Geometry of Heteroclinic Cascades in Scalar Parabolic Differential Equations

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Abstract

We investigate the geometrical properties of the attractor for semilinear scalar parabolic PDEs on a bounded interval with Neumann boundary conditions. Using the nodal properties of the stationary solutions which are determined by an ordinary boundary value problem, we obtain crucial information about the long-time behavior for the full PDE. Especially, we prove a criterion for the intersection of strong- stable and unstable manifolds in the finite dimensional Morse-Smale flow on the attractor.

Contents

Introduction

The one-dimensional reaction-diffusion equation

$$u_t = u_{xx} + f(u, u_x, x), \quad x \in [0,1],\ t \in \mathbf{R}^+$$

with Neumann boundary conditions is one of the best understood examples for an infinite dimensional dynamical system. Due to a lot of special properties, such as the gradient-structure or the linearization being of Sturm-Liouville-Type, one can not only prove under certain conditions the existence of a global attractor, but can even give a geometrical description of the flow on this finite dimensional attractor. The main tool for such investigations, beside the general geometric theory for semilinear parabolic equations introduced by D.Henry [Hen81], are the nodal properties of the stationary solutions. It was recognized by G.Fusco and C.Rocha that one can describe these nodal properties in terms of a permutation of the equilibrium solutions [FuRo91]. An astonishing fact is that this permutation which is determined already by the stationary equation, an ordinary differential equation, can provide crucial information about the long-time behavior of the full PDE.

The attractor consists of the unstable manifolds of all equilibria. These are partitioned into sets of heteroclinic solutions, connecting to the same equilibrium. The main problem in the geometrical description is to understand the structure of these sets and how they are assembled in the attractor. A first important step in this direction was a result of C.Rocha and B.Fiedler [FR96], which gives a criterion for these sets to be empty or not. But as an inspection of several examples shows, their result is not sufficient to provide a satisfactory geometrical description of the attractor.

In this paper we first give an overview over earlier results which are important for a geometrical description of the attractor. Then, we prove a result, giving more insight in the structure of the sets of connecting orbits by taking into account the decomposition of stable and unstable manifolds according to different exponential rates. We give a criterion in terms of the nodal properties of the stationary solutions, deciding exactly in which strong-stable and strong-unstable manifold the heteroclinic connections between given equilibria take place. This information seems also to be crucial for the understanding of the attractor as a whole. The main ingredient of the proof is a detailed study of the permutation of equilibria.

Finally we discuss the consequences of our results in some examples where the shape of the attractor can be shown constructively by a method of G.Fusco and C.Rocha [FuRo91]. Moreover, we suggest a new concept for the equivalence of attractors.

1 General properties of the semiflow and existence of a global attractor

Global existence and dissipativity

The main subject of our study are semilinear parabolic equations of the form

$$u_t = u_{xx} + f(x, u, u_x), \qquad 0 < x < 1, \quad f \in C^2 \tag{1}$$

with Neumann boundary conditions

$$u_x(0, t) = u_x(1, t) = 0.$$

Equations of this type generate on the Hilbert-space X of x-profiles in $H^2([0, 1])$, satisfying the boundary conditions, a local C^1-semiflow (see [Hen81])

$$(t, u_0) \mapsto u(t) = u(t, \ .\ ) \in X.$$

Assuming additional conditions on f, as for example

$$f(x, u, 0) \cdot u < 0$$

for large $|u|$, and

$$\partial_x f(x, u, v) + \partial_u f(x, u, v) \cdot v \leq 0$$

for large $|v|$, the semiflow is point-dissipative in the sense of [BV89]. That means, there is a ball in X, absorbing any trajectory $u(t)$ for some $t > t_0(u_0)$. This yields of course global existence and, in addition, the existence of a global attractor with finite Hausdorff-dimension. The global attractor is the maximal compact invariant set, is connected, and attracts all bounded sets. It consists of all trajectories which are defined also for all negative times and which are uniformly bounded.

These results can be shown by general theorems, applying also to a variety of much more complicate equations (see [BV89]). For our equation however, there are some additional properties allowing moreover for a more detailed description of the attractor.

Gradient structure

A fundamental property of the equation was already recognized by Zelenyak in [Zel68]: There exists a Lyapunov-functional

$$V(u) = \int_0^1 g(x, u, u_x) dx,$$

decaying strictly along non-constant solutions. This implies that any α- or ω-limit set of a trajectory in the attractor consists of a single stationary solution. Thus the attractor A consists of the set of equilibria E and of heteroclinic orbits, connecting two equilibria.

Due to this result, the description of the attractor can be looked at as a problem of calculus of variation and one can apply for instance Morse-theory.

The linearized equation and Matano's principle

We define on X the linear operator

$$L(t)u := a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u$$

with coefficients $a, b, c \in C^1$. First, we look at the eigenvalue problem for the linearization of (1) at an equilibrium $v \in E$:

$$\lambda u = L_v u := u_{xx} + f_p(x, v(x), v_x(x))u_x + f_u(x, v(x), v_x(x))u$$

Obviously, L_v does not depend on t. This equation has the form of a Sturm-Liouville problem and hence the following properties (see e.g. [Har64]):

- All eigenvalues are real and algebraically simple.
- Only finitely many eigenvalues are positive.

$$\lambda_0 > \lambda_1 > \dots \lambda_i \geq 0 > \lambda_{i+1} > \dots$$

- The eigenfunction $\phi_n(v)$, corresponding to the n-th eigenvalue $\lambda_n(v)$ has exactly n zeroes.

For simplicity, we make now a further general assumption: Let all equilibria $v \in E$ be hyperbolic, i.e there is no eigenvalue $\lambda = 0$. Although main parts of the theory can also be carried out in the non-hyperbolic case (see e.g. [Hen85]), we make this generic assumption on f to avoid technicalities.

We call the finite number $i(v)$ of positive eigenvalues of the linearization at v Morse-index. It gives the dimension of the unstable manifold at v.

Now, we consider a linear equation of the form

$$w_t = L(t)w$$

with Neumann boundary conditions. Equations of this type arise as variational equations along solutions of the nonlinear equation (1). But simple calculations show that also the difference of any two solutions of the nonlinear equation (1) satisfies an equation of this form.

For solutions $w(t) \neq 0$ of this equation, we have as a very important tool for our investigations [Mat82]:

Matano's principle: *For any $u(\,.\,) \in X$, let $z(u)$ be the number of strict sign changes of the x-profile $u(\,.\,)$. Then we have*

- *$z(w(t))$ is finite for all $t > 0$*
- *$z(w(t))$ drops strictly at each multiple zero $w(t_0, x_0) = w_x(t_0, x_0)$.*

Note that the zero-number z is well defined on X, since $X \subset H^2$ is embedded into C^1. Since it can be applied to tangent vectors, as well as to differences of solutions, this theorem governs the local and global behaviour in the same way. Therefore, together with the zero-number property of the Sturm-Liouville eigenfunctions, it plays the central role in all the theorems, concerning the special structure of the attractor. However, this shows that we cannot expect to obtain in the same way similar results for higher space dimensions $x \in \Omega \subset \mathbf{R}^n$.

2 Permutations of equilibria and meandric curves

As we will see, the Morse-index $i(v)$ and the zero-numbers $z(v-w)$ for equilibria $v, w \in E$ play the central role in the description of the flow on the attractor. We will show now an easy method to compute them for all equilibria $v, w \in E$, considering only the stationary equation

$$u_{xx} + f(u, u_x, x) = 0$$

for $x \in [0,1]$. If we solve this ordinary differential equation with Neumann boundary conditions, we get the set of equilibria $E = \{v_1, \ldots, v_N\}$. In addition these equilibria are ordered according to their value at both boundaries. Therefore, if

$$v_1(0) < v_2(0) < \cdots < v_{N-1}(0) < v_N(0),$$

we can define a permutation according to the values at $x = 1$:

$$v_{\pi(1)}(1) < v_{\pi(2)}(1) < \cdots < v_{\pi(N-1)}(1) < v_{\pi(N)}(1)$$

This permutation π was introduced in [FuRo91] and extensively used in [FR96a]. An important feature of this permutation can be seen as follows: Consider all trajectories $u(x),\ x \in [0,1]$, which satisfy the first boundary condition $u_x(0) = 0$. This is a one-parametric family of curves, parametrized by $u(0) = \alpha$ and forms a smooth surface $S(x, \alpha)$ in the extended phase space (u, u_x, x) (see figure 1).

The intersection points of the curve $S(1, \alpha) := \gamma(\alpha)$ with the straight line $\{x = 1, u_x = 0\}$ lie on trajectories satisfying also the boundary condition at $x = 1$. The order of these solutions along the curve $\gamma(\alpha)$ is the same as along the line $\{x = 0,\ u_x = 0\}$. Thus, the permutation π of the equilibria is determined only by $\gamma(\alpha)$. This curve has no self-intersections and hence the permutation π is a so called planar or meandric permutation. These permutations were first described by V.I. Arnol'd in [Arn88]. They give rise to a lot of interesting questions and have been studied also from a pure combinatorial point of view (see [LS92],[LS93],[R84]).

One can easily prove that the condition for the equilibria to be hyperbolic makes the corresponding intersection point of the curve $\gamma(\alpha)$ transversal. Moreover the dissipativity condition on f leads to $\gamma(\alpha) \to \pm\infty$ for $\alpha \to \pm\infty$.

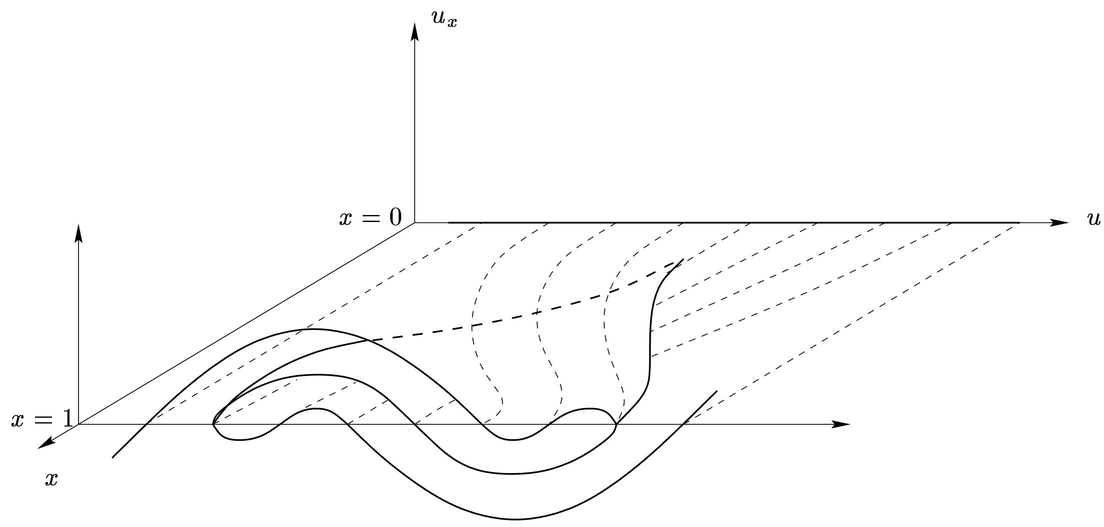


Figure 1: surface $S(x, \alpha)$

Hence, there has to be an odd number of hyperbolic equilibria. Obviously the permutation determines the curve up to a diffeomorphism of the phase plane. So, for simplicity we can assume all the transversal intersections to be orthogonal.

