple* $\iff v_b \in \mathrm{conv}(v_a, v_c)$ with $\mathrm{conv}(v_a, v_c j) := \{(1-\lambda)v_a + \lambda v_c \,|\, \lambda \in [0,1]\}$
- A *convexity triple free string of elements of X* is any $(v_0, v_1) \in X^2$ and any $(v_0, \ldots, v_n) \in X^{n+1}$ with $1 < n \in \mathbb{N}$ such that for all $i \in \{1, \ldots, n-1\}$ it holds that (v_{i-1}, v_i, v_{i+1}) is **no** convexity triple.
- Let $S_F(X) := \{V \in X^{n+1} \,|\, n \in \mathbb{N}$ and V is a convexity triple free string of elements of $X\}$.
- It holds $S_F(\mathbb{R}) = S_A$ with S_A being the set of *alternating strings:* introduced in Brokate–Sprekels 1996, i.e.:

$$S_A = \{(v_0, v_1, \ldots, v_n) \in \mathbb{R}^{n+1} \,|\, n \geq 1, \\ (v_i - v_{i-1})(v_{i+1} - v_i) < 0, \quad \forall\, 1 \leq i < n\}.$$

Lemma

Every function $G : S_F(X) \to Y$ *generates a hysteresis operator* $\mathcal{H}_G^{\mathrm{Map}} : \mathrm{Map}_{\mathrm{pw.ma.}}([0,T];X) \to \mathit{Map}([0,T];Y)$ *by mapping* $u \in \mathrm{Map}_{\mathrm{pw.ma.}}([0,T];X)$ *to the function* $\mathcal{H}_G^{\mathrm{Map}}[u] : [0,T] \to Y$ *being defined by considering the standard monotaffinicity partition* $0 = t_0 < t_1 < \cdots < t_n = T$ *of* $[0,T]$ *for* u *and defining*

$$\mathcal{H}_G^{\mathrm{Map}}[u](t) = G\left(u(t_0), u(t)\right), \forall\, t \in [t_0, t_1],$$

$$\mathcal{H}_G^{\mathrm{Map}}[u](t) = G\left(u(t_0), u(t_1-), u(t)\right), \quad \forall \quad t \in]t_1, t_2],$$

$$\mathcal{H}_G^{\mathrm{Map}}[u](t) = G\left(u(t_0), u(t_1-), \ldots, u(t_{i-1}-), u(t)\right),$$

$$\forall \quad t \in]t_{i-1}, t_i], \quad 3 \le i \le n.$$

K. 2012a, K. 2012b, K. 2013:

Theorem

For ***every hysteresis operator*** $\mathcal{B} : \mathrm{C}_{\mathrm{pw.ma.}}([0,T];X) \to \mathrm{Map}([0,T];Y)$ *there* ***exists a unique function*** $G : S_F(X) \to Y$ *such that* $\mathcal{B} = \mathcal{H}_G^C$ *with* $\mathcal{H}_G^C$ *being the restriction of* $\mathcal{H}_G^{\mathrm{Map}}$ *to* $\mathrm{C}_{\mathrm{pw.ma.}}([0,T];X)$
(G can be determined by considering appropriate piecewise affine function representing the strings.)

(Result in Brokate-Sprekels 1996 for operators with scalar input and output:
$X = \mathbb{R}, Y := \mathbb{R}, S_F(X) \mapsto S_A, \mathrm{C}_{\mathrm{pw.ma.}}([0,T];X) \mapsto \mathrm{C}_{\mathrm{pm}}([0,T]),$
"monotaffinicity" $\mapsto$ "monotonicity",

K. 2013:

Corollary

For every $G : S_F(X) \to Y$ *is holds that* $\mathcal{H}_G^{\mathrm{Map}}$ *is the restriction to* $\mathrm{Map}_{\mathrm{pw.ma.}}([0,T];X)$ *of the arclength BV-extension (see Recupero 2011, talk Recupero) of* $\mathcal{H}_G^C$.

