Global existence result for rate-independent processes in viscous solids with heat transfer

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Mathematics for key technologies

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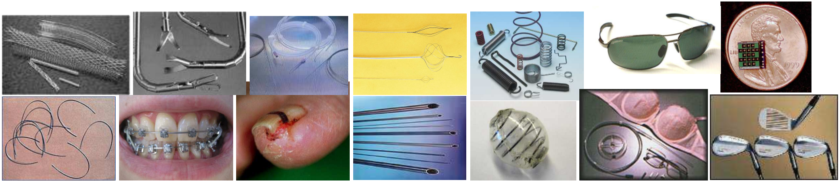
Shape-Memory Alloys are used today in real life:



WHY?

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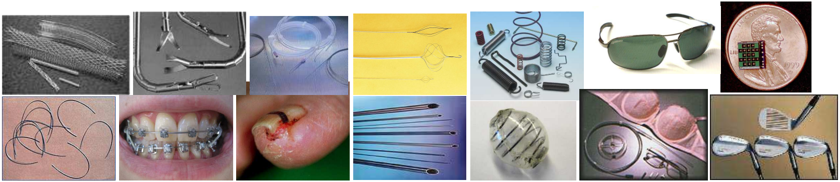

WHY? Because their have the following properties:

- shape memory under heating and cooling
- superelastic properties under mechanical loading
- hysteretic behavior for damping of vibrations

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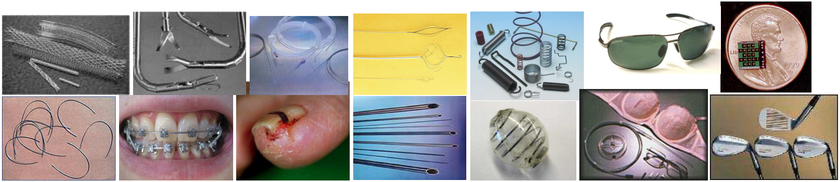

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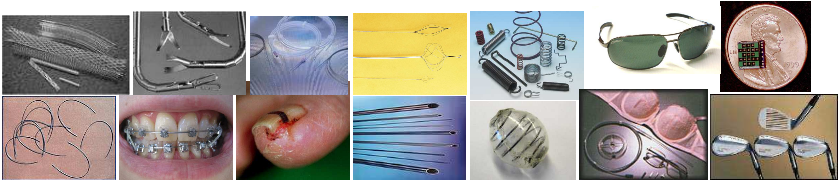

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AIM: Find good mathematical models (analysis and numerics)
$\Longrightarrow$ Souza-Auricchio model for shape-memory alloys[1]

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State variables

$u : \Omega\times(0,T) \to \mathbb{R}^3$ displacement

$z : \Omega\times(0,T) \to \mathbb{R}^{3\times 3}_{\text{dev}}$ phase indicator

$\theta : \Omega\times(0,T) \to \mathbb{R}$ temperature

Applied field

$\ell : \Omega\times(0,T) \to \mathbb{R}^3$ mech. load.

We consider the problem $(\mathbf{P_{uz\theta}})$

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u)-z)+\alpha\theta \boldsymbol{I}+\mathbb{L}\mathrm{e}(\dot u)) = \ell \qquad \text{momentum equilibrium equ.}$$

$$\partial\Psi(\dot z)+\mathbb{M}\dot z-\mathbb{E}(\mathrm{e}(u)-z)+\mathrm{D}_z H_1(z)+\theta \mathrm{D}_z H_2(z)-\nu\Delta z \ni 0 \qquad \text{flow rule}$$

$$c(\theta)\dot\theta-\operatorname{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)$$
$$= \mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)+\theta(\alpha \mathrm{tr}(\mathrm{e}(\dot u))+\mathrm{D}_z H_2(z):\dot z)+\Psi(\dot z)+\mathbb{M}\dot z:\dot z \qquad \text{heat equ.}$$

$$u_{|\partial\Omega} = 0, \quad \nabla z\cdot\eta_{|\partial\Omega} = 0, \quad \kappa\nabla\theta\cdot\eta_{|\partial\Omega} = 0 \qquad \text{boundary cond.}$$

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \theta(\cdot,0) = \theta^0 \qquad \text{initial cond.}$$

The problem can be rewritten in terms of enthalpy by employing the so-called enthalpy transformation ($\mathbf{P}_{\mathbf{uz}\theta}$)

$$g(\theta) = \vartheta \stackrel{\text{def}}{=} \int_0^\theta c(s)\,\mathrm{d}s$$

