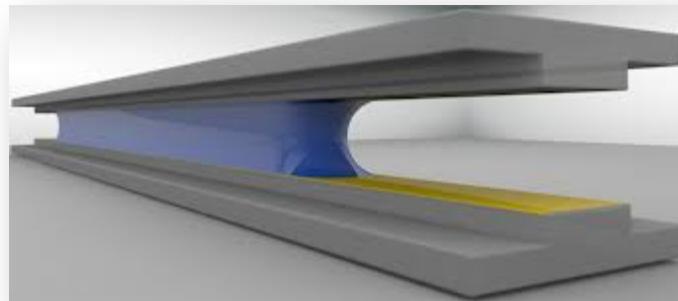


# Investigation of losses due to hysteresis in capillary effect



Dr. Ram Iyer  
Department of Mathematics and Statistics  
Texas Tech University

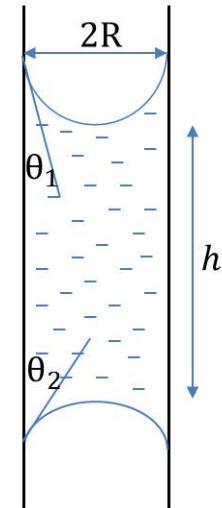
# Capillary Action

## Capillary Action or Capillarity

is the ability of a liquid to flow in narrow spaces without the assistance of, and in opposition to, external forces like gravity.

Three effects contribute to capillary action:

- adhesion of the liquid to the walls of the confining solid (or solids),
- meniscus formation,
- low Reynolds number fluid flow.

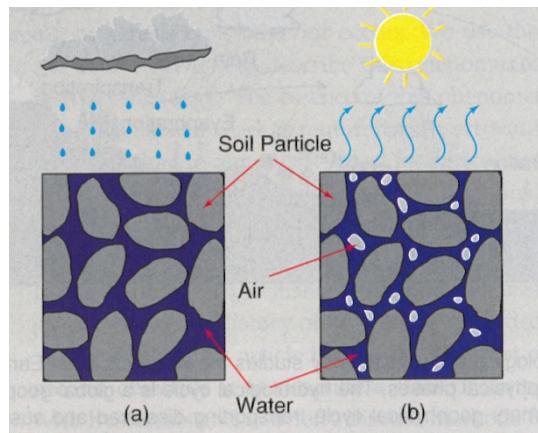


Straw with water

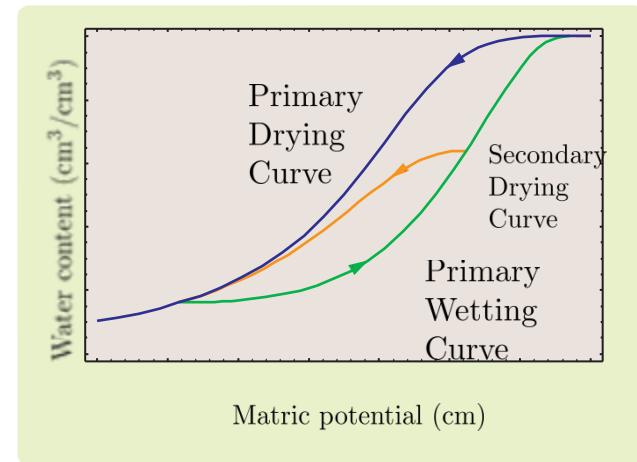


Capillarity plays a major role in soil science, plant biology, contact lens design, and chemical industry.

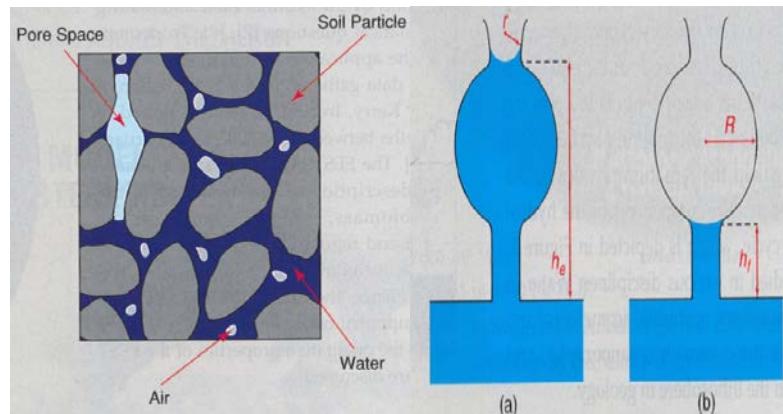
# Capillary action in Terrestrial Hydrology



Wetting and Drying of soil



Soil-water wetting and drying characteristic



Soil consists of Capillary tubes of Various sizes

Rate-Independent Hysteresis in Terrestrial Hydrology - Appelbe, Pokrovskii, Rachinskii et al., IEEE Control Systems Magazine, 2009

# Capillary Action in Contact Lens - Eye system

- \* A contact lens is subjected to forces from both the tear film as well as those of an eye blink
- \* These forces cause rotational and translational lens movement
- \* Position of the contact lens may have different location on the cornea that affects the visual acuity



A patient wearing a contact lens

# Motivation

A lens with **good rotational and translational stability** are necessary to provide good vision correction after blinks and fast eye movement.

The energy losses due to competing mechanisms in capillarity are not well understood.

## Motivation:

We started studying the capillarity with a view to identify the major mechanism providing dissipation of energy.