For the hysteresis operator $\mathcal{A} : \mathrm{Map}([0,T];X) \to \mathrm{Map}([0,T];X)$ defined by

$$\mathcal{A}[u](t) := \begin{cases} 0_X, & \text{if} \quad \exists\, s \in [0,t] : u(s) = 0_X, \\ u(t), & \text{otherwise.} \end{cases}$$

it holds:

- The restriction of $\mathcal{A}$ to $\mathrm{C}_{\mathrm{pw.ma.}}([0,T];X)$ is equal to $\mathcal{H}_G^C$ with

$$G(v_0,\dots,v_n) = \begin{cases} v_n, \ \text{if}\ \forall\, i \in \{1,\dots,n\} : \\ \qquad\qquad (v_{i-1}, 0_X, v_i) \text{ is no convexity triple}, \\ 0_X, \ \text{otherwise}, \end{cases}$$

- but the restriction of $\mathcal{A}$ to $\mathrm{Map}_{\mathrm{pw.ma.}}([0,T];X)$ is not equal to $\mathcal{H}_G^{\mathrm{Map}}$.

Definition

Let $(x_a, y_a, r_a), (x_b, y_b, r_b), (x_c, y_c, r_c) \in X^2 \times \{0,1\}$ be given. $(x_a, y_a, r_a), (x_b, y_b, r_b), (x_c, y_c, r_c)$ is denoted as *convexity triple containing triple of elements of $X^2 \times \{0,1\}$* *i.e.* as *CTC triple*, if $x_b = y_b$ and (y_a, x_b, x_c) is a convexity triple.

Definition

a) Let

$$\begin{aligned} & S^{2,b}(X) \\ :=& \Big\{ \big((x_0, y_0, r_0), \ldots, (x_n, y_n, r_n)\big) \in \big(X^2 \times \{0,1\}\big)^{n+1} \ \Big| \\ & \quad n \in \mathbb{N},\, x_n = y_n, r_n = 1, \\ & \quad \forall\, i = 1, \ldots, n : (y_{i-1} = x_i \Longrightarrow r_{i-1} = 1) \Big\}. \end{aligned}$$

b) $\big((x_0, y_0, r_0), \ldots, (x_n, y_n, r_n)\big) \in S^{2b}(X)$ is denoted as *CTC triple free string* if $n = 1$ or if $n > 1$ and it holds for all $i \in \{1, \ldots, n-1\}$ that .

$$\big((x_{i-1}, y_{i-1}, r_{)}, (x_i, y_i), (x_{i+1}, y_{i+1})\big) \text{ is no CTC triple.}$$

Definition

Let some function $u : [0,T] \to X$ be given

Let some $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ be given.

u is *monotaffine on* $]t_1,t_2]$ $:\Longleftrightarrow$ $u(t_1+)$ exits and for $v : [0,T] \to X$ with $v(t) = u(t)$ for all $t \neq t_1$ $v(t_1) = u(t_1+)$ if holds that v is monotaffine on $[t_1,t_2]$

Definition

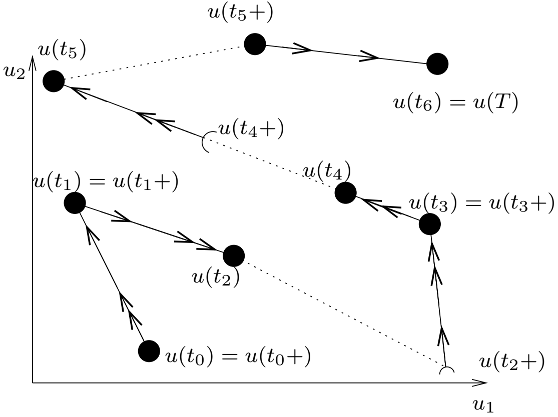
a) A function $u : [0,T] \to X$ is denoted as *piecewise left-open, right-closed monotaffine-continuous (pw. lo. rc. monotaffine-continuous)* if there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that u is monotaffine and continuous on $]t_i, t_{i+1}]$ for all $i = 0, \ldots, n-1$. In this case, the decomposition is denoted as *lo. rc. monotaffinicity continuity decomposition of* $[0,T]$ *for* u.

b) Let $\mathrm{Map}_{\mathrm{pw},*}([0,T], X)$ be the set of all pw. lo. rc. monotaffine-continuous functions from $[0,T]$ to X.