For each equilibrium solution $v(x)$ one can define a winding-number $i(v)$ in the following way: Count the positive clockwise half-twists of a tangent vector to the surface, which is orthogonal to x and moves along the solution $v(x),\ x \in [0,1]$. Classical Sturm-Liouville theory shows that this winding-number is equal to the Morse-index of the equilibrium (see e.g. [Har64]). At the other hand one can compute $i(v)$ easily from the permutation π: By dissipativity the first and last equilibrium both have index zero. The following formula shows, how one can track the changes of the winding-number along the curve $\gamma(\alpha)$:

$$i(v_m) = \sum_{j=1}^{m-1} (-1)^{j+1} \quad \mathrm{sgn}(\pi^{-1}(j+1) - \pi^{-1}(j)) \tag{2}$$

Note that for subsequent equilibria the winding-number changes by ± 1.

A similar formula can be derived for the zero-number $z(v_m - v_n)$ of the difference of two equilibria. The winding-number $i(v_m)$ gives the zero-number of the difference between v_m and a trajectory on the surface $S(x, \alpha)$ infinitesimally close to v_m. Consider now the number of clockwise half-twists of a line between v_m and a point, moving along the curve $\gamma(\alpha)$ from v_m to v_n. Adding this number to $i(v_m)$, one obtains the zero-numbers $z(v_m - v_n)$ for all v_n.

These changes of $z(v_m - \,.\,)$ along the curve $\gamma(\alpha)$ are expressed in the following formula:

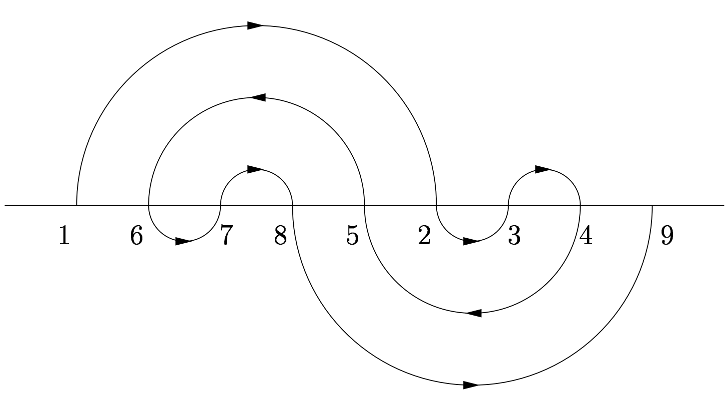
$$z(v_m - v_n) = i(v_m) \quad + \quad \frac{1}{2}\Big((-1)^n \mathrm{sgn}(\pi^{-1}(n) - \pi^{-1}(m)) - 1\Big) \\ + \sum_{j=m+1}^{n-1} (-1)^j \mathrm{sgn}(\pi^{-1}(j) - \pi^{-1}(m)) \qquad (3)$$

Note that for fixed v_m and subsequent v_n, the zero-number $z(v_m - v_n)$ changes by ± 1, according to the change of $i(v_n)$ or remains constant. For a detailed proof of these formulas and some useful tricks for practical computation see [FR96a].

Example: Starting with the meandric permutation

$$\pi = (2\,6)(4\,8)(3\,7),$$

one obtains the meandric curve:



Computing the Morse-indices leads to the vector

$$(i(v_1), \ldots, i(v_9)) = (0, 1, 0, 1, 2, 1, 0, 1, 0).$$

Note that along an arc from left to right in the upper half-plane and along an arc from right to left in the lower half-plane, the Morse-index grows and in all other cases decreases by one.

The zero-numbers can be arranged in a symmetric matrix, and it is reasonable to put the vector of Morse-indices in the diagonal where the zero number is not defined.

$$(z(v_n - v_m))_{n,m} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For more details about the combinatorics see [Wo95].

3 Geometry of the attractor

In this chapter we summarize the main concepts and results for a geometrical description of the attractor. In our presentation we especially emphasize the importance of strong stable and unstable manifolds which are in our opinion essential for an understanding of the geometry of the attractor. For most of the results we don't give detailed proofs, but only some hints and references.

As a general assumption, we need dissipativity of f and hyperbolicity of all equilibria.

3.1 Invariant manifolds

Due to the simple and real eigenvalues of the Sturm-Liouville operators in the linearization, we get by standard theorems (see [Hen81][BF86]) in each hyperbolic equilibrium a hierarchy of stable and unstable manifolds:

Theorem 3.1: *Let v be an equilibrium with Morse-index $i(v) = n$. Then we have the (strong-)unstable manifolds*

$$W_1^u(v) \subset W_2^u(v) \subset \cdots \subset W_n^u(v) = W^u(v), \tag{4}$$

where each $W_j^u(v)$ has the dimension j and is tangent in v to the span $\langle \phi_0, \ldots, \phi_{j-1} \rangle$ of the first j eigenfunctions. Analogously we have the infinite dimensional (strong-)stable manifolds

$$\cdots \subset W_{n+2}^s(v) \subset W_{n+1}^s(v) \subset W_n^s(v) = W^s(v). \tag{5}$$

Here, each $W_k^s(v)$ has codimension k and is tangent to the span $\langle \phi_k, \phi_{k+1}, \ldots \rangle$ of all but the first k eigenfunctions.

An important property of the limit behavior of solutions for the linearized equation has been found by D.Henry in [Hen85], using Matano's principle. Again there are two applications: one for tangent vectors and the variational equation, the other one for the difference of two heteroclinic orbits, connecting the same equilibria.

Proposition 3.2: *(1) Let $u(t,\,.\,)$ be a heteroclinic orbit connecting from v to w. Then for a difference*

$$w_\Delta(t,\,.\,) := u(t+\Delta,\,.\,) - u(t,\,.\,)$$

or tangent vector to the orbit

$$w_\delta(t,\,.\,) := u_t(t,\,.\,)$$

we have the following limits: If $u(t,\,.\,) \in W^u_{j+1}(v) \setminus W^u_j(v)$, *then*

$$\lim_{t\to-\infty} \frac{w_{\Delta,\delta}(t,\,.\,)}{\|w_{\Delta,\delta}(t,\,.\,)\|} = \pm\phi_j(v)$$

Analogously, if $u(t,\,.\,) \in W^s_k(w) \setminus W^s_{k+1}(w)$, *then*

$$\lim_{t\to\infty} \frac{w_{\Delta,\delta}(t,\,.\,)}{\|w_{\Delta,\delta}(t,\,.\,)\|} = \pm\phi_k(w)$$

(2) Let $u_{1,2}(t,\,.\,)$ *be two heteroclinic orbits, both connecting from* v *to* w, *i.e.* $u_{1,2}(t,\,.\,) \in W^u_j(v) \cap W^s_k(w)$ *for some* j, k. *Then for the difference*

$$w_\Delta(t,\,.\,) := u_1(t,\,.\,) - u_2(t,\,.\,)$$

or a tangent vector

$$w_\delta(t,\,.\,) \in T_{u(t)}W^u_j(v) \quad \textit{or res.} \quad T_{u(t)}W^s_k(w),$$

we have the following limits:

$$\lim_{t\to-\infty} \frac{w_{\Delta,\delta}(t,\,.\,)}{\|w_{\Delta,\delta}(t,\,.\,)\|} = \pm\phi_h(v)$$

for some $h < j$, *and*

$$\lim_{t\to\infty} \frac{w_{\Delta,\delta}(t,\,.\,)}{\|w_{\Delta,\delta}(t,\,.\,)\|} = \pm\phi_l(w)$$

for some $l \geq k$. *The convergence is in* $C^1[0,1]$.

This limit behaviour of solutions is well known in finite dimensional dynamical systems with simple real eigenvalues in the linearization. As an additional feature, we can use here the nodal properties of the Sturm-Liouville eigenfunctions ϕ_n, together with Matano's principle.

Proposition 3.2 is the main tool for the proof of the two following important theorems. Using the version for differences of solutions, one gets a theorem, relating the invariant manifolds with the zero-numbers of the equilibria (For a detailed proof see e.g. [BF86]).

Theorem 3.3: *Let again v be an equilibrium with Morse-index $i(v) = n$. Then we have in the unstable manifolds*

$$z(v - u_j(t)) < j \leq n$$

for $u_j(t) \in W^u_j(v) \setminus \{v\}$ and in the stable manifolds

$$z(v - u_k(t)) \geq k \geq n$$

for $u_k(t) \in W^s_k(v) \setminus \{v\}$.

The tangent vector version of proposition 3.2 is the main ingredient in the proof of a generalized Morse-Smale property:

Theorem 3.4: *All intersections of (strong-)stable and (strong-)unstable manifolds are transversal. Hence,*

$$W^u_j(v) \ \overline{\pitchfork}\ W^s_k(w) =: C_{j,k}(v,w)$$

is a embedded submanifold and, if it is not empty, of dimension $j - k$.

For the intersection

$$W^u(v) \cap W^s(w) =: C(v,w),$$

containing all connections between v and w, transversality has been proved already in [Hen85] and [An86]. In [FuRo91] the method of their proof was shown to cover also the generalized version.

If $C_{j,k}(v,w)$ is one-dimensional, which is the case in particular for $C(v,w)$, when $i(v) - i(w) = 1$, the situation can be shown to be quite simple:

Lemma 3.5: *For $j = k + 1$ the set*

$$W^u_j(v) \ \overline{\pitchfork}\ W^s_k(w)$$

is empty or consists of one single trajectory $u(t, .)$ and the values at the boundaries $u(t,0)$ and $u(t,1)$ are monotonic functions in t.

Proof: We repeat exactly the arguments of [BF89], where in lemma 3.5 the same result has been proved for $C(v,w)$, when $i(v)-i(w)=1$.