Definitions:

- $\zeta(\vartheta) \stackrel{\text{def}}{=} g^{-1}(\vartheta)$ if $\vartheta \geq 0$ and $\zeta(\vartheta) \stackrel{\text{def}}{=} 0$ otherwise
- $\kappa^c(\mathrm{e}(u), z, \vartheta) \stackrel{\text{def}}{=} \frac{\kappa(\mathrm{e}(u), z, \zeta(\vartheta))}{c(\zeta(\vartheta))}$

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$-\,\mathrm{div}(\mathbb{E}(\mathrm{e}(u)-z)+\alpha\zeta(\vartheta)\boldsymbol{I}+\mathbb{L}\mathrm{e}(\dot{u})) = \ell$	momentum equilibrium equ.
$\partial\Psi(\dot{z})+\mathbb{M}\dot{z}-\mathbb{E}(\mathrm{e}(u)-z)+\mathrm{D}_z H_1(z)+\zeta(\vartheta)\mathrm{D}_z H_2(z)-\nu\Delta z\ni 0$	flow rule
$\dot{\vartheta}-\,\mathrm{div}(\kappa^c(\mathrm{e}(u), z, \vartheta)\nabla\vartheta)$ $= \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u})+\zeta(\vartheta)(\alpha\mathrm{tr}(\mathrm{e}(\dot{u}))+\mathrm{D}_z H_2(z){:}\dot{z})+\Psi(\dot{z}) + \mathbb{M}\dot{z}{:}\dot{z}$	heat equ.
$u_{\vert\partial\Omega} = 0, \quad \nabla z\cdot\eta_{\vert\partial\Omega} = 0, \quad \kappa^c\nabla\vartheta\cdot\eta_{\vert\partial\Omega} = 0$	boundary cond.
$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \vartheta(\cdot,0) = \theta^0$	initial cond.

- The dissipation potential $\Psi(z)$:
  - $\exists C^{\Psi} > 0 \forall\gamma \geq 0\ \forall z \in \mathbb{R}^{3\times3}_{\mathrm{dev}}:\ \Psi(\gamma z) = \gamma\Psi(z)$ and $0 \leq \Psi(z) \leq C^{\Psi}|z|$
  - $\forall z_1, z_2 \in \mathbb{R}^{3\times3}_{\mathrm{dev}}:\ \Psi(z_1{+}z_2) \leq \Psi(z_1) + \Psi(z_2)$
- The hardening functionals H_i, $i = 1, 2$:
  - $\exists c^{H_1}, C^{H_i}_{zz} > 0\ \forall z \in \mathbb{R}^{3\times3}_{\mathrm{dev}}:\ H_1(z) \geq c^{H_1}(|z|^2{-}1)$ and $|\mathrm{D}^2_z H_i(z)| \leq C^{H_i}_{zz}$
- The elastic tensor $\mathbb{E}$, $\mathbb{L}$ and $\mathbb{M}$:
  - $\exists c^{\mathbb{E}} > 0\ \forall z \in \mathrm{L}^2(\Omega) : c^{\mathbb{E}}\|z\|^2_{\mathrm{L}^2(\Omega)} \leq \int_\Omega \mathbb{E}z{:}z\,\mathrm{d}x$ and $\mathbb{E}(\cdot), \frac{\partial\mathbb{E}_{i,j}(\cdot)}{\partial x_k} \in \mathrm{L}^\infty(\Omega)$
  - $\exists c^{\mathbb{L}}, C^{\mathbb{L}} > 0\ \forall z \in \mathbb{R}^{3\times3}_{\mathrm{sym}}:\ c^{\mathbb{L}}|z|^2 \leq \mathbb{L}z{:}z \leq C^{\mathbb{L}}|z|^2$
  - $\exists c^{\mathbb{M}}, C^{\mathbb{M}} > 0\ \forall z \in \mathbb{R}^{3\times3}_{\mathrm{dev}}:\ c^{\mathbb{M}}|z|^2 \leq \mathbb{M}z{:}z \leq C^{\mathbb{M}}|z|^2$
- The external loading $\ell \in \mathrm{H}^1(0, T; \mathrm{L}^2(\Omega))$
- The heat capacity c and the conductivity κ^c:
  - $c : [0, \infty) \to [0, \infty)$ is continuous
  - $\beta_1 \geq 2\ \exists c^c > 0\ \forall\theta \geq 0:\ c^c(1{+}\theta)^{\beta_1 - 1} \leq c(\theta)$
  - $\exists c^{\kappa^c}, C^{\kappa^c} > 0 : \kappa^c(e, z, \vartheta)v{\cdot}v \geq c^{\kappa^c}|v|^2$ and $|\kappa^c(e, z, \vartheta)| \leq C^{\kappa^c}$.