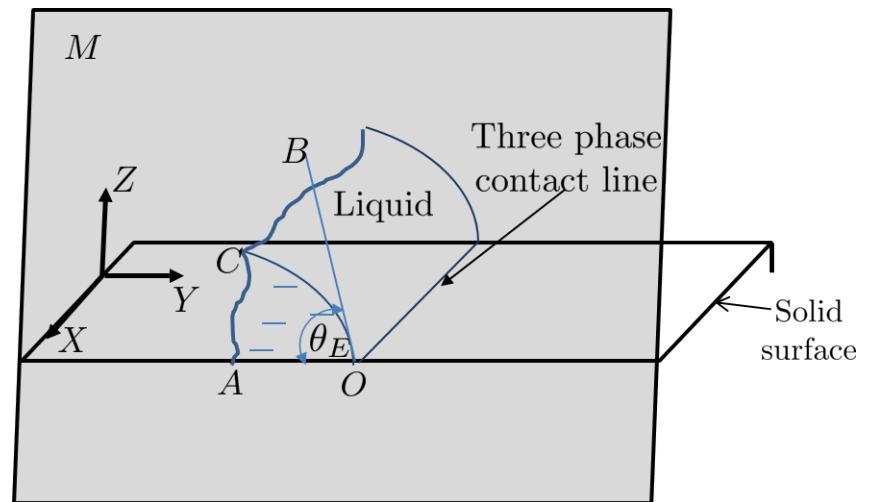


A patient wearing a contact lens

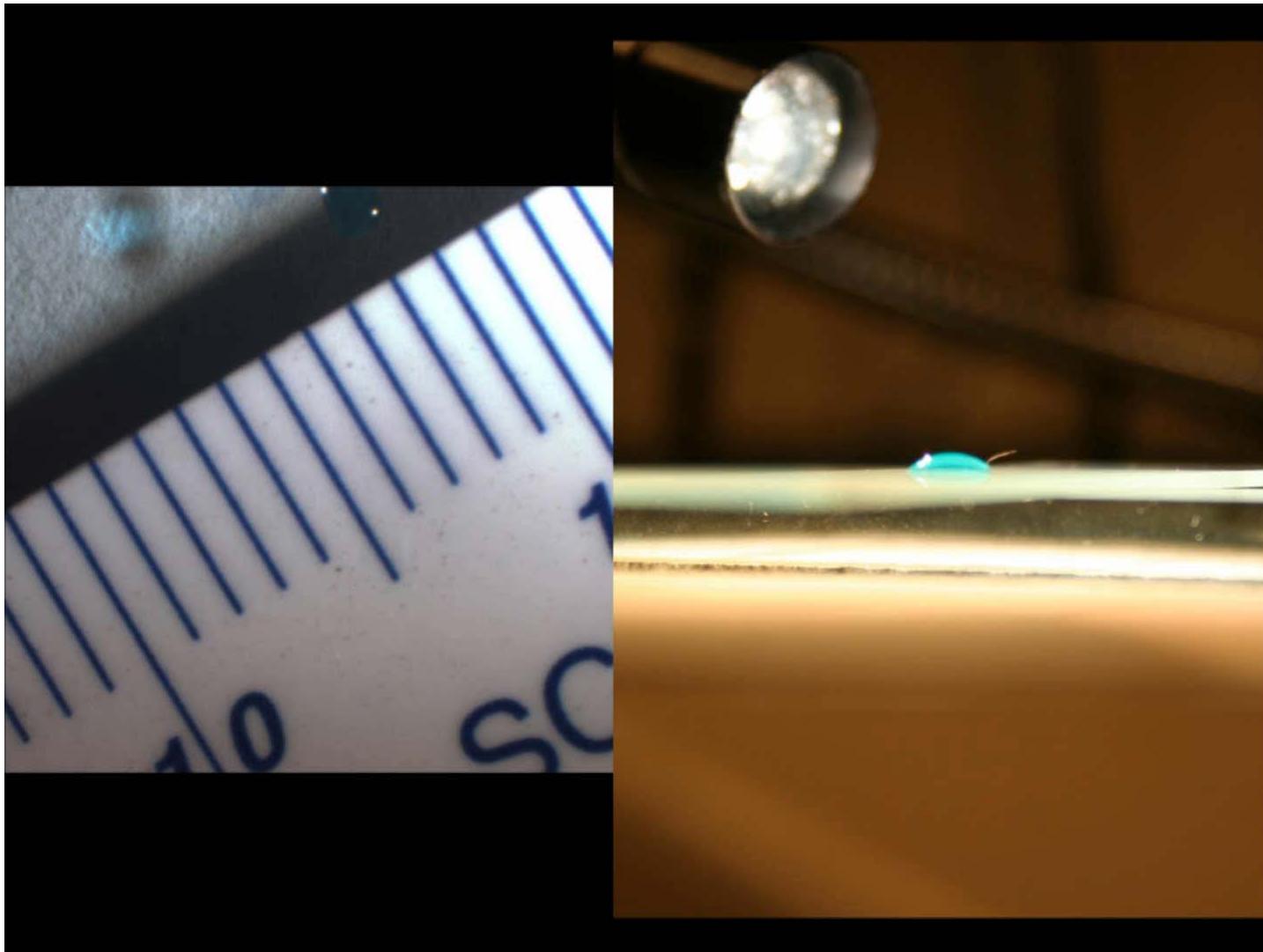
# Energy dissipation

(1) - viscosity of the fluid

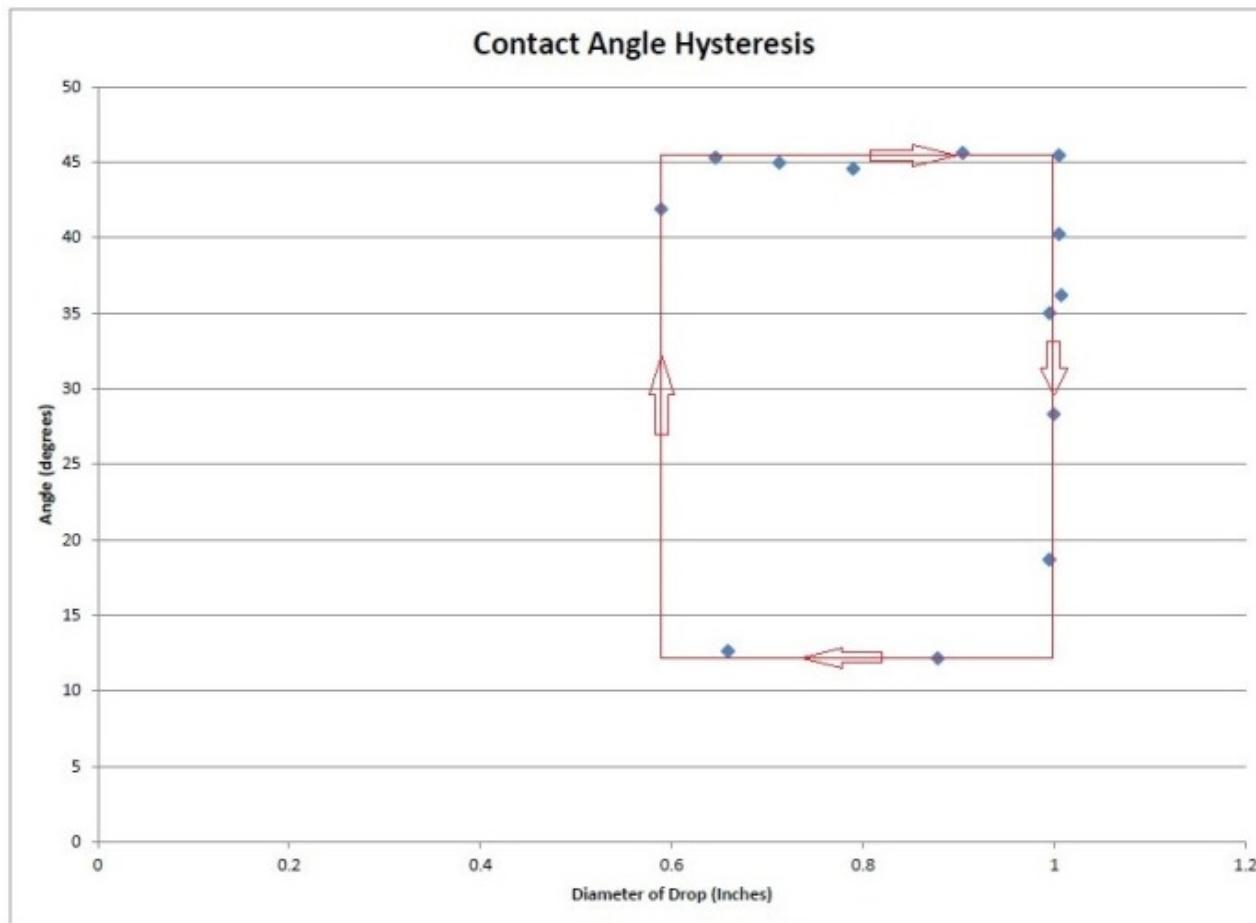
(2) - motion of the contact line



# Contact angle hysteresis



# Contact angle hysteresis



D - Diameter of the contact circle

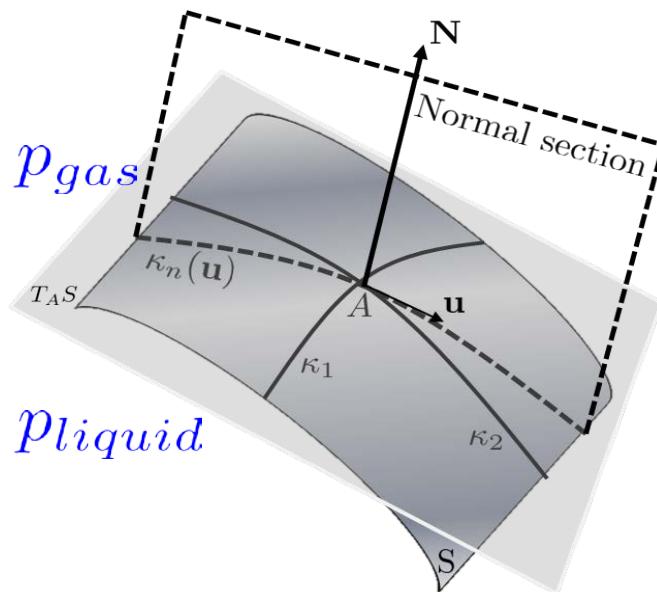
# Capillary surfaces

Capillary surfaces are interfaces that occur when one material is a liquid and the other is a non-miscible liquid or a gas.



# Model for the capillary surface

\* Young-Laplace equation



$$\delta p = \gamma (\kappa_1 + \kappa_2) = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = 2\gamma H$$

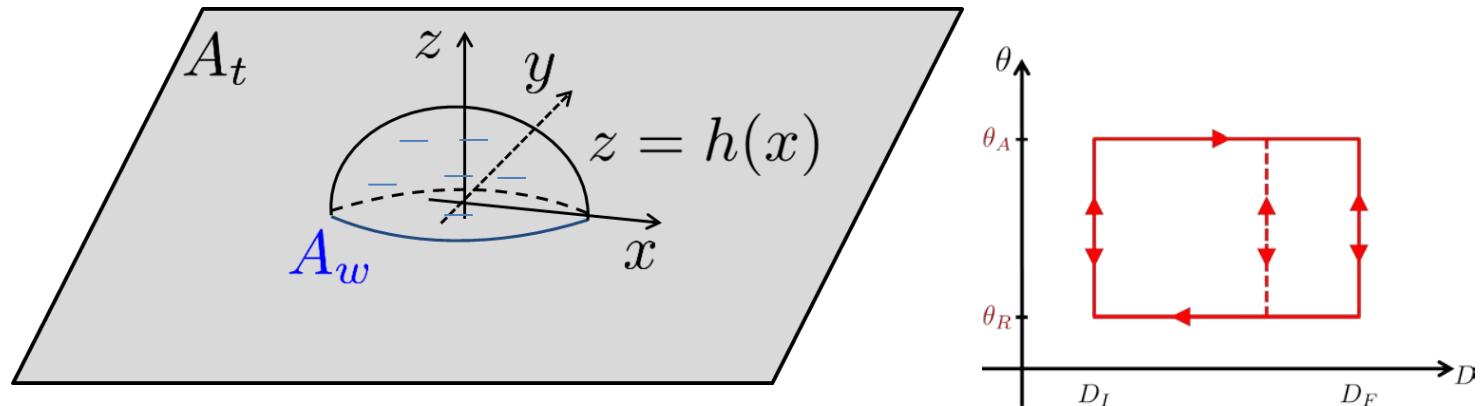
# Model for the capillary surface

## \* Calculus of variations approach

Minimize the **total potential energy** associated with the liquid bridge with a volume constraint

- Surface energy  $\mathcal{E}_s = \int_S \gamma dS$
- Wetting energy  $\mathcal{E}_w = \int_{\partial S} \gamma \beta_i dS$
- Potential energy due to gravity  $\mathcal{E}_g = \int_{\Omega} \rho g \mathbf{z} dV$

# Meniscus profile of a stationary droplet



$$\mathcal{E}_{tot} = (A_t - A_w)\gamma_{SG} + A_w\gamma_{SL} + \gamma_{LG} \int_0^R 2\pi x \sqrt{1 + h'^2(x)} dx - \bar{p} \int_0^R 2\pi x h(x) dx$$

