Vibrations with unilateral constraints: an overview of M. Schatzman's contributions
Part II: deformable bodies

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State variables

$u : \Omega\times[0,T] \to \mathbb{R}$ displacement

Applied field

$\ell : \Omega\times[0,T] \to \mathbb{R}$ density forces

Notations:

- $\Omega \stackrel{\text{def}}{=} [0,\infty)\times\mathbb{R}^{d-1}$ and $\Sigma \stackrel{\text{def}}{=} \{0\}\times\mathbb{R}^{d-1}$
- $x \stackrel{\text{def}}{=} (x_1, x')$ with $x' \stackrel{\text{def}}{=} (x_2, \dots, x_d)$
- $K \stackrel{\text{def}}{=} \left\{v \in \mathrm{H}^1_{\text{loc}}(\Omega\times[0,\infty)) : v|_{\{0\}\times\mathbb{R}^{d-1}} \geq 0\right\}$

We consider the following problem **(DI)**:

$u_{tt} - \Delta u = \ell,\ x \in \Omega,\ t > 0$	wave equation
$0 \leq u(0,x',t) \perp -u_{x_1}(0,x',t) \geq 0$	unilateral bdry conditions
$u(\cdot,0) = u_0$ and $u_t(\cdot,0) = u_1$	Cauchy initial data

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We consider the following problem **(DI)**:

$$u_{tt} - \Delta u = \underbrace{\ell}_{\in \mathrm{H}^{3/2}(\Omega\times[0,T])}, \; x \in \Omega, \; t > 0 \qquad \text{wave equation}$$

$$0 \leq u(0,x',t) \perp -u_{x_1}(0,x',t) \geq 0 \qquad \text{unilateral bdry conditions}$$

$$u(\cdot,0) = \underbrace{u_0}_{\in \mathrm{H}^1_0(\Omega)\cap \mathrm{H}^{3/2}(\Omega)} \quad \text{and} \quad u_t(\cdot,0) = \underbrace{u_1}_{\in \mathrm{H}^{1/2}(\Omega)} \qquad \text{Cauchy initial data}$$

Theorem (Existence and uniqueness results[1])

There exists $u \in \mathrm{L}^\infty(0, T; \mathrm{H}^{3/2}(\Omega)) \cap \mathrm{W}^{1,\infty}(0, T; \mathrm{H}^{1/2}(\Omega))$ *a unique solution to problem* **(DI)**, *which satisfies the identity, for all* $\tau \in [0, T]$,

$$\frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) \mathrm{d}x = \int_\Omega \left(|u_1|^2 + |\nabla u_0|^2\right) \mathrm{d}x + \int_0^\tau \int_\Omega \ell \cdot u_t \, \mathrm{d}x \, \mathrm{d}t.$$

Idea of the proof: The problem **(DI)** can be reduced to a problem on the boundary, which involves the operator $\mathcal{A}$ defined by

$$u_{tt} - \Delta u = \ell,\ x \in \Omega,\ t > 0,$$
$$u_0 = u_1 = 0,\ x \in \Omega$$
$$u(0, x', t) = v(x', t),\ x' \in \Sigma \quad \text{and} \quad (\mathcal{A}v)(x', t) = u_{x_1}(0, x', t)$$

We can show that $\mathcal{A} : \mathrm{L}^2((0, T) \times \mathbb{R}^{d-1}) \to \mathrm{L}^2((0, T) \times \mathbb{R}^{d-1})$ is positive

$\Longrightarrow$ Existence and uniqueness results

[1] Lebeau, Schatzman. *J. Diff. Equ.*, 1984.

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We consider the following problem **(DI)**:

$u_{tt} - \Delta u - \alpha\Delta u_t = \ell,\ \alpha > 0,\ x\in\Omega,\ t>0$	damped wave equation
$0 \leq u(0,x',t) \perp (u_{x_1} + \alpha u_{x_1 t})(0,x',t) \geq 0$	unilateral bdry conditions
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$$u_{tt} - \Delta u - \alpha\Delta u_t = \underbrace{\ell}_{\in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))}, \alpha > 0,\ x \in \Omega,\ t > 0 \quad \text{damped wave equation}$$

$$0 \leq u(0, x', t) \perp (u_{x_1} + \alpha u_{x_1 t})(0, x', t) \geq 0 \quad \text{unilateral bdry conditions}$$

$$u(\cdot, 0) = \underbrace{u_0}_{\in \mathrm{H}^2(\Omega)} \quad \text{and} \quad u_t(\cdot, 0) = \underbrace{u_1}_{\in \mathrm{H}^1(\Omega)} \quad \text{Cauchy initial data}$$

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The weak formulation **(VI)** is given by