The momentum equilibrium equation and the flow rule $(\mathrm{P_{uz}})$

Assumption: $\theta \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$, $q=\beta_1\bar{q}$, $p\in\big[4,\min(\beta_1\bar{p},6)\big]$, $\bar{q}>4$, $\bar{p}=2$

$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u)-z)+\alpha\theta\boldsymbol{I}+\mathbb{L}\mathrm{e}(\dot{u}))=\ell,$	momentum equilibrium equ.
$\partial\Psi(\dot{z})+\mathbb{M}\dot{z}-\mathbb{E}(\mathrm{e}(u)-z)+\mathrm{D}_zH_1(z)+\theta\mathrm{D}_zH_2(z)-\nu\Delta z\ni 0$	flow rule
$u(\cdot,0)=u^0\in\mathrm{H}^1(\Omega)$ and $z(\cdot,0)=z^0\in\mathrm{H}^1(\Omega)$	initial cond.
$u_{\vert\partial\Omega}=0$ and $\nabla z\cdot\eta_{\vert\partial\Omega}=0$	boundary cond.

Definitions:

- $\forall(u,v)\in(\mathrm{H}^1(\Omega))^2:\ \langle\mathcal{A}u,v\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)}\overset{\text{def}}{=}\int_\Omega\nu\mathbb{M}^{-1}\nabla u{:}\nabla v\,\mathrm{d}x$
  $\Longrightarrow$ $\mathcal{A}$ generates an analytic semigroup on $\mathrm{L}^2(\Omega)$, which extends to a C^0-semigroup of contractions on $\mathrm{L}^p(\Omega)$. We denote by $\mathcal{A}_p$ (resp. $\mathcal{A}_{\frac{p}{2}}$) the realization of its generator in $\mathrm{L}^p(\Omega)$ (resp. $\mathrm{L}^{p/2}(\Omega)$)
- $\mathrm{X}_{q,p}(\Omega)\overset{\text{def}}{=}(\mathrm{L}^p(\Omega),\mathcal{D}(\mathcal{A}_p))_{1-\frac{2}{q},\frac{q}{2}}\cap(\mathrm{L}^{p/2}(\Omega),\mathcal{D}(\mathcal{A}_{\frac{p}{2}}))_{1-\frac{1}{q},q}$

The momentum equilibrium equation and the flow rule ($\mathbf{P_{uz}}$)

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Theorem (Existence result for ($\mathbf{P_{uz}}$))

There exists $(u,z) \in \mathrm{H}^1(0, T; \mathrm{H}^1_0(\Omega)\times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0, T; \mathrm{H}^1_0(\Omega)\times \mathrm{H}^1(\Omega))$ *a solution to* ($\mathbf{P_{uz}}$). *Moreover if* $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *then* $\dot{z} \in \mathrm{L}^q(0, T; \mathrm{L}^2(\Omega))$ *and* $z \in \mathrm{L}^q(0, T; \mathrm{H}^2(\Omega)) \cap \mathrm{C}^0([0, T], \mathrm{X}_{q,p}(\Omega))$.

Idea of the proof: Use the Yosida approximation and classical results for PDE and ODE.

The uniform convexity of $H_1 \Rightarrow \exists h_1 \in \mathrm{C}^2(\mathbb{R}^{3\times3}_{\mathrm{dev}};\mathbb{R})$ such that

$$\exists C^{H_1} > 0\, \forall z \in \mathbb{R}^{3\times3}_{\mathrm{dev}} :\ H_1(z) \stackrel{\mathrm{def}}{=} C^{H_1}|z|^2 + h_1(z).$$

Moreover we have

$$\exists C^{h_1} > 0\, \forall z_1, z_2 \in \mathbb{R}^{3\times3}_{\mathrm{dev}} :\ |\mathrm{D}_z h_1(z_1) - \mathrm{D}_z h_1(z_2)| \le C^{h_1}|z_1 - z_2|.$$

Proposition (Uniqueness result ($\mathbf{P_{uz}}$))

The problem ($\mathbf{P_{uz}}$) *admits a unique solution.*

Idea of the proof: The uniqueness result relies on a uniform convexity argument inspired from Mielke&Theil[2] combined with the Grönwall's lemma.