$A_t$ - total surface area

$A_w$ - wetted area

$R$ - droplet radius

$$\bar{p} := p_{liq} - p_{atm}$$

# First variation:

$$\delta\mathcal{E}_{tot} = [\gamma_{SL} - \gamma_{SG} + \gamma_{LG}\sqrt{1+h'^2(R)}] R \delta R + \gamma_{LG} \left( \frac{h'(R)}{\sqrt{1+h'^2(R)}} \right) R \delta h(R) -$$

$$\int_0^R \left( \gamma_{LG} \left( \frac{xh'(x)}{\sqrt{1+h'^2(x)}} \right)' + \bar{p} x \right) \delta h dx = 0.$$

End-point condition  $h(R) = 0$  yields:  $\delta h(R) + h'(R) \delta R = 0$

$$\frac{\bar{p}}{\gamma_{LG}} = \frac{1}{x} \left( \frac{-xh'}{\sqrt{1+h'^2}} \right)' = \frac{1}{x} \frac{h'}{\sqrt{1+h'^2}} + \frac{h''}{\sqrt{(1+h'^2)^3}} \quad \forall x \in [0, R]$$

$$\frac{\gamma_{SG} - \gamma_{SL}}{\gamma_{LG}} = \cos \theta \quad x = R$$

$\theta$ - contact angle at  $x = R$

# Experimental data

Contact angle ( $^{\circ}$ )	$h_{act}(0) \times 10^{-3}(m)$	$R \times 10^{-2}(m)$	$\bar{p} (Nm^{-2})$
41.9	3.1	1.50	3.86
45.3	2.5	1.64	2.65
45.0	3.3	1.81	2.85
44.6	3.9	2.01	2.71
45.6	4.1	2.30	2.19
45.4	4.0	2.55	1.76
40.3	4.0	2.55	1.75
36.2	3.6	2.55	1.59
28.3	3.6	2.54	1.60
18.7	2.4	2.53	1.09
12.2	1.5	2.23	0.88
12.6	1.2	1.67	1.25
41.9	3.1	1.50	3.86