Suppose there are two different connecting orbits $u_1(t,.)$ and $u_2(t,.)$. Then for each t_1 with $u_1(t_1,0)$ between $v(0)$ and $w(0)$, there is at least one t_2 such that

$$u_1(t_1,0)=u_2(t_2,0).$$

Due to the Neumann boundary condition, $u_1(t+t_1)-u_2(t+t_2)$ has a multiple zero at $t=0$, and by Matano's principle,

$$z(u_1(t+t_1)-u_2(t+t_2))$$

has to drop there. At the other hand by Proposition 3.2, the normalized difference $u_1(t+t_1)-u_2(t+t_2)$ converges for $t\to-\infty$ to some eigenvector $\phi_h(v)$ with $h<j$ and for $t\to\infty$ to some eigenvector $\phi_l(w)$ with $l\geq k$. Since $j=k+1$, this means that for all t we should have constant

$$z(u_1(t+t_1)-u_2(t+t_2))=k.$$

This contradiction proves that there can be at most one connection $u(t,.)$. With the same argument we can prove the monotonicity of this unique connection at the boundaries: Suppose there are $t_1\neq t_2$ with $u(t_1,0)=u(t_2,0)$. Then again

$$z(u(t+t_1)-u(t+t_2))$$

has to drop and to be constant at the same time. The same is true at the boundary $x=1$.

□

This lemma allows a simple conclusion: Assume, we have $v\searrow w$ with $i(v)-i(w)=1$ and $v(0)<w(0)$. Then

$$i(v) \text{ is even } \iff w(1)<v(1)$$

$$i(v) \text{ is odd } \iff v(1)<w(1).$$

This means, we can decide from the permutation of the equilibria whether the connecting orbit is tangent to $+\phi_{i(v)}(v)$ or $-\phi_{i(v)}(v)$. This observation will become fundamental in our argumentation in chapter 4.

An immediate consequence of theorem 3.3 was observed already in [BF89]: heteroclinic connections can appear only under certain conditions on the zero-numbers.

Lemma 3.6 (Morse-blocking): *Assume,*

$$C_{j,k}(v,w) \neq \emptyset$$

then we have

$$i(w) \leq k \leq z(v-w) < j \leq i(v).$$

Proof: The inequalities $i(w) \leq k$ and $j \leq i(v)$ are the conditions, under which there exist nonempty manifolds $W^u_j(v)$ and $W^s_k(w)$. Let now $u(t)$ be a trajectory in $C_{j,k}(v,w)$. By theorem 3.3 we have $k \leq z(w-u(t))$ and $z(v-u(t)) < j$. Since

$$z(v-u(t)) \to z(v-w) \text{ for } t \to \infty$$

and

$$z(w-u(t)) \to z(w-v) \text{ for } t \to -\infty,$$

we get by Matano's principle immediately the claimed inequality.

□

There is also a second necessary condition for the existence of heteroclinic connections which we want to discuss at the end of this chapter.

3.2 Boundaries of connecting sets and the cascading principle

We introduce on the set E of equilibria the relation

$$v \searrow w,$$

saying that there exists a heteroclinic connection from v to w.

By the gradient structure of the semiflow, this relation is obviously antisymmetric. By theorem 3.4 also the Morse index of source and target of a heteroclinic connection has to decay and is something like a "discrete potential". The transitivity of the relation is a well known fact for Morse-Smale systems (see e.g. [PS70]).

We can define now order-intervals in the usual way:

$$E(v,w) := \{\tilde{v} \in E | v \overset{\searrow}{=} \tilde{v} \overset{\searrow}{=} w\}$$

Note that by our definition also $v, w \in E(v,w)$. The closure $\overline{C(v,w)}$ and the boundary $\delta(C(v,w))$ in the topology of X are closed invariant sets and, of course, contained in the attractor. So, they consist also only of equilibria and heteroclinic orbits between them. The following lemma shows the relation between the order-intervals and the sets of heteroclinic connections.

Lemma 3.7: *(1) For any equilibrium $\tilde{v} \in E$, we have*

$$\tilde{v} \in \delta(C(v,w)) \Longleftrightarrow \tilde{v} \in E(v,w).$$

(2) For any heteroclinic trajectory $u(t)$ in the attractor, we have

$$u(t) \in \overline{C(v,w)} \Longleftrightarrow \lim_{t \to \pm\infty}(u(t)) \in E(v,w).$$

This can be proved straightforward, considering sequences in $C(v,w)$ which converge to the boundary (see [FR96a]).

The following definition and theorem 3.8 will show a further important property of the partial order $(E, \searrow)$, which was called in [FR96a] the *cascading principle.*

Definition: *A sequence of $n+1$ equilibria $v = v_0, v_1, \dots, v_n = w$ is called a cascade from v to w, if*

$$i(v_{h-1}) - i(v_h) = 1 \quad \text{and} \quad v_{h-1} \searrow v_h$$

for all $h \in \{1, \dots, n\}$.

Theorem 3.8: *There exists a heteroclinic connection $v \searrow w$, if and only if there exists a cascade from v to w.*

That the existence of a cascade implies a direct connection follows from the transitivity of the relation $(E, \searrow)$. The converse has been proved in [FR96a], using again the nodal properties.

3.3 Establishing heteroclinic connections

As we have seen before, heteroclinic connections can occur only under certain conditions on the zero numbers. So, it would be important to know, whether there are also sufficient conditions for the existence of a heteroclinic connection. In [FR96a] this problem has been solved in the case $i(v) - i(w) = 1$, using Conley-index technique:

Theorem 3.9: *For $i(v) - i(w) = 1$, there is a heteroclinic connection $v \searrow w$, if and only if*

- $z(v - w) = i(w)$.
- $z(v - \tilde{w}) \neq z(w - \tilde{w})$ *for all* $\tilde{w}$ *between* v *and* w *at* $x = 0$

Proof: see [FR96a].

Together with the cascading principle, this result shows, whether $C(v, w)$ is empty or not also for $i(v) - i(w) > 1$. One has to determine first all possible intermediate equilibria and then decide, whether there is a cascade or not.

However, in the case $i(v) - i(w) > 1$ this theorem gives no explicit condition in terms of $i(v), i(w)$ and $z(v - w)$ for the existence of a heteroclinic connection.

The first of the two conditions in theorem 3.9 is the restriction to the case $i(v) - i(w) = 1$ of the Morse-blocking condition in lemma 3.6. The second condition is necessary for heteroclinic connections with $i(v) - i(w) > 1$, too. But this follows only from Corollary 4.2 in the next chapter. We believe that these two conditions are already sufficient for the existence of a heteroclinic connection in the general case $i(v) - i(w) > 1$.

4 Heteroclinic connections in strong-unstable and strong-stable manifolds

In this section, we formulate and prove our main theorem.

Theorem 4.1: *Let v, w be two equilibrium solutions of (1), having a heteroclinic connection $v \searrow w$. Then the zero number $z(v-w)$ determines which strong-unstable manifolds of v and strong-stables manifold of w intersect each other and hence, where the heteroclinic connections are contained: The $(j-k)$-dimensional manifold*

$$C_{j,k}(v,w) = W^u_j(v) \pitchfork W^s_k(w)$$

is nonempty, if and only if

$$\begin{aligned} z(v-w) < \quad & j \quad \le i(v) \\ i(w) \le \quad & k \quad \le z(v-w). \end{aligned}$$

In the special case $i(v) - i(w) = 1$ the statement of this theorem is contained in the theorem 3.9 of Fiedler and Rocha. The "only if" part in theorem 4.1 is an immediate consequence of of the Morse-blocking lemma 3.6.

Of course, we have due to (4)

$$C_{j,k}(v,w) \subseteq C_{j+1,k}(v,w)$$

and due to (5)

$$C_{j,k}(v,w) \subseteq C_{j,k-1}(v,w),$$

and hence the main statement of the theorem is that $z(v-w)+1$ is the minimal j and $z(v-w)$ the maximal k, for which $C_{j,k}(v,w) \neq \emptyset$. So, with regard to lemma 3.5 we can formulate equivalently:

Corollary 4.2: *For each pair of connected equilibria $v \searrow w$, there is a unique connection in $C_{h+1,h}(v,w)$ with $h := z(v-w)$. The values at the boundaries are monotonic functions in t.*

For the purpose of establishing of heteroclinic connections, Fiedler and Rocha used in [FR96a] Conley-index theory. This seems to be not applicable in this case. For $i(v) - i(w) > 1$ there are a lot of intermediate equilibria involved, making it impossible for us to find a suitable isolating neighbourhood. Instead of this, we use in addition some combinatorial and topological arguments. One mainly has to understand the case $i(v) - i(w) = 2$. Then it needs some technicalities to figure out, how the arguments there can be used to prove by induction also the general case.

Lemma 4.3: *Let $v \searrow w$ and $i(v) - i(w) = 2$. Then there exist exactly two equilibria $\tilde{v}_{1,2}$ such that*

$$v \searrow \tilde{v}_{1,2} \searrow w.$$

In the case $z(v - w) = i(w)$ the source v lies between the intermediate equilibria $\tilde{v}_{1,2}$ at both boundaries and the target w does not. So, if we assume $v(0) < w(0)$, we get

$$\tilde{v}_1(0) < v(0) < \tilde{v}_2(0) < w(0).$$

In the case $z(v - w) = i(w) + 1$ the target w lies between the intermediate equilibria $\tilde{v}_{1,2}$ at both boundaries and the source v does not. So, if we assume again $v(0) < w(0)$, we get

$$v(0) < \tilde{v}_1(0) < w(0) < \tilde{v}_2(0).$$

Proof: Due to theorem 3.8, there exists an equilibrium $\tilde{v}$ with

$$v \searrow \tilde{v} \searrow w.$$

Let $u_1(t, .)$ be the orbit connecting v to $\tilde{v}$, and $u_2(t, .)$ the orbit connecting $\tilde{v}$ to w. From Lemma 3.5 we know that the values at the boundaries $u_1(t, 0), u_1(t, 1)$ and $u_2(t, 0), u_2(t, 1)$ are monotonic functions of t. Moreover, if the index $i(v)$ of the source equilibrium is odd, the movement at both boundaries has the same direction; if the index is even, the movement at $x = 0$ is in opposite direction to the movement at $x = 1$. Of course, we have $i(v) = i(\tilde{v}) + 1$ and one of the indices $i(v), i(\tilde{v})$ is odd, the other even. This implies, if $u_1(t, 0)$ and $u_2(t, 0)$ move in the same direction then $u_1(t, 1)$ and

$u_2(t,1)$ move in opposite directions (and vice versa). So, there are two possible types of intermediate equilibria $\tilde{v}$: Either for $x = 0$ or for $x = 1$ we have

$$v(x) < \tilde{v}(x) < w(x),$$

assuming that $v(x) < w(x)$. We show now that there can exist at most one equilibrium of each type. So, assume there are $\tilde{v}_{1,2}$ with

$$v(0) < \tilde{v}_{1,2}(0) < w(0)$$

and, if we assume without loss of generality $i(v)$ to be even,

$$\tilde{v}_1(1) < v(1), \quad \tilde{v}_2(1) < v(1).$$

Of course, we can assume that $\tilde{v}_1(1) < \tilde{v}_2(1)$. Now, let again $u_1(t,.)$ and $u_2(t,.)$ be the orbits connecting v to $\tilde{v}_1$ and $\tilde{v}_1$ to w respectively. Then at the boundary $x = 0$ we find by the mean value theorem a time t, where either $u_1(t,0)$ or $u_2(t,0)$ is equal to $\tilde{v}_2(0)$. At the other boundary, $x = 1$, we get for both, $u_1(t,1)$ and $u_2(t,1)$ a time, where they coincide with $\tilde{v}_2(1)$. The Neumann boundary conditions force each of these points to be a double zero of the difference between $\tilde{v}_2$ and the heteroclinic connection and hence, by Matano's Principle, the respective zero number has to drop at least three times. So we get

$$z(v - \tilde{v}_2) - z(\tilde{v}_2 - w) \geq 3,$$

which is an obvious contradiction to the fact that $z(v - \tilde{v}_2) = i(\tilde{v}_2) = i(v) - 1$ and $z(\tilde{v}_2 - w) = i(w) = i(v) - 2$.