[2] Mielke, Theil. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 2004.

Lemma

The mapping $\vartheta \mapsto (u, z)$ *is continuous from* $\mathrm{L}^{\bar{q}}(0, T; \mathrm{L}^{\bar{p}}(\Omega))$ *into* $\mathrm{H}^1(0, T; \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0, T; \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega))$.

Idea of the proof: Once again the uniform convexity and the Grönwall's lemma are used.

$\rightsquigarrow$ A bounded set of $\mathrm{L}^{\bar{q}}(0, T; \mathrm{L}^{\bar{p}}(\Omega))$ by the mapping $\widetilde{\vartheta} \mapsto \theta \mapsto (u, z)$ is a bounded subset of $\mathrm{H}^1(0, T; \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0, T; \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega))$

AIM: Prove further regularity results for the solutions (u, z).

Notations:

- $\forall r > 1: \ \mathrm{V}^r(\Omega; \mathbb{R}^3) \stackrel{\text{def}}{=} \left\{ u \in \mathrm{L}^2(\Omega; \mathbb{R}^3) : \ \nabla u \in \mathrm{L}^r(\Omega; \mathbb{R}^{3\times 3}) \right\}$
- $\forall r \geq 2: \ \mathrm{V}^r_0(\Omega; \mathbb{R}^3) \stackrel{\text{def}}{=} \left\{ u \in \mathrm{V}^r(\Omega; \mathbb{R}^3) : \ u_{|\partial\Omega} = 0 \right\}$
- $\forall u \in \mathrm{V}^r(\Omega; \mathbb{R}^3): \ \|u\|_{\mathrm{V}^r(\Omega)} \stackrel{\text{def}}{=} \|u\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^r(\Omega)}$

Assumption: $u^0 \in \mathrm{V}^p_0(\Omega; \mathbb{R}^3)$

Lemma (Regularity results for u)

We have $\mathrm{e}(u)$ *belonging to* $\mathrm{W}^{1,q}(0,T;\mathrm{L}^p(\Omega))$ *and* $\theta \mapsto \mathrm{e}(\dot{u})$ *is continuous from* $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^p(\Omega))$ *into* $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^p(\Omega))$ *and maps any bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ *into a bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$.

Idea of the proof: We interpret the momentum equilibrium equation in $(\mathbf{P_{uz}})$ as an ODE for u in an appropriate Banach space.

Lemma (Regularity results for z)

We have $\dot{z}$ *and* Δz *belonging to* $\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))$ *and* $z \in \mathrm{C}^0([0,T],\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))$ *and* $\theta \mapsto (\dot{z},\Delta z, z)$ *maps any bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ *into a bounded subset of* $(\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))\cap\mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega)))^2\times(\mathrm{C}^0([0,T];\mathrm{X}_{q,p}(\Omega))\cap\mathrm{L}^q(0,T;\mathrm{H}^2(\Omega)))$.

Idea of the proof: We use here the embedding theorems and the maximal regularity results for parabolic systems[3].

[3] Hieber, Rehberg. *SIAM J. Math. Anal.*, 2008.

The enthalpy equation $(\mathbf{P}_\vartheta)$

$$\dot\vartheta - \operatorname{div}(\widetilde\kappa^c \nabla\vartheta) = f \qquad \text{heat equ.}$$

$$\kappa\nabla\theta\cdot\eta_{|\partial\Omega} = 0 \qquad \text{boundary cond.} \qquad \vartheta(\cdot,0) = \vartheta^0 \qquad \text{initial cond.}$$

Assumptions:

- $\vartheta^0 \in \mathrm{H}^1(\Omega)$ and $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$
- $\exists c^{\kappa^c} C^{\kappa^c} > 0\ \forall v \in \mathbb{R}^3:\ \widetilde\kappa^c(x,t)v\cdot v \geq c^{\kappa^c}|v|^2$ and $|\widetilde\kappa^c(x,t)| \leq C^{\kappa^c}$

Theorem (Existence and uniqueness for $(\mathbf{P}_\vartheta)$)