# Numerical Results

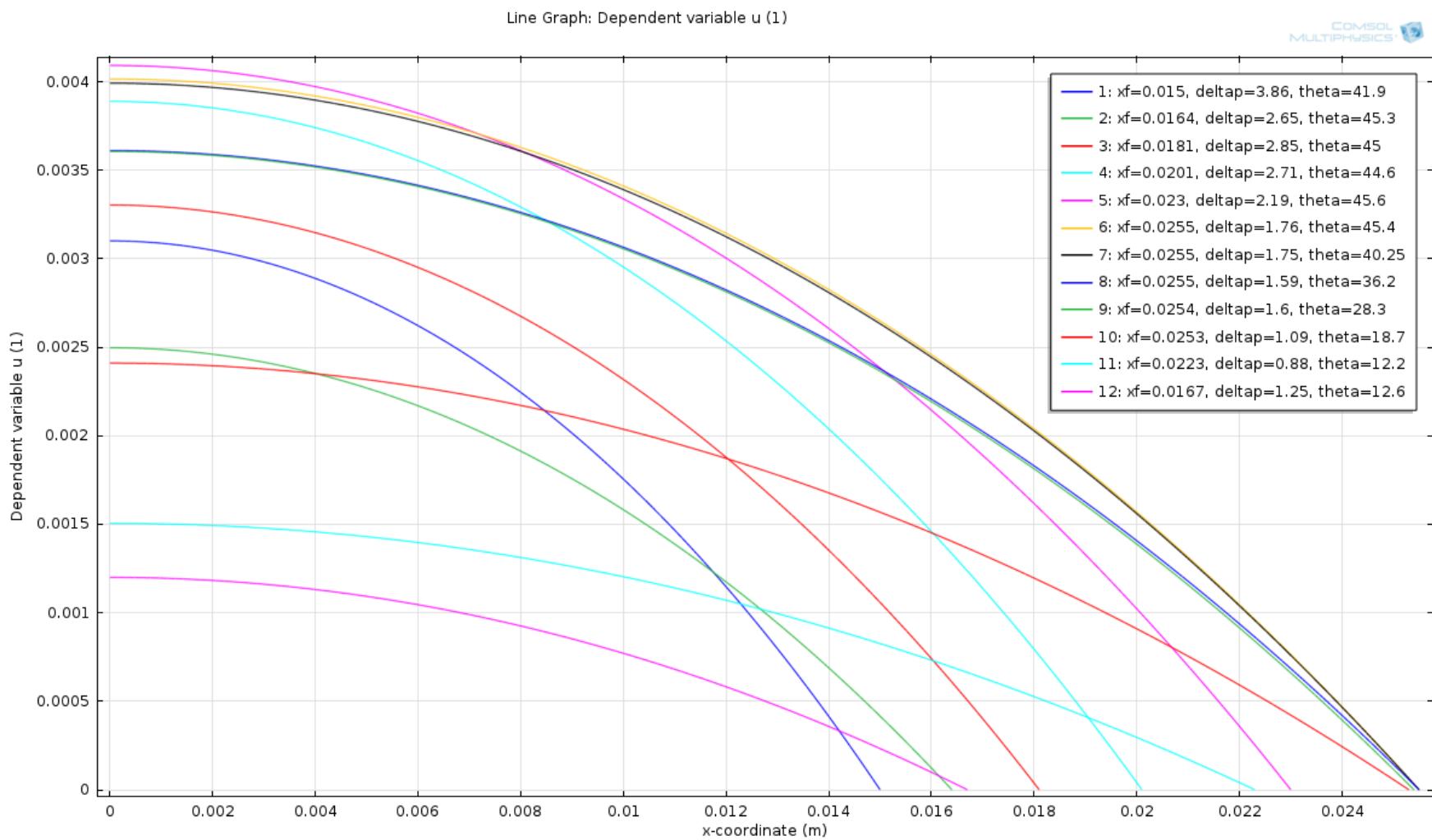
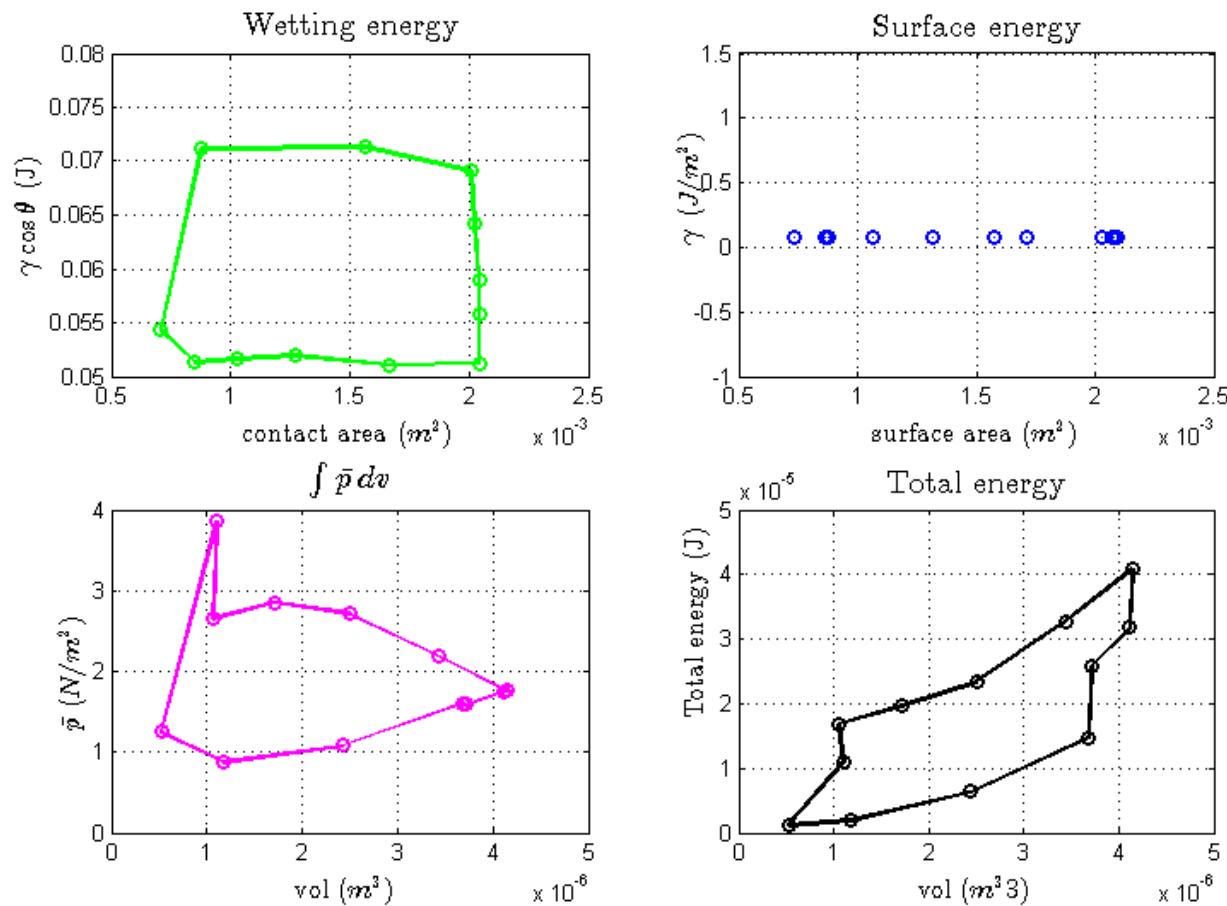