Therefore there can be indeed at most one intermediate equilibrium of each type. In ([BF89], lemma 3.6) it has been proved that for $i(v) - i(w) = 2$ there exist at least two intermediate equilibria. So we have exactly one of each type.

Now, let $\tilde{v}_1$ be the intermediate equilibrium with

$$v(0) < \tilde{v}_1(0) < w(0).$$

If we again assume $i(v)$ to be even, $z(v - \tilde{v}_{1,2}) = i(v) - 1$ is odd, and so $v(0) < \tilde{v}_1(0)$ implies $v(1) > \tilde{v}_1(1)$.

Since $z(v-w)$ can be equal to $i(w)$ or $i(w)+1$ (see lemma 3.6), it is respectively even or odd. So the assumption $v(0) < w(0)$ implies $v(1) < w(1)$ or $w(1) < v(1)$ respectively. Taking into account that $\tilde{v}_2(1)$ has to be between $v(1)$ and $w(1)$, and putting everything together, we get for $z(v-w) = i(w)$

$$\tilde{v}_2(0) < v(0) < \tilde{v}_1(0) < w(0)$$

and for $z(v-w) = i(w)+1$

$$v(0) < \tilde{v}_1(0) < w(0) < \tilde{v}_2(0).$$

If $i(v)$ is odd, one can perform easily analogous calculations, leading to the same results. Assuming $i(v)$ to be even, one obtains at the boundary $x = 1$ for $z(v-w) = i(w)$

$$\tilde{v}_1(1) < v(1) < \tilde{v}_2(1) < w(1)$$

an for $z(v-w) = i(w)+1$

$$\tilde{v}_1(1) < w(1) < \tilde{v}_2(1) < v(1).$$

If $i(v)$ is odd, we get here the reversed inequalities.

□

In the following lemma we show, how one can establish a connection in $W^u_{i(v)-1}(v)$, if $v(0)$ lies between $\tilde{v}_1(0)$ and $\tilde{v}_2(0)$, or in $W^s_{i(w)+1}(w)$, if $w(0)$ lies between $\tilde{v}_1(0)$ and $\tilde{v}_2(0)$.

Lemma 4.4: *Let again be $v \searrow w$ and $i(v) - i(w) = 2$. If we denote $k := z(v-w)$, then we have:*

$$W^u_{k+1}(v) \pitchfork W^s_k(w) = C_{k+1,k}(v,w) \neq \emptyset$$

Proof: First, we look at the case $k = i(w)$.
Take the intersection of the manifold $W^u(v) = W^u_{k+2}(v)$ with a small neighbourhood $\mathcal{N}_\varepsilon(v)$ of v. The manifold $W^u_{k+1}(v)$ has codimension 1 in $W^u(v)$ and hence $(W^u(v) \setminus W^u_{k+1}(v)) \cap \mathcal{N}_\varepsilon(v)$ consists of two connected components. According to proposition 3.2, these components contain parts of trajectories,

having in v the tangent vector $\pm\phi_{k+1}(v)$ respectively. Obviously the sign of the tangent vector reflects the direction of the monotonic movement at the boundaries, which we described in lemma 3.5. So, taking into account lemma 4.3, there exists in each connected component exactly one trajectory, connecting v to one of the intermediate equilibria $\tilde{v}_1$ and $\tilde{v}_2$.

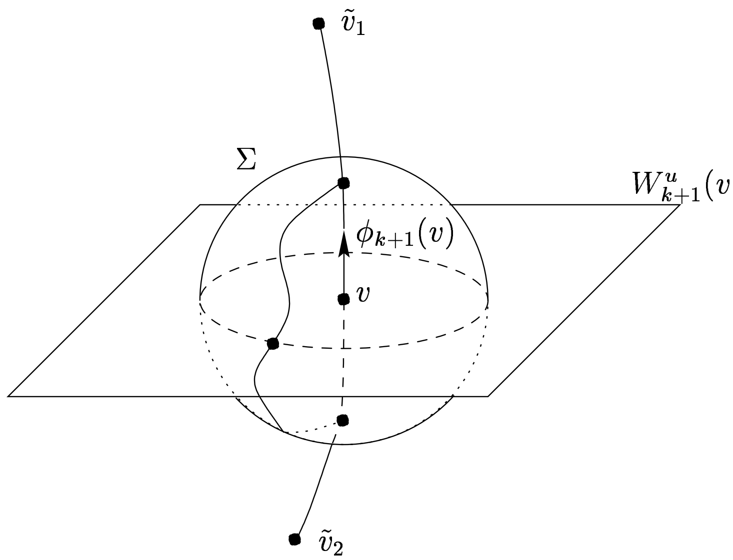


Figure 2: Linearized flow near v in $W^u(v)$ for $z(v-w) = i(w)$

Now, we consider a small sphere $\Sigma \in W^u(v)$ around v, which is transversal to the flow. The intersection

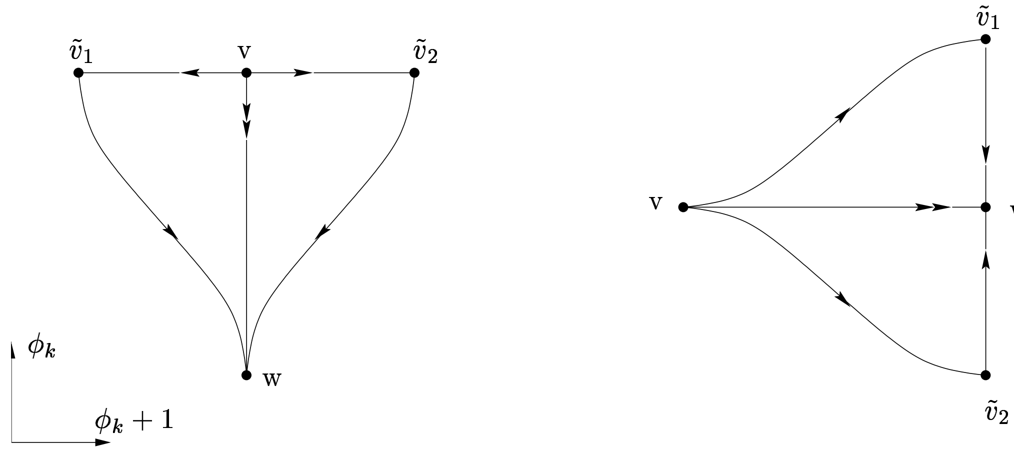
$$C(v,w) \cap \Sigma := \Sigma(v,w)$$

is a one-dimensional submanifold. According to lemma 3.7, the limit points of $\Sigma(v,w)$ belong to trajectories, connecting from v to one of the intermediate equilibria $\tilde{v}_1$ and $\tilde{v}_2$.

The fact that any connection between an intermediate equilibrium $\tilde{v}_{1,2}$ and w is unique shows, together with the λ-Lemma, that there is a one-to-one correspondence between boundary points of $\Sigma(v,w)$ and intermediate equilibria. So, there are exactly two limit points, corresponding to the connections $v \searrow \tilde{v}_1$ and $v \searrow \tilde{v}_2$. They have to be connected by an arc in $\Sigma(v,w)$, which has obviously to intersect $W^u_{k+1}(v)$ and we get the desired connection in $C_{k+1,k}(v,w)$.

In the case $z(v-w) = i(w)+1$ one has to carry out the same construction in a neighbourhood of w.

Projecting onto the two involved eigenvectors ϕ_k and ϕ_{k+1} which can be chosen equal for all four equilibria, we get, up to reflection, the following schematic pictures:



$z(v-w) = i(w)$ $\qquad\qquad$ $z(v-w) = i(w) + 1$

Figure 3: 2-dimensional manifolds of heteroclinic connections

□

To use similar arguments also in the case $i(v) - i(w) > 2$, we need first an analog of lemma 4.3. To this end, we study now in detail the order intervals $E(v, w)$.

Definition 4.5: *(1) In an order-interval $E(v, w)$ we denote the sets of equilibria with equal Morse-indices by*

$$E_h(v, w) = \{e \in E(v, w) | i(e) = h\}.$$

(2) For two connected equilibria $v \searrow w$ with $i(v) = n$ and $i(w) = m$ we call a cascade

$$v = v_n \searrow v_{n-1} \searrow \ldots \searrow v_{m+1} \searrow v_m = w$$

monotonous, if either for all $m < h \leq n$

$$v_h(0) < v_{h-1}(0)$$

or

$$v_{h-1}(0) < v_h(0).$$

(3) We call a cascade $(v_h)_{h=n\ldots m}$, *connecting from* v *to* w *maximal, if for any three consecutive equilibria in the cascade with* $m < h < n$

$$v_{h+1} \searrow v_h \searrow v_{h-1},$$

the second intermediate equilibrium (cf. lemma 4.3) $\tilde{v}_h \in E_h(v_{h+1}, v_{h-1})$ *satisfies:*

$$\tilde{v}_h(0) < v_h(0).$$

If there is always $v_h(0) < \tilde{v}_h(0)$, *we call the cascade* $(v_h)_{h=n\ldots m}$ *minimal.*

For simplicity, the subscript here always indicates the Morse-index of the corresponding equilibrium.