Find $u \in K$ such that for all $v \in K$ and for all $\tau \in (0,\infty)$, we have

$$\int_\Omega (u_t(v-u))\Big|_0^\tau \mathrm{d}x - \int_0^\tau \int_\Omega u_t(v_t-u_t)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega (\nabla u+\alpha\nabla u_t)(\nabla v-\nabla u)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\tau \int_\Omega \ell(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

Penalized Problem (PP) (here $r^- \stackrel{\text{def}}{=} -\min(r,0)$)

$u^\epsilon_{tt} - \Delta u^\epsilon - \alpha \Delta u^\epsilon_t = \ell,\ \alpha > 0\, x \in \Omega,\ t > 0$	damped wave equation
$(u^\epsilon_{x_1} + \alpha u^\epsilon_{x_1 t})(0, x', t) = (u^\epsilon(0, x', t))^- / \epsilon$	normal compliance cond.[2]
$u^\epsilon(\cdot, 0) = u_0$ and $u^\epsilon_t(\cdot, 0) = u_1$	Cauchy initial data

[2] Martins, Oden. *Nonlinear Anal.*, 1988.

[3] Jarušek. *Czechoslovak Math. J.*, 1996.

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Theorem (Existence and uniqueness results[3])

There exists a unique weak solution $u^\epsilon \in \mathrm{H}^1_{\mathrm{loc}}(\Omega \times [0, \infty))$ *of the problem* **(PP)** *such that* $\nabla u^\epsilon_t \in \mathrm{L}^2_{\mathrm{loc}}([0, \infty); \mathrm{L}^2(\Omega))$.

Idea of the proof: Use Galerkin method.

[2] Martins, Oden. *Nonlinear Anal.*, 1988.

[3] Jarušek. *Czechoslovak Math. J.*, 1996.

Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates give us

- $u^\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $u^\epsilon_t \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\nabla u^\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\nabla u^\epsilon_t \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\Delta u^\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$

Moreover $u_t \in \mathrm{C}^0([0,\infty);\mathrm{L}^2(\Omega))$ equipped with the weak topology.

Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_\Omega u_t^\epsilon \varphi(v-u^\epsilon)\Big|_0^\tau \mathrm{d}x - \int_0^\tau \int_\Omega u_t^\epsilon (\varphi(v-u^\epsilon))_t \,\mathrm{d}x\,\mathrm{d}t$$
$$-\frac{1}{\epsilon}\int_0^\tau \int_\Sigma \left(\left((u^\epsilon)^-\right)^2 \varphi\right) \mathrm{d}x'\,\mathrm{d}t - \frac{1}{\epsilon}\int_0^\tau \int_\Sigma \left((u^\epsilon)^- \varphi v\right) \mathrm{d}x'\,\mathrm{d}t$$
$$+\int_0^\tau \int_\Omega (\nabla u^\epsilon + \alpha \nabla u_t^\epsilon)\nabla(\varphi(v-u^\epsilon))\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_\Omega \ell \varphi(v-u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

where $\varphi \in \mathrm{C}_0^\infty(\bar{\Omega} \times [0,\infty))$ taken its values in $[0,1]$.

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There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_\Omega u_t^\epsilon \varphi(v-u^\epsilon)\big|_0^T \,\mathrm{d}x - \int_0^T \int_\Omega u_t^\epsilon (\varphi(v-u^\epsilon))_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega (\nabla u^\epsilon + \alpha \nabla u_t^\epsilon)\nabla(\varphi(v-u^\epsilon))\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_\Omega \ell \varphi(v-u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

where $\varphi \in \mathrm{C}_0^\infty(\bar{\Omega} \times [0,\infty))$ taken its values in $[0,1]$.

Remark: Nothing is known about uniqueness.

AIM: Characterize the trace spaces

Recall that $u(0,x',t) \in \mathrm{H}^{a,b}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty))$ for $(a,b)\in\mathbb{R}^2$

$$\Updownarrow$$

$$|\xi|^a\widehat{u}(0,\xi,\omega) \in \mathrm{L}^2(\mathbb{R}^d) \text{ and } |\omega|^b\widehat{u}(0,\xi,\omega) \in \mathrm{L}^2(\mathbb{R}^d)$$

- $\xi \stackrel{\text{def}}{=} (\xi_2,\ldots,\xi_d)^\mathsf{T}$: the dual variable to $x' = (x_2,\ldots,x_d)^\mathsf{T}$
- ω : the dual variable to t
- $\widehat{u}(0,\xi,\omega)$: the Fourier transform of $u(0,x',t)$

Lemma (Regularity of the trace[4])

Let u^ϵ be the solution of **(PP)**. *Then we may extract a subsequence, still denoted by u^ϵ such that*

$$u^\epsilon(0,x',t) \rightharpoonup u(0,x',t) \quad \textit{weakly} \quad \textit{in} \quad \mathrm{H}^{1/2,5/4}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty)).$$

Moreover u is a strong solution of **(DI)**.