Fig: Meniscus profiles

# Numerical Results



Energy loss  $\approx 26 \mu J + 3.5 \mu J = [29.5 \mu J]$

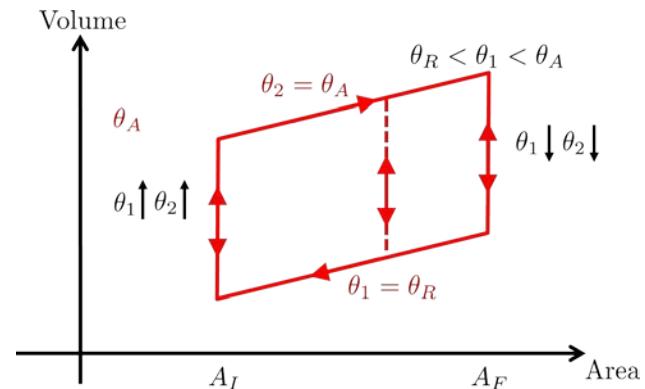
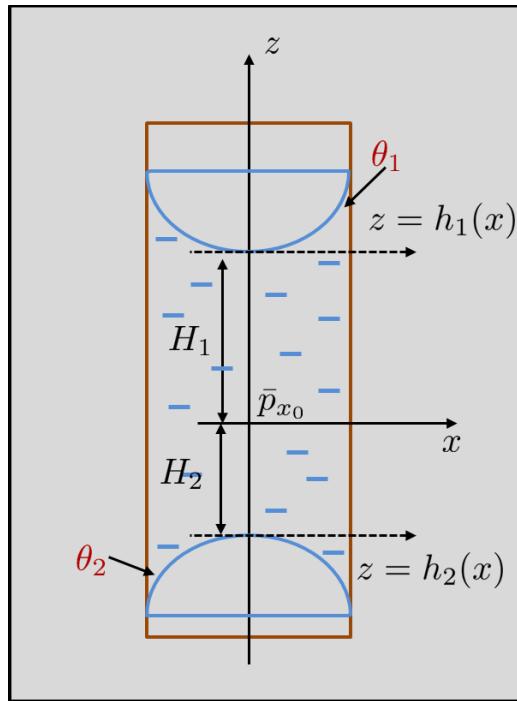
Wet. en + Vol. en

$$\begin{aligned}
\mathcal{E}_{tot} &= -A_w \gamma_{LG} \cos \theta + \gamma_{LG} \int_0^R 2\pi x \sqrt{1 + h'^2(x)} dx - \bar{p} \int_0^R 2\pi x h(x) dx \\
&= \text{Wetting energy} + \text{Surface energy} - \text{Volume energy}
\end{aligned}$$

### Conclusions for drop on plate:

- Excellent match between data from experiment and predictions from model observed.
- Loss due to adhesion/wetting energy term are significantly greater than loss due to volume energy term.

# Contact angle hysteresis in a capillary column



$$\begin{aligned} \mathcal{E}_{tot} = & A_t \gamma_{SG} + 2\pi x_f \int_{H_2+h_2(x_f)}^{H_1+h_1(x_f)} (\gamma_{SL}(z) - \gamma_{SG}) dz + \gamma_{LG} \int_{x_0}^{x_f} 2\pi x \sqrt{1 + h_1'^2(x)} dx + \gamma_{LG} \int_{x_0}^{x_f} 2\pi x \sqrt{1 + h_2'^2(x)} dx - \\ & \int_{x_o}^{x_f} \int_{H_2+h_2(x)}^{H_1+h_1(x)} 2\pi x (\bar{p}(x_0) - \rho g z) dz dx \end{aligned}$$

# First variation:

★  $\delta h_1(x)$

$$-\gamma_{LG} \left( \frac{xh'_1(x)}{\sqrt{1 + h'^2_1(x)}} \right)' - (\bar{p}_{x_0})x + \rho g x(h_1(x) + H_1) = 0$$

★  $\delta H_1$

$$H_1 = \frac{2\gamma_{LG} \cos \theta_1 x_f}{\alpha} + \frac{\bar{p}_{x_0} x_f^2}{\alpha} - \frac{2\rho g}{\alpha} \int_{x_o}^{x_f} x h_1(x) dx$$

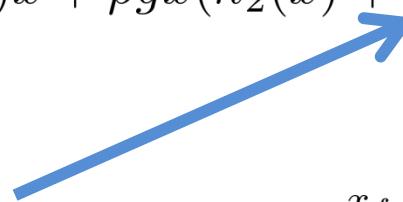
★  $\delta h_1(x_f)$

$$\frac{\gamma_{SG} - \gamma_{SL}(H_1 + h_1(x_f))}{\gamma_{LG}} = \cos \theta_1$$

where  $\alpha = \rho g x_f^2$

# First variation:

★  $\delta h_2(x)$

$$-\gamma_{LG} \left( \frac{xh'_2(x)}{\sqrt{1 + h'^2_2(x)}} \right)' - (\bar{p}_{x_0})x + \rho g x(h_2(x) + H_2) = 0$$


★  $\delta H_2$

$$H_2 = \frac{2\gamma_{LG} \cos \theta_2 x_f}{\alpha} + \frac{\bar{p}_{x_0} x_f^2}{\alpha} - \frac{2\rho g}{\alpha} \int_{x_o}^{x_f} x h_2(x) dx$$

★  $\delta h_2(x_f)$

$$\frac{\gamma_{SG} - \gamma_{SL}(H_2 + h_2(x_f))}{\gamma_{LG}} = \cos \theta_2$$