Note that by the finiteness of $E(v, w)$, any cascade can be transformed into a maximal or minimal one, making step by step all triples maximal or minimal, respectively. For monotonous maximal or minimal cascades, it is now easy to compute the zero-numbers:

Lemma 4.6: *(1) Let* v, w *be connected equilibria with* $v(0) < w(0)$. *If there exists a monotonous maximal cascade* $(v_h)_{h=n\ldots m}$ *connecting* v *to* w, *then for all* $m \leq h_{1,2} \leq n$, *we have*

$$z(v_{h_1} - v_{h_2}) = \min\{h_1, h_2\};$$

in particular

$$z(v - w) = i(w).$$

The same holds true for a monotonous minimal cascade, if $w(0) < v(0)$.

(2) If there exists a monotonous minimal cascade $(v_h)_{h=n\dots m}$ *in the case* $v(0) < w(0)$*, then for all* $m \leq h_{1,2} \leq n$

$$z(v_{h_1} - v_{h_2}) = \max\{h_1, h_2\} - 1;$$

in particular

$$z(v - w) = i(v) - 1.$$

The same holds true for a monotonous maximal cascade, if $w(0) < v(0)$.

Note, that there are always two versions: one for $v(0) < w(0)$, and one for the opposite case. For simplicity, from now on we confine ourselves to the case $v(0) < w(0)$ and leave it to the reader, to find for each statement an analogous reflected version.

Proof: For two connected equilibria $v \searrow w$ with $l := i(v) - i(w) = 1$, both statements (1) and (2) are by lemma 3.6 trivially satisfied. For $l = 2$, the statement follows immediately from lemma 4.3. Therefore, we can assume v, w to be a counterexample for (1) with minimal $l > 2$.

Of course $(v_h)_{h=n\dots m+1}$ and $(v_h)_{h=n-1\dots m}$ are monotonous maximal cascades, and we can apply induction. So, the only thing we have to prove is

$$z(v - w) = i(w) = m.$$

As we have already

$$z(v - v_{m+1}) = i(v_{m+1}) = m + 1 \tag{6}$$

and $z(v - u_{m+1}(t))$ cannot grow along the trajectory $u_{m+1}(t)$, connecting from v_{m+1} to w, we get immediately

$$m \leq z(v - w) \leq m + 1. \tag{7}$$

The first part of this inequality is due to lemma 3.6.

We assumed $l > 2$ and so there exist v_{n-1} and v_{n-2} different from w in the monotonous maximal cascade. In the triple $v \searrow v_{n-1} \searrow v_{n-2}$, we have by induction

$$z(v - v_{n-2}) = n - 2.$$

and with the second equilibrium $\tilde{v}_{n-1} \in E_{n-1}(v, v_{n-2})$ also

$$\tilde{v}_{n-1}(0) < v(0) < v_{n-1}(0) < v_{n-2}(0) < w(0). \tag{8}$$

Obviously, $\tilde{v}_{n-1} \searrow v_{n-2} \searrow \ldots \searrow w$ is a monotonous cascade. We will show it to be maximal, too. The only triple, we have to check is $\tilde{v}_{n-1} \searrow v_{n-2} \searrow v_{n-3}$. We assume in contradiction to maximality that there is $\tilde{v}_{n-2} \in E(\tilde{v}_{n-1}, v_{n-3})$ with

$$v_{n-2}(0) < \tilde{v}_{n-2}(0).$$

At $x = 0$, we have the order

$$\tilde{v}_{n-1}(0) < v(0) < v_{n-1}(0) < v_{n-2}(0) < v_{n-3}(0) < \tilde{v}_{n-2}(0).$$

Now we consider the order at $x = 1$. For simplicity, we assume $i(v) = n$ to be odd. Otherwise, all inequalities must be reversed. For $E(v, v_{n-2})$, we get due to lemma 4.3

$$v_{n-2}(1) < \tilde{v}_{n-1}(1) < v(1) < v_{n-1}(1).$$

Since $v_{n-2}(1) < \tilde{v}_{n-1}(1)$, we get for $E(\tilde{v}_{n-1}, v_{n-3})$:

$$v_{n-2}(1) < v_{n-3}(1) < \tilde{v}_{n-2}(1) < \tilde{v}_{n-1}(1)$$

These two inequalities imply together $v_{n-3}(1) < v_{n-1}(1)$. In contradiction to that, we get for $E(v_{n-1}, v_{n-3})$

$$v_{n-2}(1) < v_{n-1}(1) < \tilde{\tilde{v}}_{n-2}(1) < v_{n-3}(1),$$

where $\tilde{\tilde{v}}_{n-2}$ is the second equilibrium in $E_{n-2}(v_{n-1}, v_{n-3})$. Thus we can apply induction to the cascade $\tilde{v}_{n-1} \searrow v_{n-2} \searrow \ldots \searrow w$ and we obtain

$$z(\tilde{v}_{n-1} - w) = z(v_{n-1} - w) = i(w) = m. \tag{9}$$

This equation allows us to determine the position of w at $x = 1$: Due to (8), v_{n-1} and $\tilde{v}_{n-1}$ are at the same side of w at $x = 0$. From (9) we can conclude that this is the case at $x = 1$, too. So we get either

$$\tilde{v}_{n-1}(1) < v(1) < v_{n-1}(1) < w(1)$$

or

$$w(1) < \tilde{v}_{n-1}(1) < v(1) < v_{n-1}(1)$$

or one of these inequalities reflected. But never w lies between v and v_{n-1} at the boundary. So we obtain

$$z(v-w) \equiv z(v_{n-1}-w) \mod 2.$$

Together with (7), this proves that indeed $z(v-w) = m$.

To prove (2), we need again only to show that for a minimal counterexample

$$z(v-w) = i(v) - 1.$$

Analogously we conclude immediately from Matano's principle that

$$i(v) - 1 \le z(v-w) \le z(v_{n-1}-w) = i(v) - 2.$$

Applying now lemma 4.3 to the sequence

$$v_{m+2} \searrow v_{m+1} \searrow w$$

leads in the same way as above to the stated result.

□

Now, we have to find monotonous cascades in an arbitrary order-interval $E(v,w)$.

Definition 4.7: *In each order-interval $E(v,w)$ there is a maximal and a minimal element $v^{\max}$ and $v^{\min}$ such that*

$$v^{\min}(0) \le \tilde{v}(0) \le v^{\max}(0)$$

holds for all $\tilde{v} \in E(v,w)$.

Lemma 4.8: *For two connected equilibria v, w with $v(0) < w(0)$, we have (as long as the argument of the function z is not identically zero):*

$$\begin{aligned} z(v-v^{\max}) &= z(w-v^{\max})+1 = & z(v-w) = i(v^{\max}) \\ z(v-v^{\min}) &= z(w-v^{\min})+1 = & z(v-w)+1 = i(v^{\min}) \end{aligned}$$

Moreover there are monotonous maximal cascades

$$v \searrow \ldots \searrow v^{\max} \quad \textit{and} \quad v^{\max} \searrow \ldots \searrow w$$

and monotonous minimal cascades

$$v \searrow \ldots \searrow v^{\min} \quad \textit{and} \quad v^{\min} \searrow \ldots \searrow w.$$

Proof: For $l =: i(v) - i(w) = 1$ we have, due to our general assumption $v(0) < w(0)$, that $v^{\min} = v$ and $v^{\max} = w$ and the statement follows from theorem 3.9.

So, let again v, w be a counterexample with minimal $l > 1$. Now we have to distinguish several cases:

First, we assume that $v^{\min} = v$ and $v^{\max} = w$. Let $v_{n-1}^{\max}$ be the maximal element in $E_{n-1}(v, w)$. By assumption we have

$$v(0) < v_{n-1}^{\max}(0) < w(0).$$

Now we apply induction to $E(v_{n-1}^{\max}, w)$. Since $v^{\max} = w$ is maximal in $E(v_{n-1}^{\max}, w)$, too, we obtain a monotonous maximal cascade

$$v_{n-1}^{\max} \searrow v_{n-2} \searrow \ldots \searrow w.$$

Comparing the cascade $v \searrow v_{n-1}^{\max} \searrow v_{n-2}$ with lemma 4.3 leads to a contradiction: For the second intermediate equilibrium $\tilde{v}_{n-1} \in E_{n-1}(v, v_{n-2})$, we have either

$$\tilde{v}_{n-1}(0) < v(0) < v_{n-1}^{\max}(0) < v_{n-2}(0),$$

contradicting the minimality of v in $E(v, w)$, or

$$v(0) < v_{n-1}^{\max}(0) < v_{n-2}(0) < \tilde{v}_{n-1}(0),$$

contradicting the maximality of $v_{n-1}^{\max}$.

Next, we want to look at the cases where either $v^{\min} = v$ or $v^{\max} = w$. They can be treated analogously, and we confine ourselves to the case $v^{\max} = w, \quad v^{\min} \neq v$.

Since we suppose $v^{\min} \neq v$, we can apply induction to $E(v^{\min}, w) = E(v^{\min}, v^{\max})$. But here the minimal and maximal elements are $v^{\min}$ and

$v^{\max}$, such that we are in the preceding case in our proof. So, we conclude that

$$i(v^{\min}) - i(v^{\max}) = 1$$

and hence

$$i(v^{\min}) - 1 = i(w) = i(v^{\max}) = z(v^{\min} - w).$$

Applying induction to $E(v, v^{\min})$ provides a monotonous minimal cascade $v \searrow \ldots \searrow v^{\min}$ and

$$z(v - v^{\min}) = i(v^{\min}).$$

It only remains to show the existence of a maximal monotonous cascade $v \searrow \ldots \searrow w$. This would imply, using lemma 4.6, that

$$z(v - w) = z(v - v^{\max}) = i(v^{\max})$$

and the proof is complete. Therefore, let $\tilde{v}^{\max}$ be the maximal element in $E(v, w) \setminus \{w\}$. If $\tilde{v}^{\max} = v$, then $E(v, v^{\min}) = \{v, v^{\min}\}$ and

$$i(v) - i(w) = 2.$$

In this case lemma 4.3 proves the existence of the desired maximal monotonous cascade immediately. In any other case we can apply induction to $E(v, \tilde{v}^{\max})$ and $E(\tilde{v}^{\max}, w)$. Obviously, $\tilde{v}^{\max}$ must be contained in some maximal cascade in $E(v, w)$. But the two parts $v \searrow \ldots \searrow \tilde{v}^{\max}$ and $\tilde{v}^{\max} \searrow \ldots \searrow w$ of this cascade are maximal, too. Since $\tilde{v}^{\max}$ is maximal in $E(v, \tilde{v}^{\max})$, and w is maximal in $E(\tilde{v}^{\max}, w)$, both parts are monotonous. Hence the whole cascade is monotonous.