[4] P., Schatzman. *SIAM J. Math. Anal.*, 2009.

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- Introduce $\bar{u}$ solution of **(DI)** with the Dirichlet boundary conditions

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- Introduce $\bar{u}$ solution of **(DI)** with the Dirichlet boundary conditions
- Let $v^\epsilon \overset{\text{def}}{=} \mathrm{e}^{-\nu t}(u^\epsilon - \bar{u})$, $\nu > 0$, be a solution of

$$(\nu+\partial_t)^2 v^\epsilon - (1+\alpha(\nu+\partial_t))\Delta v^\epsilon = 0, \quad x \in \Omega, \quad t > 0$$

$$(1+\alpha(\nu+\partial_t))v^\epsilon_{x_1}(0, x', t) = \mathrm{e}^{-\nu t}\bar{g} - (v^\epsilon(0, x', t) + \mathrm{e}^{-\nu t}\bar{u}(0, x', t))^- / \epsilon$$

$$v^\epsilon(\cdot, 0) = 0 \quad \text{and} \quad v^\epsilon_t(\cdot, 0) = 0$$

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$$v^\epsilon(\cdot,0) = 0 \quad \text{and} \quad v^\epsilon_t(\cdot,0) = 0$$

- Use the partial Fourier transform in x' and t

$$\widehat{v}^\epsilon_{x_1x_1}(x_1,\xi,\omega) = \widehat{\lambda}^2\widehat{v}^\epsilon(x_1,\xi,\omega) \quad \text{where} \quad \widehat{\lambda} \stackrel{\text{def}}{=} \sqrt{|\xi|^2 + \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}}$$

$$\rightsquigarrow \widehat{v}^\epsilon(x_1,\xi,\omega) = \widehat{a}^\epsilon \mathrm{e}^{\widehat{\lambda}x_1}$$

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$$(1+\alpha(\nu+\partial_t))v^\epsilon_{x_1}(0,x',t) = \mathrm{e}^{-\nu t}\bar{g} - (v^\epsilon(0,x',t)+\mathrm{e}^{-\nu t}\bar{u}(0,x',t))^-/\epsilon$$
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$$\rightsquigarrow \widehat{v}^\epsilon(x_1,\xi,\omega) = \widehat{a}^\epsilon \mathrm{e}^{\widehat{\lambda}x_1}$$

- Use the boundary conditions

$$((1+\alpha(\nu+\partial_t))v^\epsilon_{x_1}\widehat{)}(0,\xi,\omega) = \widehat{\lambda}_1\widehat{v}^\epsilon(0,\xi,\omega) \text{ where } \widehat{\lambda}_1 \stackrel{\text{def}}{=} (1+\alpha(\nu+i\omega))\widehat{\lambda}$$

$$\rightsquigarrow \lambda_1 * v^\epsilon(0,x',t) = \mathrm{e}^{-\nu t}\bar{g} + (v^\epsilon(0,x',t)+\mathrm{e}^{-\nu t}\bar{u}(0,x',t))^-/\epsilon$$

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Notations:

- ▷ $\varepsilon_{ij}(u) \stackrel{\text{def}}{=} (u_{j,x_i} + u_{i,x_j})/2$ and $a^n_{ijkl} \stackrel{\text{def}}{=} \lambda^n \delta_{ij}\delta_{kl} + 2\mu^n \delta_{ik}\delta_{jl}$ for $n = 0, 1$
- ▷ $K \stackrel{\text{def}}{=} \left\{ v \in \mathbf{H}^1(\Omega \times (0,\tau)) : \nabla v_t \in \mathbf{L}^2(\Omega \times (0,\tau)), v(0,\cdot) \leq 0 \right\}$

We consider the following problem **(DI)**:

$$\rho u_{tt} - a^n_{ijkl}\partial_j \varepsilon_{kl}(u) - a^n_{ijkl}\partial_j \varepsilon_{kl}(u_t) = \ell, \quad x \in \Omega, \quad t > 0$$

$$0 \geq u_1 \perp a^0_{11kl}\varepsilon_{kl}(u) + a^1_{11kl}\varepsilon_{kl}(u_t) \leq 0 \quad \text{on} \quad \Sigma \times [0, \infty)$$

$$a^0_{12kl}\varepsilon_{kl}(u) + a^1_{12kl}\varepsilon_{kl}(u_t) = 0 \quad \text{and} \quad a^0_{13kl}\varepsilon_{kl}(u) + a^1_{13kl}\varepsilon_{kl}(u_t) = 0$$