where  $\alpha = \rho g x_f^2$

# First variation:

Observe that  $\delta H_1 + \delta H_2$  yields

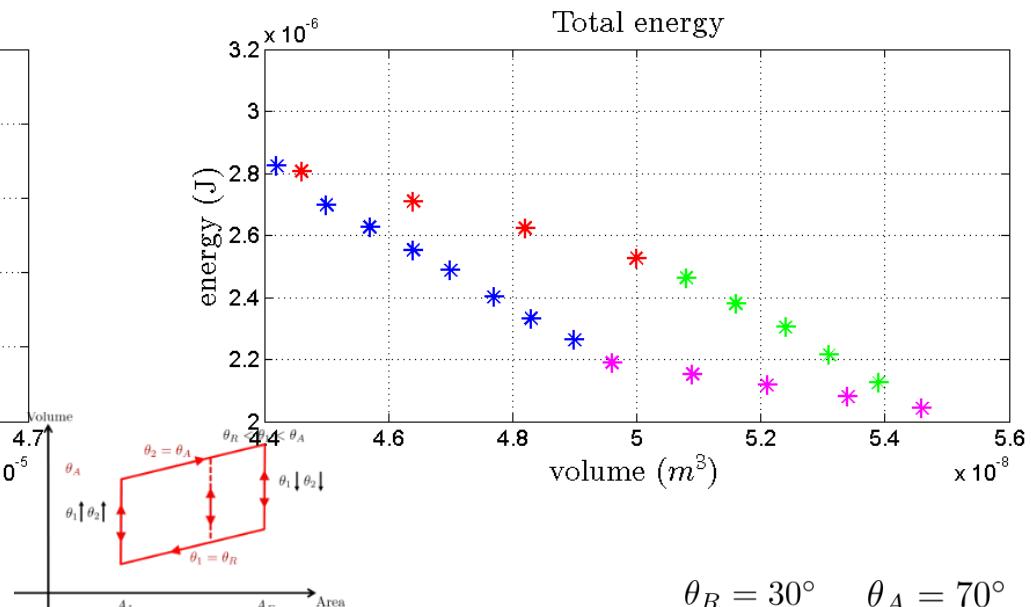
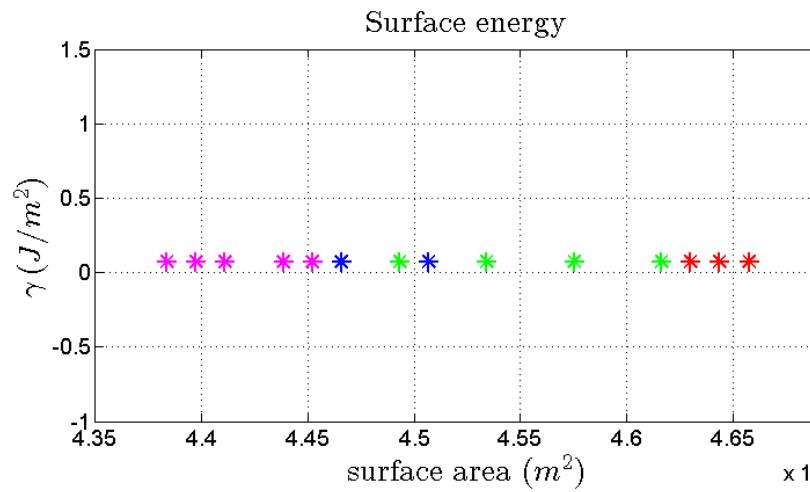
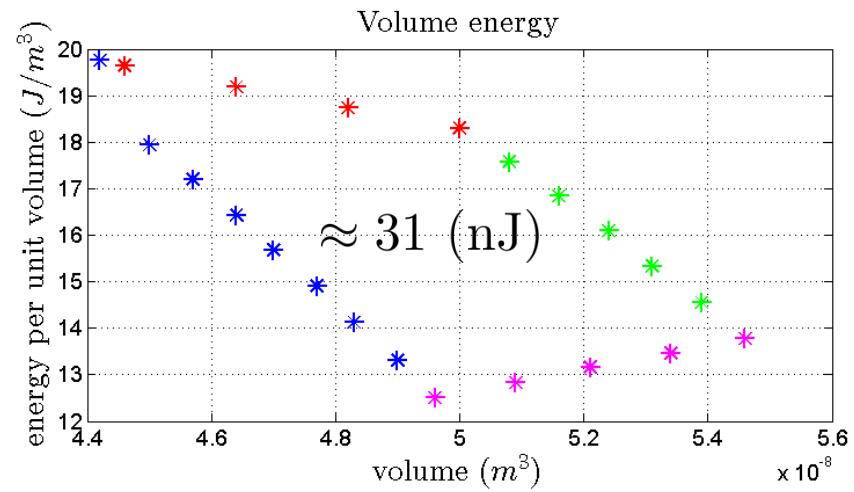
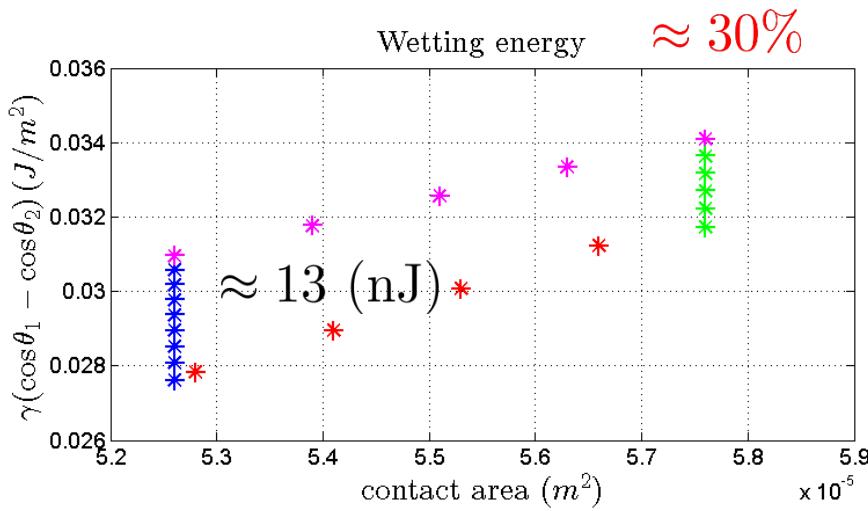
$$\gamma_{LG}(-\cos \theta_1 + \cos \theta_2)x_f + \rho g \int_{x_0}^{x_f} x(h_1(x) + H_1 - h_2(x) - H_2) dx = 0$$


Force balance: Weight of liquid = Adhesion force

Numerical computations:

- \* Boundary conditions are of **Dirichlet or Neumann** type depending on the case considered.
- \* The coupled PDEs together with constraints and boundary conditions are solved using COMSOL.