Finally, we assume that $v^{\min} \neq v$ and $v^{\max} \neq w$. Then we get

$$v^{\min}(0) < v(0) < w(0) < v^{\max}(0). \tag{10}$$

Now, by induction, we can apply the lemma to $E(v, v^{\max})$. The maximal element is here of course again $v^{\max}$, and we get a monotonous maximal cascade $v \searrow \ldots \searrow v^{\max}$ and

$$z(v - v^{\max}) = i(v^{\max}). \tag{11}$$

Applying induction to $E(v^{\max}, w)$, we have to use the reflected version of the lemma, because $w(0) < v^{\max}(0)$. So, we obtain a monotonous maximal cascade $v^{\max} \searrow \ldots \searrow w$ and

$$z(w - v^{\max}) + 1 = i(v^{\max}). \tag{12}$$

Using now Matano's principle, we get from (11)

$$z(v - w) \leq z(v - v^{\max}) = i(v^{\max})$$

since $z(v - u_1(t))$ cannot grow along a trajectory $u_1(t)$ from $v^{\max}$ to w.

Due to (10), $w(0)$ lies between $v(0)$ and $v^{\max}(0)$. Therefore, along a trajectory $u_2(t)$ from v to $v^{\max}$, the zero-number $z(w - u_2(t))$ must drop at least by one . The Neumann boundary conditions guaranty a double zero of this difference for some t. So, we get from (12)

$$z(w - v) > z(w - v^{\max}) + 1 = i(v^{\max}).$$

Putting the last two inequalities together proves the claimed equation for $v^{\max}$:

$$z(v - w) = z(v - v^{\max}) = z(w - v^{\max}) + 1 = i(v^{\max})$$

In the same way one can prove the equation for $v^{\min}$ and show the existence of the corresponding monotonous minimal cascades.

□

Now we are able to start the proof of theorem 4.1:

Proof of theorem 4.1: We will proceed again by induction over $l = i(v) - i(w)$. In the case $l = 1$ the statement is trivial. So, we can assume $v \searrow w$ to be a counterexample with minimal $l > 1$. Again, we may assume without loss of generality that $v(0) < w(0)$.

In the case $v^{\max} \neq w$ and $v^{\min} \neq v$, we get

$$i(v) - i(v^{\max}) < l \quad \text{and} \quad i(v^{\min}) - i(w) < l.$$

So, we can apply induction to $E(v, v^{\max})$ and $E(v^{\min}, w)$. This gives us connections in

$$C_{h+1,h}(v, v^{\max}) \quad \text{and} \quad C_{h+1,h}(v^{\min}, w),$$

where due to lemma 4.8

$$h = z(w - v) = z(v - v^{\max}) = z(w - v^{\min}).$$

Note that $W^u_{h+1}(v)$ and $W^s_h(w)$ meet transversely $C_{h+1,h}(v, v^{\max})$ and $C_{h+1,h}(v^{\min}, w)$ which are, due to lemma 3.7, contained in the boundary of $C(v, w)$. Due to transversality, they meet also the interior of $C(v, w)$ and we have in this case

$$C_{h+1,h}(v, w) \neq \emptyset.$$

Now, we investigate the case $v^{\max} = w$. Let $v^{\max}_{m+1}$ be the maximal element in $E_{m+1}(v, w)$. But $v^{\max}_{m+1}$ is due to lemma 4.8 contained in a monotonous maximal cascade, and hence maximal in $E(v, v^{\max}_{m+1})$, too. Therefore we have

$$z(v - v^{\max}_{m+1}) = m + 1$$

So, by induction we get immediately a connection in $C_{m+2,m+1}(v, v^{\max}_{m+1})$. By the usual transitivity argument (see [PS70]), the 2-dimensional manifold $C_{m+2,m}(v, w)$ is nonempty, too. So, with some slight modifications we can argue now as in lemma 4.3 and 4.4 where we proved the theorem for 2-dimensional manifolds $C(v, w)$. First, we intersect a small sphere around v in $W^u_{m+2}(v)$, which is transversal to the flow, with $C_{m+2,m}(v, w)$. The resulting one-dimensional manifold has a connected component with a limit point, lying on the unique orbit in $C_{m+2,m+1}(v, v^{\max}_{m+1})$. The second limit point of this connected component is lying on another orbit, connecting from v to an intermediate equilibrium $\tilde{v}_{m+1}$. Due to theorem 3.3 and 3.4, we have indeed

$$z(v - \tilde{v}_{m+1}) = i(\tilde{v}_{m+1}) = m + 1.$$

If we have in addition

$$\tilde{v}_{m+1}(0) < v(0) < v^{\max}_{m+1}(0), \tag{13}$$

we can complete the proof in analogy to lemma 4.4. Since we chose $v^{\max}_{m+1}$ maximal in $E_{m+1}(v, w)$, we have

$$\tilde{v}_{m+1}(0) < v^{\max}_{m+1}(0) < w(0).$$

But the same argument as in the proof of lemma 4.3 shows that there can be at most one equilibrium $v_{m+1} \in E_{m+1}(v, w)$ between v and w at $x = 0$, satisfying also

$$z(v - v_{m+1}) = m + 1.$$

This proves (13) and hence the statement of the theorem in the case $v^{\max} = w$. The other case $v^{\min} = v$ can again be treated analogically.

$\square$

5 Examples, outlook and discussion

Whereas Fiedler and Rocha (Theorem 3.6, 3.7 and [FR96a]) gave some general results about the existence of heteroclinic connections, the problem was attacked in an earlier approach by Fusco and Rocha [FuRo91] in a very constructive way: They tried to construct the attractor by a sequence of pitchfork bifurcations, starting from a single attracting equilibrium.

They noticed that the replacement of a single intersection point by three consecutive intersection points in the meandric curve corresponds to a pitchfork bifurcation of the respective equilibrium:

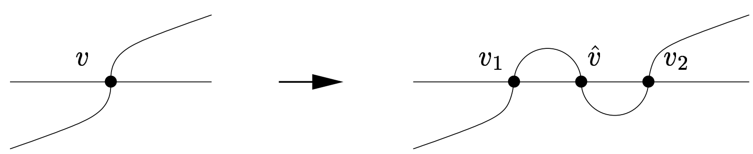


Figure 4: supercritical pitchfork bifurcation in the meandric curve

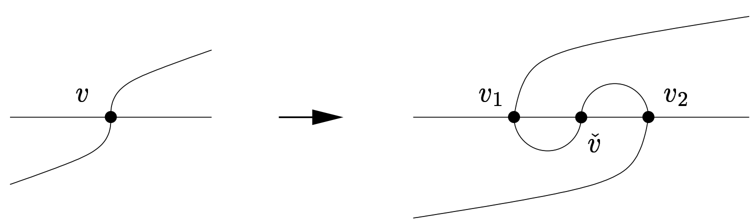


Figure 5: subcritical pitchfork bifurcation in the meandric curve

Note that the Morse-indices of the bifurcating equilibria are due to formula (2)

$$\begin{aligned} i(v_{1,2}) &= i(v) \\ i(\hat{v}) &= i(v) + 1 \\ i(\check{v}) &= i(v) - 1 \end{aligned}$$

There are two new connections $\hat{v} \searrow v_{1,2}$, or $v_{1,2} \searrow \check{v}$. But one can also keep track of all other connections:
In the supercritical case, all connections in $W^u(v)$ are also present in $W^u(v_{1,2})$, as well as in $W^u_{i(v)}(\hat{v})$. $W^u(\hat{v}) \setminus W^u_{i(v)}(\hat{v})$ contains additional only the new

connections $\hat{v} \searrow v_{1,2}$. The stable manifold $W^s(\hat{v})$ contains exactly the connections in $W^s_{i(v)+1}(v)$ and in each of the manifolds $W^s(v_{1,2})$ is in addition one part of the connections which have been in $W^s(v) \setminus W^s_{i(v)+1}(v)$.

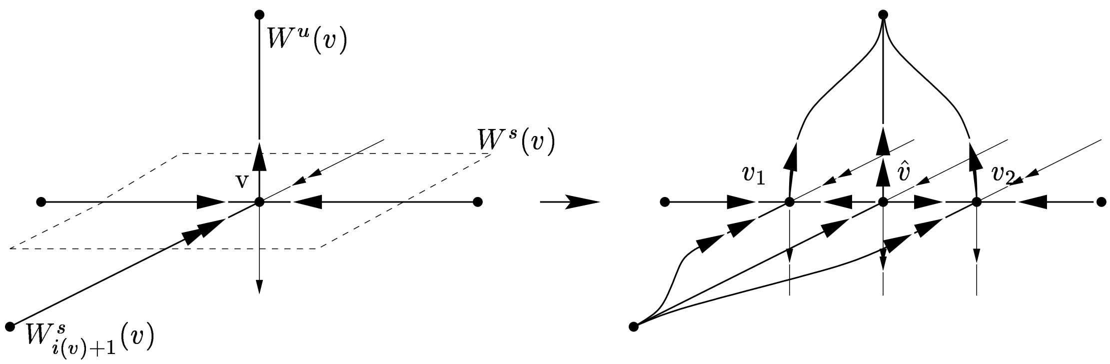


Figure 6: supercritical pitchfork bifurcation and some heteroclinic connections

In the subcritical case one can figure out the connections in a similar way. For details, see [FuRo91].

So, this method gives direct geometrical insight in the structure of the attractor. But the method is applicable only to those attractors which can be generated by a sequence of pitchfork bifurcations. Naturally this property should be decided, looking at the meandric curve.

There has been presented already in [Ro90] a single example of a meandric curve and a corresponding nonlinearity f which does not belong to this class.

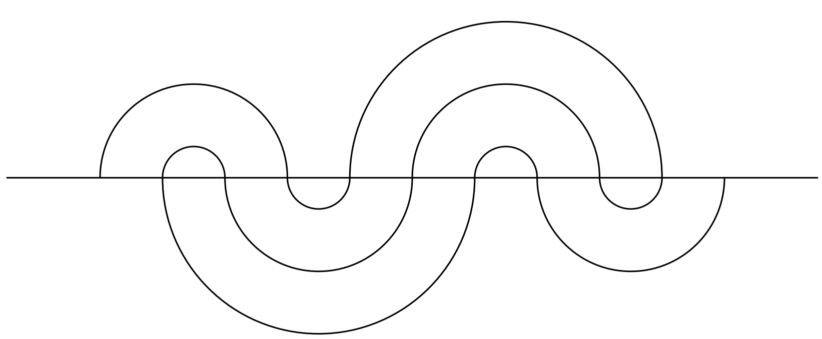


Figure 7: "nonpitchforkable" meandric curve

In this example there are no three intersection points consecutive both along the line and along the curve. A list of examples in [Fi94] shows that one needs at least eleven equilibria to construct such an example. The bifurcation which has to be understood additionally in these examples is the saddle-node bifurcation. But there, one has no a priori information how the new arising equilibria are connected with all other equilibria.