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The weak formulation **(VI)** is given by

Find $u \in K$ such that for all $v \in K$ and for all $\tau \in (0,\infty)$, we have

$$\int_0^\tau \int_\Omega \rho u_{tt}\cdot(v-u)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega a^0_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega a^1_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v-u)\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^\tau \int_\Omega \ell\cdot(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

Penalized Problem (PP) (here $r^+ \stackrel{\text{def}}{=} \max(r,0)$)

$$\rho u^\epsilon_{tt} - a^n_{ijkl}\partial_j \varepsilon_{kl}(u^\epsilon) - a^n_{ijkl}\partial_j \varepsilon_{kl}(u^\epsilon_t) = \ell, \quad x \in \Omega, \quad t > 0$$

$$a^0_{11kl}\varepsilon_{kl}(u^\epsilon) + a^1_{11kl}\varepsilon_{kl}(u^\epsilon_t) = -(u^\epsilon_1)^+/\epsilon \quad \text{on} \quad \Sigma \times [0,\infty)$$

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Theorem (Existence and uniqueness results[5])

There exists a unique weak solution $u^\epsilon \in \mathbf{H}^1_{\mathrm{loc}}([0, \infty) \times \Omega)$ *of the problem* **(PP)** *such that* $\nabla u^\epsilon_t \in \mathbf{L}^2_{\mathrm{loc}}([0, \infty); \mathrm{L}^2(\Omega))$.

Idea of the proof: Use Galerkin method.

[5] Jarušek. *Czechoslovak Math. J.*, 1996.

Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)**.

Sketch of the proof: A-priori estimates and Korn's inequality lead to

- $u^\epsilon \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $u^\epsilon_t \in \mathbf{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\nabla u^\epsilon \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $a^n_{ijkl}\partial_j\varepsilon_{kl}(u) \in \mathbf{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$ for $n = 0, 1$
- $\nabla u^\epsilon_t \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$

Moreover $u^\epsilon \longrightarrow u$ in $\mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$.

Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_\Omega \rho u_t^\epsilon \cdot (\varphi(v - u^\epsilon))|_0^\tau \, dx - \int_0^\tau \int_\Omega \rho u_t^\epsilon \cdot (\varphi(v - u^\epsilon))_t \, dx \, dt$$

$$+ \int_0^\tau \int_\Omega (a^0_{ijkl} \varepsilon_{kl}(u^\epsilon) \varepsilon_{ij}(u^\epsilon) + a^1_{ijkl} \varepsilon_{kl}(u_t^\epsilon) \varepsilon_{ij}(u^\epsilon))(\varphi(v_i - u_i^\epsilon)) \, dx \, dt$$

$$- \frac{1}{\epsilon} \int_0^\tau \int_\Sigma ((u_1^\epsilon)^+)^2 \varphi \, dx' \, dt + \frac{1}{\epsilon} \int_0^\tau \int_\Sigma (u_1^\epsilon)^+ \varphi v_1 \, dx' \, dt$$

$$= \int_{Q_\tau} \ell \cdot (\varphi(v - u^\epsilon)) \, dx \, dt$$

where $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ taken its values in $[0, 1]$.

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$$+ \int_0^\tau \int_\Omega (a^0_{ijkl}\varepsilon_{kl}(u^\epsilon)\varepsilon_{ij}(u^\epsilon) + a^1_{ijkl}\varepsilon_{kl}(u_t^\epsilon)\varepsilon_{ij}(u^\epsilon))(\varphi(v_i - u_i^\epsilon))\,\mathrm{d}x\,\mathrm{d}t$$
$$\geq \int_{Q_\tau} \ell \cdot (\varphi(v-u^\epsilon))\,\mathrm{d}x\,\mathrm{d}t$$

where $\varphi \in \mathrm{C}_0^\infty(\bar\Omega \times [0,\infty))$ taken its values in $[0,1]$.

Remark: Nothing is known about uniqueness.

Lemma (Regularity of the trace[6])

Let $u^\epsilon = (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)^\mathsf{T}$ *be the solution of* **(PP)**. *Then we may extract a subsequence, still denoted by* u_1^ϵ *such that*

$$u_1^\epsilon(0, x', t) \rightharpoonup u_1(0, x', t) \quad \text{weakly} \quad \text{in} \quad \mathrm{H}_{\mathrm{loc}}^{1/2,5/4}(\mathbb{R}^{d-1} \times [0, \infty)).$$

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Remark: We were unable to prove that the energy loss is purely viscous[7] for the damped wave equation and the evolution of a Kelvin-Voigt material.

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Thank you for your attention !

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