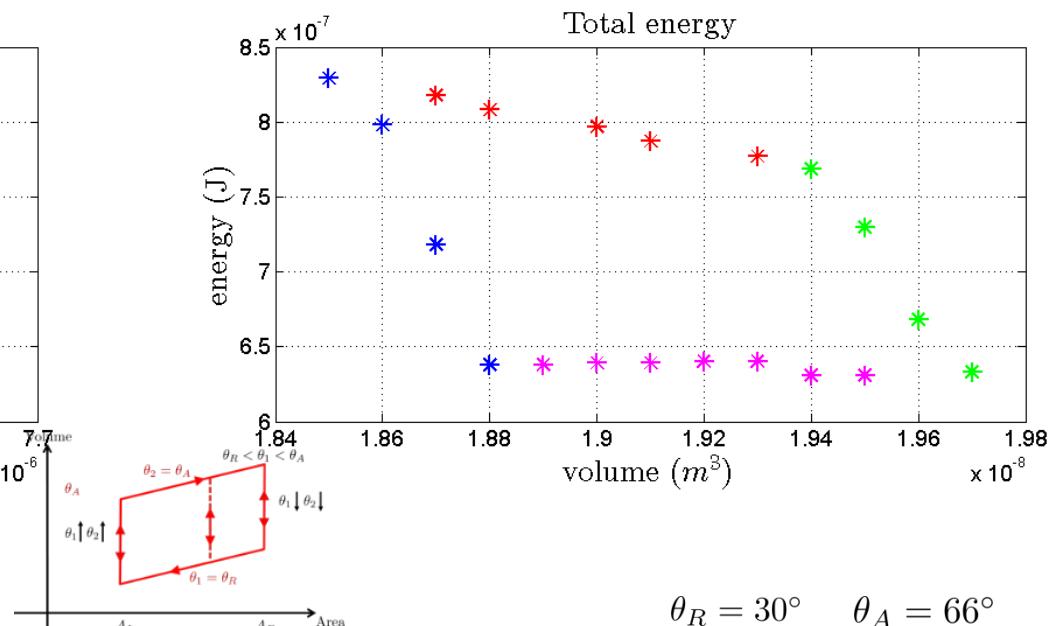
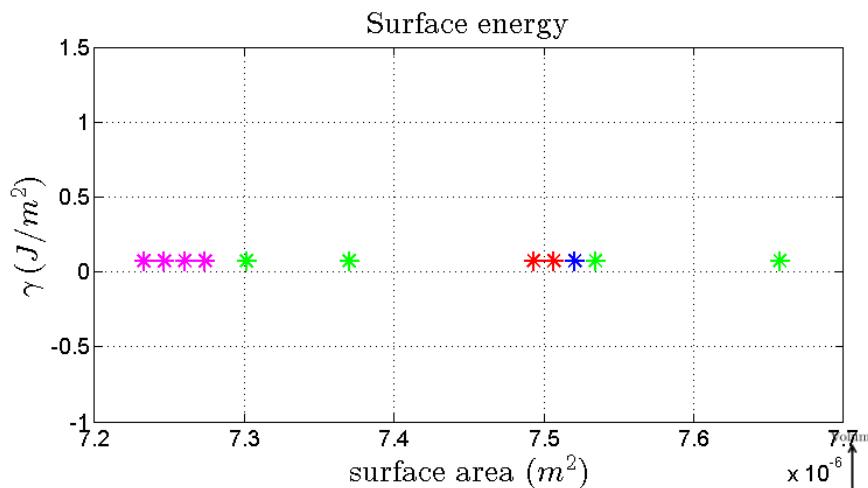
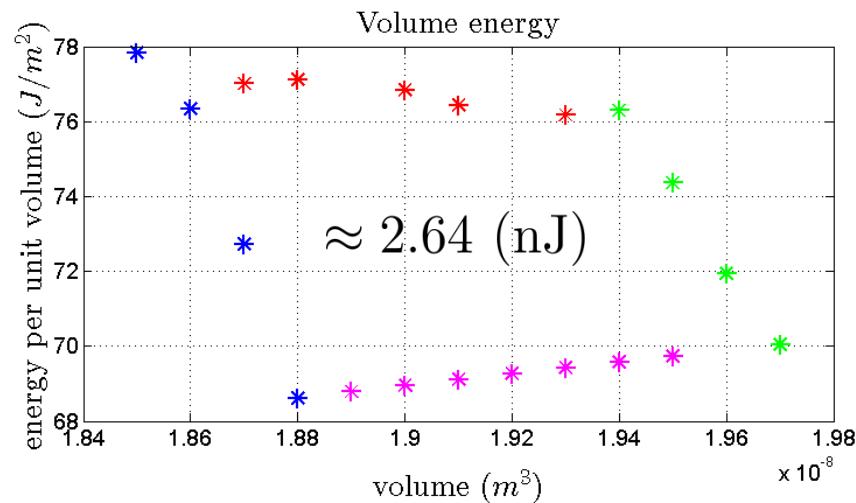
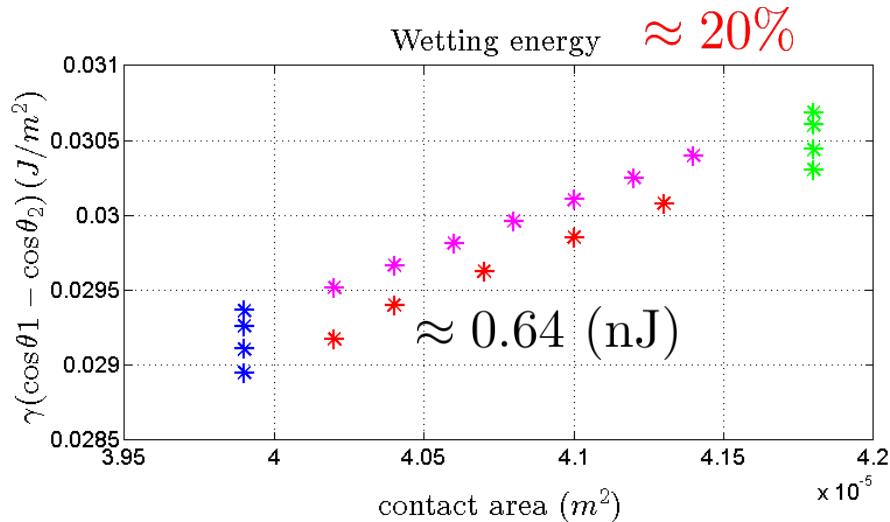
# Numerical Results



Capillary tube diameter = 5 mm

$$\theta_R = 30^\circ \quad \theta_A = 70^\circ$$

# Numerical Results



Capillary tube diameter = 1 mm

# Conclusions

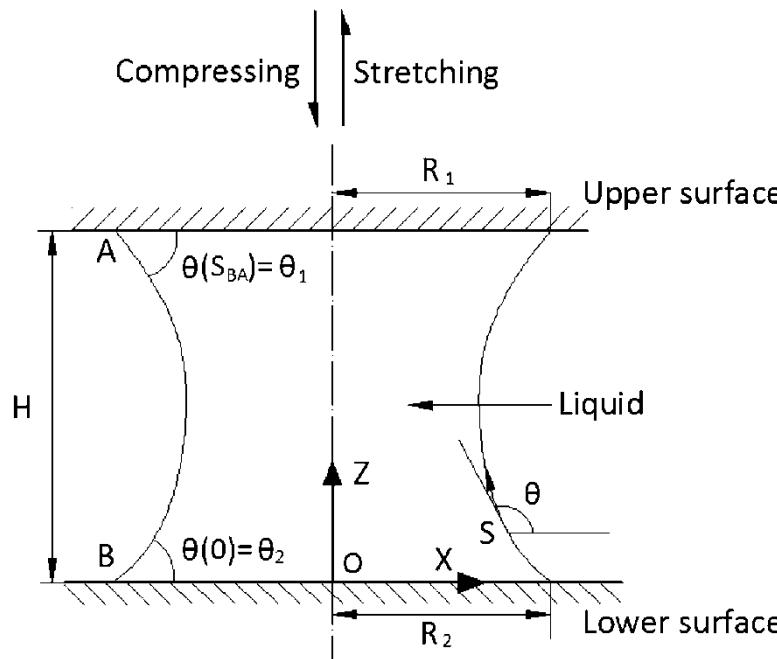
- Hysteresis energy loss seems to be significantly larger (factor of 100) than energy loss due to viscosity for comparable dimensions(Athukorallage and Iyer, HMM 2013)
- **Consequence:**
  - Even though contact line motion is very small (0.025 mm in 5 mm tube and 0.15 mm in 1 mm tube), modeling it is important.
  - For a RGP contact lens of 9 mm diameter, edge thickness is  $\sim 0.25$  mm, and tear size  $\sim 1.5$  mm. Modeling line motion is very important in this application.
  - Fluid flow could be modeled using Darcy equation instead of Navier-Stokes or Stokes equation.
- **Unexpected result:** The hysteresis loss due to wetting decreases with diameter of capillary tube as a percentage of total hysteresis losses.

# Prior work - Contact lens motion

- Hayashi, T.T., (1977), Mechanics of Contact Lens Motion:
  - neglects the hydrostatic pressure variation due to gravity and the capillary effect.
- Moriarty, J.A., Terrill, E.L, (1996), Mathematical Modeling of the Motion of Hard Contact Lenses:
  - gravitational force acting on the fluid is neglected in the analysis.
- Chauhan, A. and Radke, C. (2002). Settling and deformation of a thin elastic shell on a thin fluid layer lying on a solid surface:
  - assume the liquid pressure at the lens edge to be atmospheric pressure.