In this context, it was an important question for which class of permutations there exist nonlinearities f, realizing them as permutations of the equilibria solutions.

Theorem 5.1: *For each planar permutation* π *with* $2N+1$ *intersection points and* $\pi(1)=1, \quad \pi(2N+1)=2N+1$*, there exists a corresponding* f_π *if and only if*

$$i(v) \geq 0 \quad \textit{for all} \quad v.$$

We call these permutations *Morse-permutations*. For a proof of the theorem, see [FR98]. Some combinatorial aspects of this problem are treated already in [Wo95].

This theorem allows not only to prove the existence of infinitely many "nonpitchforkable" attractors but also opens the possibility for a complete classification of all attractors in terms of the admissible permutations. But the relation between the permutations and the corresponding attractors is not yet completely understood. At the one hand, we have:

Theorem 5.2: *If for two nonlinearities* f_1 *and* f_2 *the corresponding permutations are equal, then the attractors* A_1 *and* A_2 *are* C^0*-orbit equivalent.*

For a proof see [FR96b].

At the other hand there are a lot of different permutations leading to C^0-orbit equivalent attractors:

Transforming u into $-u$ leads to a reversed numbering of the equilibria and a rotation by the angle π of the meandric curve. Transforming x into $1-x$ interchanges the two boundaries and hence the role of the line and the curve in the meandric permutation. It is not difficult to see that this means exactly to take the inverse permutation. These transformations have already been mentioned in [FR96a].

But unfortunately there are also more complicated transformations. Imagine for instance a meandric curve composed by two subsequent curves. Now, you can pass to the inverse permutation only in one of the two parts. For a restricted class of permutations, all transformations of this type were investigated in [Wo95].

So, up to now it is not clear which attractors are really C^0-orbit equivalent and, moreover, whether this is the right concept for equivalence in this context.

In [FR96a] has been suggested another concept of equivalence, based exactly on the information, provided by the theory in that paper: Two attractors are called *connection equivalent*, if there is a bijection σ between the sets of equilibria

$$\sigma : E_1 \longrightarrow E_2$$

preserving Morse-indices and connections:

$$i(v) = i(\sigma(v)) \quad \text{and} \quad v \searrow w \Leftrightarrow \sigma(v) \searrow \sigma(w)$$

This concept has the advantage that one can decide on equivalence directly from the permutation, computing the Morse-indices and all connections. But it was already noticed that there are some strange examples where two totally different, not even conjugate permutations lead to equivalent attractors.

We want to study now such an example in detail, pointing out, how our theorem about connections in strong-stable and strong-unstable manifolds leads to an explanation of the problem.

Example 5.3: First we compute for the two permutations π_1 and π_2 with the method from [FR96a] all connections and represent them in connection graphs C_1 and C_2. In these graphs the horizontal level corresponds to the Morse-index and connections between consecutive levels are indicated by arrows. All other connections follow from the cascading theorem 3.8.

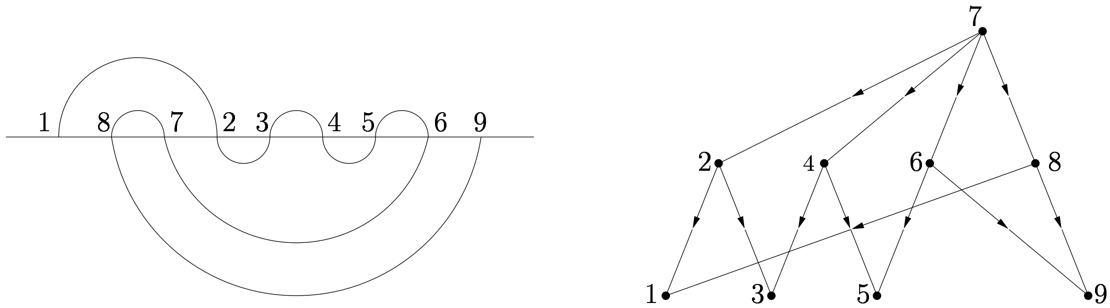


Figure 8: permutation $\pi_1 = (2\,4\,6\,8)(3\,5\,7)$ and connection graph C_1

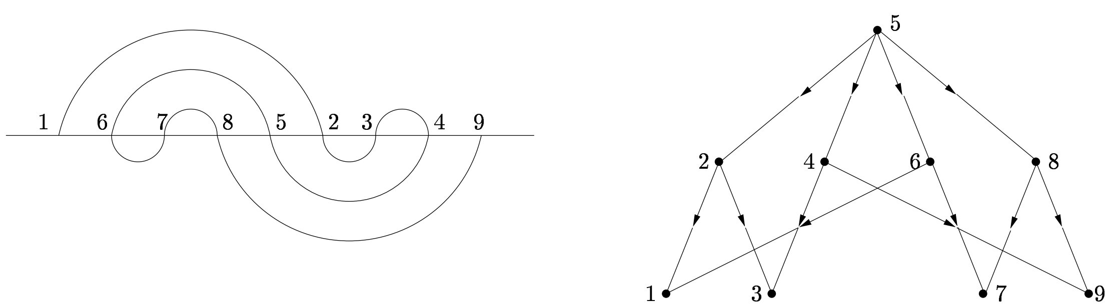


Figure 9: permutation $\pi_2 = (2\,6)(4\,8)(3\,7)$ and connection graph C_2

Note that the two connection graphs C_1 and C_2 are isomorphic via

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 9 & 8 & 5 & 6 & 7 \end{pmatrix}.$$

So, the attractors are connection equivalent in the sense of Fiedler and Rocha.

Now, we investigate the corresponding attractors A_1 and A_2, using the pitchfork method we mentioned above. First, we notice that both π_1 and π_2 can be obtained by one single subcritical bifurcation from the permutation π_0:

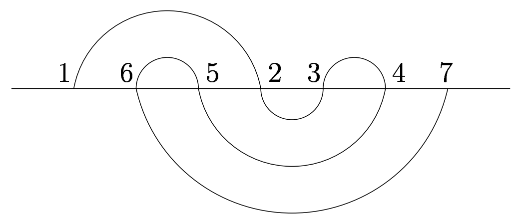


Figure 10: permutation $\pi_0 = (2\,4\,6)(3\,5)$

To get π_1, the bifurcation has to take place in equilibrium 2 or 4 of π_0; to get π_2, in equilibrium 6. Constructing the corresponding attractor A_0 by subsequent pitchfork bifurcations, one obtains the following picture:

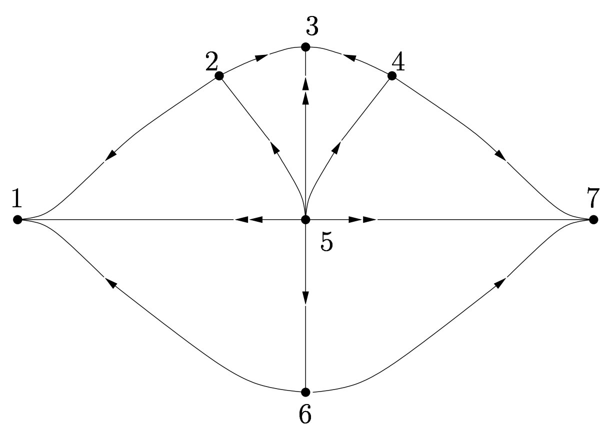


Figure 11: Attractor A_0

If we carry out the two different bifurcations leading either to A_1 or A_2, we get attractors which don't look very similar (see figures 12,13).

This can be explained as follows: Looking only at the connections $v \searrow w$ with $i(v) - i(w) = 1$, there is no difference between A_1 and A_2. But according to corollary 4.2, there are also unique trajectories, connecting equilibria with $i(v) - i(w) > 1$. In these cases, due to theorem 4.1, the zero number $z(v - w)$ decides in which invariant manifolds they are contained. But one can easily see that in this example there exists no bijection σ, satisfying also

$$z(v - w) = z(\sigma(v) - \sigma(w))$$

for all v, w. So any bijection maps connections of different type onto each other. Although both attractors are apparently even C^0-orbit equivalent,

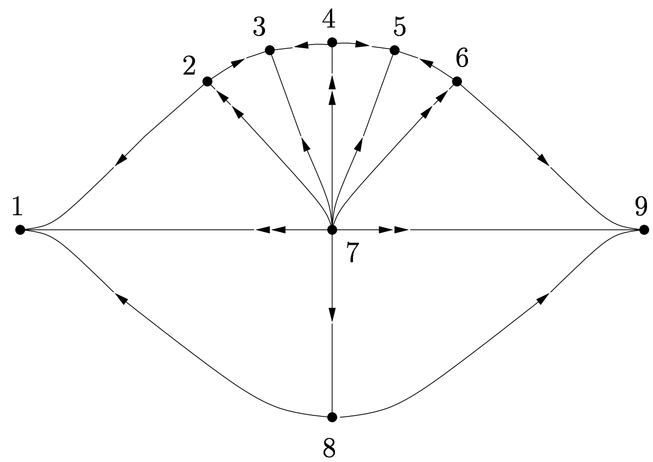


Figure 12: Attractor A_1

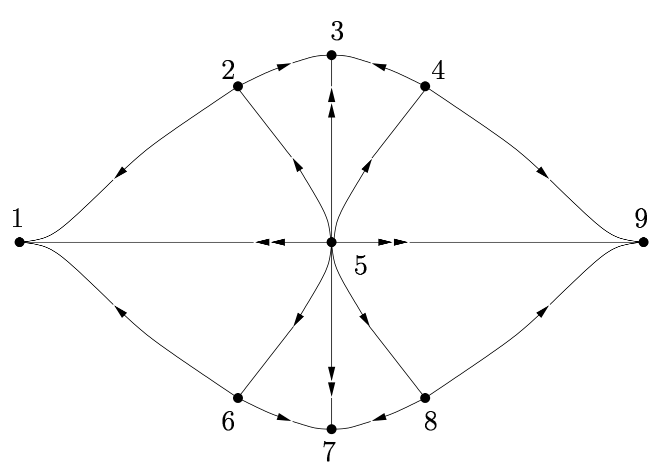


Figure 13: Attractor A_2

there can be no homeomorphism, preserving also the structure of strong-stable and strong-unstable manifolds at the equilibria. We regard this, however, together with the Sturm-Liouville property of the linearization as crucial for a description of the attractor an hence we define:

Definition 5.4: *Two attractors A_1 and A_2 are called Sturm-equivalent, if there is a bijection σ between the sets of equilibria, preserving Morse-indices, connections and zero-numbers $z(v-w)$ for connected equilibria.*

This equivalence can also be decided only from the permutation and uses apparently all information which is available on the ODE-level of the stationary equation. A combinatorial description of the classes of permutations, leading to equivalent attractors seems to be possible. Whether however, this notion of equivalence implies already C^0-orbit equivalence or even C^0-flow equivalence, remains an open question.