The problem $(\mathbf{P}_\vartheta)$ *possesses* $\vartheta \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ *a unique solution such that* $\dot\vartheta \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)'))$ *and*
$$\|\vartheta(\tau)\|^2_{\mathrm{L}^2(\Omega)} + 2c^{\kappa^c}\int_0^\tau \|\nabla\vartheta(t)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \leq \mathrm{e}^\tau\left(\|\vartheta^0\|^2_{\mathrm{L}^2(\Omega)} + \|f\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}\right).$$

The enthalpy equation ($\mathbf{P}_\vartheta$)

$\dot{\vartheta} - \operatorname{div}(\widetilde{\kappa}^c \nabla\vartheta) = f$	heat equ.		
$\kappa\nabla\theta\cdot\eta_{\vert\partial\Omega} = 0$	boundary cond.	$\vartheta(\cdot, 0) = \vartheta^0$	initial cond.

Assumptions:

▷ $\vartheta^0 \in \mathrm{H}^1(\Omega)$ and $f \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))$

▷ $\exists c^{\kappa^c} C^{\kappa^c} > 0\ \forall v \in \mathbb{R}^3 :\ \widetilde{\kappa}^c(x, t)v\cdot v \geq c^{\kappa^c}|v|^2$ and $|\widetilde{\kappa}^c(x, t)| \leq C^{\kappa^c}$

Notations:

▷ $\mathcal{W} \stackrel{\text{def}}{=} \{\vartheta \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega)) :\ \dot{\vartheta} \in \mathrm{L}^2(0, T; (\mathrm{H}^1(\Omega)'))\}$

▷ $\forall\vartheta \in \mathcal{W} :\ \|\vartheta\|_{\mathcal{W}} \stackrel{\text{def}}{=} \|\vartheta\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\vartheta\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \|\dot{\vartheta}\|_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)'))}$

$\Longrightarrow$ $\mathcal{W}$ is compactly embedded in $\mathrm{L}^{\bar{q}}(0, T; \mathrm{L}^{\bar{p}}(\Omega))$[a]

[a] Simon. *Ann. Mat. Pura Applic.*, 1987.

AIM: Existence result for the problem $(\mathbf{P_{uz\theta}})$ by using fixed-point theorem

Proposition

$\phi_\tau : \widetilde{\vartheta} \mapsto \vartheta$ is continuous from $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$.

Idea of the proof: Use the regularity results of u and z.

The image of the closed ball $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ by ϕ_τ is relatively compact in $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$ with $\|\widetilde{\vartheta}\|_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))} \leq R^\vartheta$.

$\Longrightarrow$ The problem $(\mathbf{P_{uz\vartheta}})$ possesses a local solution (u,z,ϑ) on $[0,\tau]$.

Assumption: $g(\theta^0(x)) = \vartheta^0(x) > 0$ a.e. $x \in \Omega$

Theorem (Local existence result)

The problem $(\mathbf{P_{uz\theta}})$ admits a solution on $[0,\tau]$, $\tau \in (0,T]$.

Idea of the proof: Use the Stampacchia's truncation method.

Proposition (Global energy estimate)

For any solution (u, z, ϑ) *of problem* $(\mathbf{P_{uz\theta}})$ *defined on* $[0, \tau]$, $\tau \in (0, T]$:
$\exists \widetilde{C} > 0 \ \forall \widetilde{\tau} \in [0, \tau] : \ \|u(\cdot, \widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|z(\cdot, \widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|\vartheta(\cdot, \widetilde{\tau})\|_{\mathrm{L}^1(\Omega)} \leq \widetilde{C}$.

Idea of the proof: Use the Grönwall's lemma.

Let us define

$$\bar{\tau} \stackrel{\text{def}}{=} \sup\{\tau \in (0, T] : \ \phi_\tau \text{ admits a fixed point in } \bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)\}.$$

By a contradiction argument $\bar{\tau} = T \implies$ the problem $(\mathbf{P_{uz\theta}})$ admits a global solution.

Theorem (Global existence result)

The problem $(\mathbf{P_{uz\theta}})$ *admits a global solution* (u, z, θ) *such that*
$u \in \mathrm{W}^{1,q}(0, T; \mathrm{V}^p_0(\Omega))$, $z \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\Omega) \cap \mathrm{X}_{q,p}(\Omega)) \cap \mathrm{H}^1(0, T; \mathrm{L}^2(\Omega))$,
$\dot{z}, \Delta z \in \mathrm{L}^{q/2}(0, T; \mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0, T; \mathrm{L}^{p/2}(\Omega))$ *and* $\theta \in \mathcal{W}$.

Thank you for your attention !

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