# Prior work - Liquid bridge between two plates

- Chen, Amirfazli and Tang (2013):  
Modeling liquid bridge between surfaces with CAH



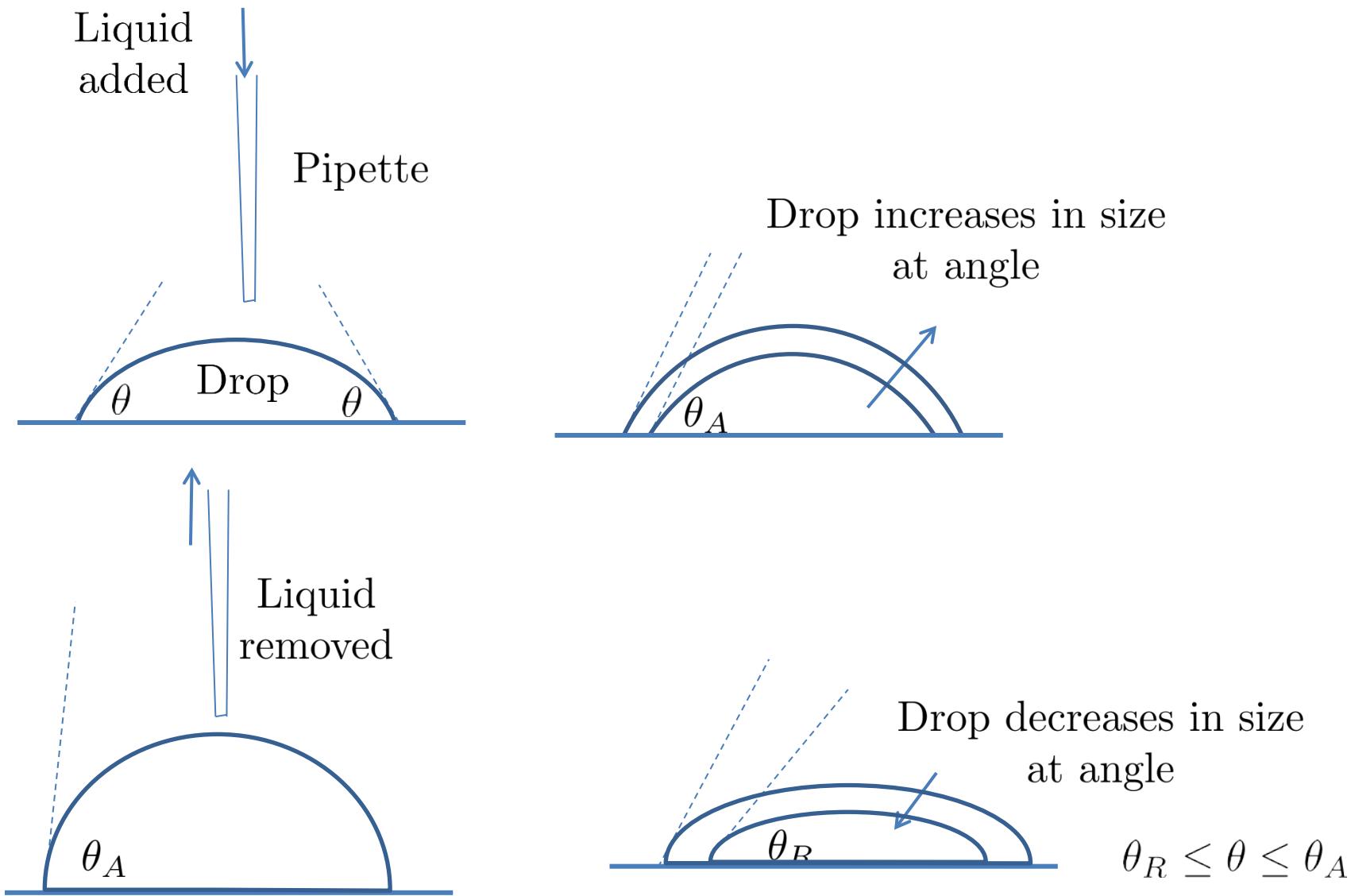
- CAH results in an energy dissipation during a stretching and compressing cycle of the surfaces.

# Prior work - Liquid bridge between two plates

- T.I. Vogel (1987):  
Stability of a liquid drop trapped between two parallel planes  
- neglects the potential energy due to gravity
- Ichikawa, Hosokawa, and Maeda (2004):  
Interface motion of capillary-driven flow in rectangular microchannel

Thank you

# Contact angle hysteresis



# Contact angle hysteresis

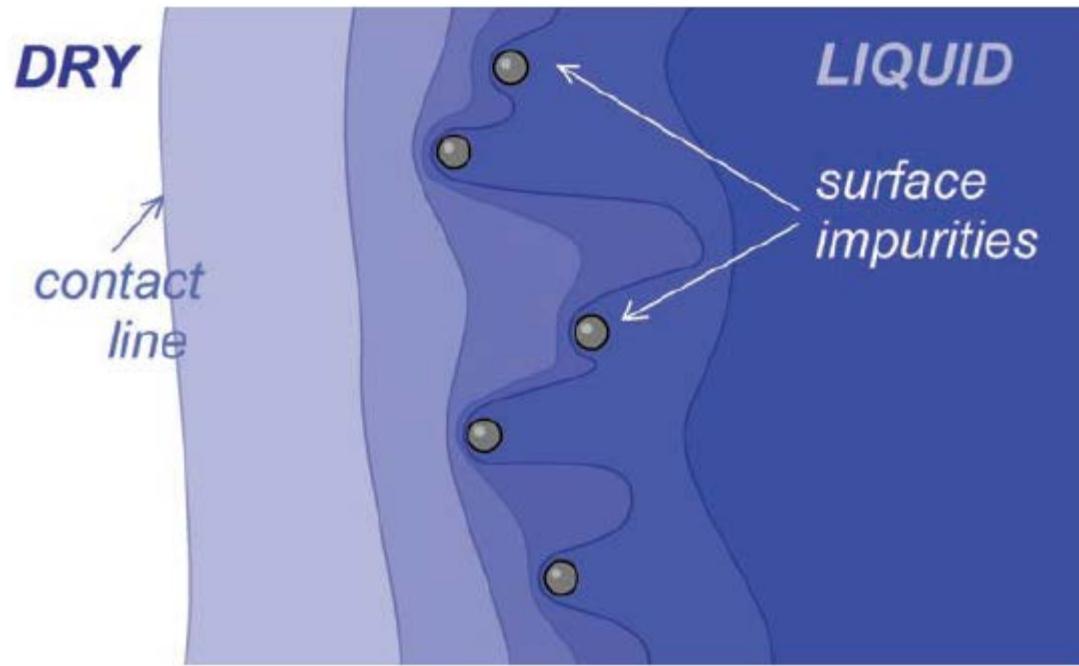


Fig: Pinning of a contact line

image: MIT OCW 18.357 Interfacial Phenomena John W. M. Bush