Further examples

The first nontrivial example is the cubic nonlinearity depending only on u which has been studied by N.Chafee and E.Infante in [CI74]:

$$f(u) = \mu^2 u(1 - u^2).$$

For $N < \mu < N + 1$, there are $2N + 1$ equilibria, one with index $N + 1$ and two for each index $0 \leq i(v) \leq N$. With increasing μ, all equilibria arise by supercritical pitchfork bifurcations of the central equilibrium. Pictures of the corresponding attractors can be found already in [Hen81].

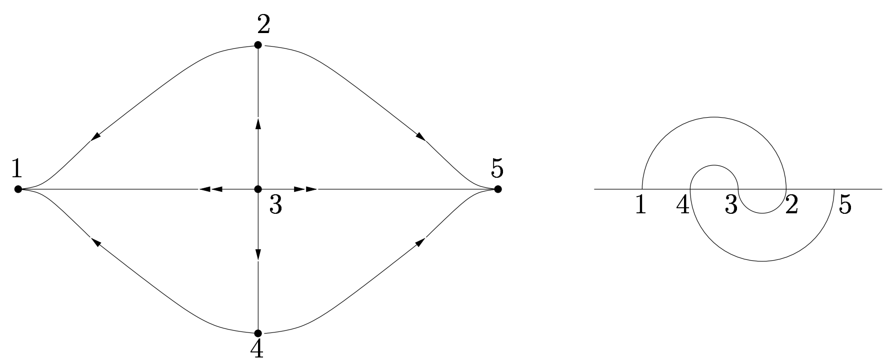
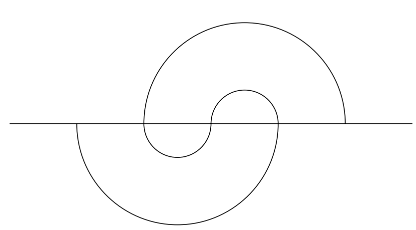


Figure 14: The Chafee-Infante attractor with 5 equilibria and the corresponding meandric curve

According to theorem 5.1, the main condition for the existence of a nonlinearity f_π realizing a given permutation π is that all winding-numbers i.e.. Morse-indices are nonnegative. So, for instance a curve



with winding-numbers $(0, -1, -2, -1, 0)$ can be realized only as a part of another curve, lifting these winding numbers to be nonnegative:

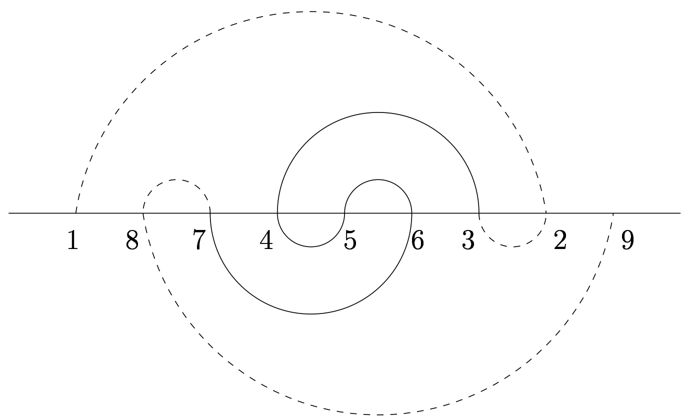


Figure 15: $\pi = (2\,8)(3\,7)(4\,6)$

Studying the corresponding attractor, one can realize that the reflected Chafee-Infante curve $\{3, 4, 5, 6, 7\}$, now with Morse-indices $(i(v_3), \ldots, i(v_7)) = (2, 1, 0, 1, 2)$, corresponds exactly to a Chafee-Infante attractor with time-reversed flow. This time-reversed Chafee-Infante attractor is repelling in some directions, and so, the whole attractor has to contain some more equilibria $\{1, 2, 8, 9\}$, corresponding to the "lifting part" in the meandric curve.

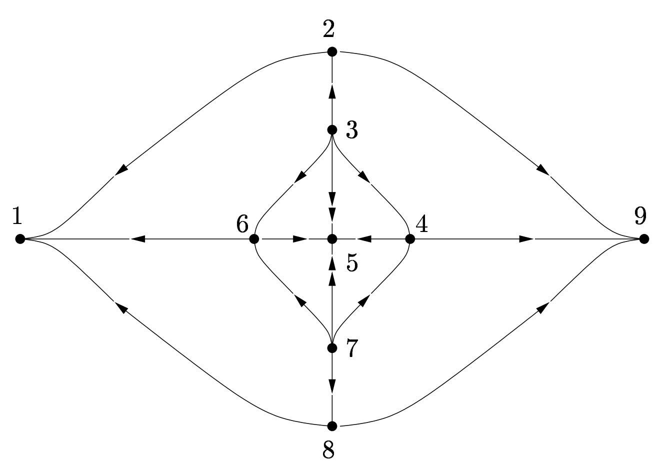


Figure 16: attractor, containing a time-reversed Chafee-Infante flow

Note that for all connections $v \searrow w$ in the original Chafee-Infante attractor we have $z(v - w) = i(w)$, whereas in the reversed attractor we have $z(v - w) = i(v) - 1$.

Starting from the reflected meandric curve, one can obtain in this way for any attractor the time reversed flow, being included in some other attractor. Of course, we can prove this only in terms of Sturm-equivalence.

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References

[An86] S. Angenent, *The Morse-Smale property for a semilinear parabolic equation*, J. of Diff. Equ. 62, (1986), 427-442.

[Arn88] V.I.Arnol'd, *A branched covering of* $\mathbf{CP^2} \to \mathbf{S^4}$*, hyperbolicity and projective topology*, Siberian Math.J. 29 (1988), 36-47.

[BF86] P. Brunovský, B. Fiedler, *Numbers of zeros on invariant manifolds in reaction-diffusion equations*, Nonlin. Analysis 10, (1986), 427-442.

[BF88] P. Brunovský, B. Fiedler, *Connecting orbits in scalar reaction-diffusion equations*, Dynamics Reported 1, (1988), 57-89.

[BF89] P. Brunovský, B. Fiedler, *Connecting orbits in scalar reaction-diffusion equations II: The complete solution*, J. of Diff. Equ. 81, (1989), 106-135.

[BV89] A.V. Babin, M.I. Vishik *Attractors in Evolutionary Equations*, Nauka, Moscow, 1989.

[CI74] N.Chafee, E.Infante *A bifurcation problem for a nonlinear parabolic problem*, J. Applicable Analysis 4 (1974), 17-37.

[FR96a] B.Fiedler, C.Rocha, *Heteroclinic orbits of semilinear parabolic equations*, J. of Diff. Equ. 125, no.1, (1996), 239-281.

[FR96b] B.Fiedler, C.Rocha, *Orbit equivalence of global attractors of semilinear parabolic equations*, preprint, FU-Berlin, (1996).

[FR98] B.Fiedler, C.Rocha, *Realization of Meander Permutations by Boundary Value Problems*, preprint, (1998).

[Fi94] B.Fiedler, *Global attractors of one-dimensional parabolic equations: sixteen examples*, Tatra Mountains Math. Publ. 4 (1994), 67-92.

[Fi96] B.Fiedler, *Do global attractors depend on boundary conditions ?*, Doc. Math. 1, (1996), 215-228.

[FuRo91] G.Fusco, C.Rocha, *A permutation related to the dynamics of a scalar parabolic PDE*, J. of Diff. Equ. 91 (1991), 111-137.

[Ha85] J.K.Hale, *Asymptotic behavior and dynamics in infinite dimensions*, Res.Notes in Math. 132, London (1985), 1-41.

[Har64] P. Hartman *Ordinary Differential Equations*, Willey: New York (1964).

[Hen81] D. Henry *Geometric theory of semilinear parabolic equations*, Lect. Notes in Math. 840, New York (1981).

[Hen85] D. Henry *Some infinite dimensional Morse-Smale systems defined by parabolic equations*, J. of Diff. Equ. 59 (1985), 165-205.

[Jo89] M.S. Jolly *Explicit Construction of an Inertial Manifold for a Reaction Diffusion Equation*, J. of Diff. Equ. 78, (1989), 220-261.

[K68] J.E.Koehler, *Folding a strip of stamps*, J. of Combin. Theory 5 (1968), 135-152.

[LS92] S.K. Lando, A.K. Zvonkin, *Meanders*, Selecta Math. Soviet. 11 (2),(1992), 117-144.

[LS93] S.K. Lando, A.K. Zvonkin, *Plane and projective meanders*, Theor.Comp.Science 117 (1993),227-241.

[Mat82] H. Matano *Nonincrease of the lap-number of a solution for a one-dimensional semi-linear parabolic equation*, J. Fac. Sci. Univ. Tokyo Sec. IA 29, (1982), 401-441.

[PS70] J. Palis, S. Smale, *Structural stability theorems*, in *Global Analysis* Proc. Symp. in Pure Math. XIV. AMS, Providence, 1970.

[Poin12] H.Poincaré, *Sur un théorème de géométrie*, Rend.del Circ.Mat.Palermo 33 (1912),375-407.

[Ro90] C.Rocha, *Properties of the attractor of a scalar parabolic PDE*, J. of Dyn. and Diff. Equ. 3, (1991), no. 4, 575-591.

[Ro88] C.Rocha, *Examples of attractors in scalar reaction-diffusion equations*, J. of Diff. Equ. 73 (1988), 178-195.

[R84] P.Rosenstiehl, *Planar permutations defined by two intersecting Jordan curves*, in: Graph Theory and Combinatorics (Academic Press, London 1984).

[Wo95] M. Wolfrum, *Attraktoren von semilinearen parabolischen Differentialgleichungen und Mäander*, Diplomarbeit, Freie Universität Berlin, 1995.

[Zel68] T.J.Zelenyak, *Stabilisation of solutions of BVP for a second order parabolic equation with one space variable*, Diff. Equ. 4, (1968), 17-22.