Definition

For $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ the *standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u* is the lo. rc. monotaffinicity continuity decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ for u, such that for all for all $i = 0,\ldots,n-1$, it holds

$$t_{i+1} = \max\left\{t \in]t_i,T] \,\middle|\, u \text{ is monotaffine and continuous on }]t_i,t]\right\}. \quad (1)$$


$u(t_5+)$
$u(t_5)$
u_2
$u(t_6) = u(T)$
$u(t_4+)$
$u(t_4)$
$u(t_1) = u(t_1+)$
$u(t_3) = u(t_3+)$
$u(t_2)$
$u(t_0) = u(t_0+)$
$u(t_2+)$
u_1

Let $F : S_F^{2,b}(X) \to Y$ be some function. The *hysteresis operator* $\mathcal{H}_F^* : \mathrm{Map}_{\mathrm{pw},*}\left([0,T],X\right) \to \mathrm{Map}\left([0,T], S_F^2(X)\right)$ *generated by F* is defined by mapping $u \in \mathrm{Map}_{\mathrm{pw},*}\left([0,T],X\right)$ to $\mathcal{H}_F^*[u] : [0,T] \to S_F^2(X)$ according to: Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u. Let $r_0, \dots, r_{n-1} \in \{0,1\}$ be defined by

$$r_i := \begin{cases} 1, & \text{if} \quad u(t_{i-1}+) \in u(]t_{i-1}, t_i[), \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, i \in \{1,\dots,n\}.$$

Let $\mathcal{H}_F^*[u] : [0,T] \to S_F^2(X)$ be defined by

$$\mathcal{H}_F^* := F\Big((u(0), u(0), 1),\ (u(0), u(0), 1)\Big),$$

$$\mathcal{H}_F^*[u](t) := F\Big((u(0), u(0+), r_0), (u(t), u(t), 1)\Big) \quad \forall\, t \in]t_0, t_1],$$

$$\mathcal{H}_F^*[u](t) := F\Big((u(t_0), u(t_0+), r_0), \dots, (u(t_{i-1}), u(t_{i-1}+), r_{i-1}),$$

$$(u(t), u(t), 1) \qquad \forall\, t \in]t_i, t_{i+1}],\ i \in \{2,\dots,n\}.$$

Theorem

a) *Let* $F : S_F^{2,b}(X) \to Y$ *be some function. Then it follows that the operator* $\mathcal{H}_F^* : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],Y)$ *is a hysteresis operator.*

b) *For every hysteresis operator* $\mathcal{B} : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],Y)$ *there exists a unique function* $F : S_F^{2,b}(X) \to Y$ *such that* $\mathcal{B} = F \circ \rho$*.*

- For the hysteresis operator $\mathcal{A} : \mathrm{Map}([0,T];X) \to \mathrm{Map}([0,T];X)$ defined by

$$\mathcal{A}[u](t) := \begin{cases} 0_X, & \text{if} \quad \exists\, s \in [0,t] : u(s) = 0_X, \\ u(t), & \text{otherwise.} \end{cases}$$

it holds:

- The restriction of $\mathcal{A}$ to $\mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ is equal to $\mathcal{H}^*_G$ with

$$G((x_0,y_0,r_0)\ldots,(x_n,y_n,r_n)\ldots,$$
$$= \begin{cases} 0_X, \text{ if } \exists\, i \in \{1,\ldots,n\} : \\ \qquad\qquad y_{i-1} \neq 0_X, \quad (y_{i-1}, 0_X, x_i) \text{ is a convexity triple,} \\ 0_X, \text{ if } \exists\, i \in \{1,\ldots,n\} : \\ \qquad\qquad y_{i-1} = 0_X, r_i = 1, \\ x_n, \text{ otherwise,} \end{cases}$$

Many thanks for your attention