Asymptotic behaviour of a micropolar fluid in a thin domain with rough boundary

L.Paoli

(joint work with M.Boukrouche)

LaMUSE, University of Lyon, France

Weierstraß-Institut für Angewandte Analysis und Stochastik

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I - Description of the problem

The **micropolar fluid** model aims to describe fluids consisting in randomly oriented (or spherical) particles suspended in a viscous medium, when the deformation of the particle is ignored.

References:

A.C.Eringen, *Theory of micropolar fluids*, J. Math. Mech. (1966)

G.Lukaszewicz, *Micropolar fluids. Theory and Applications*, Birkhäuser (1999).

Experimental studies have showned that this model better represents the behaviour of numerous fluids (eg blood or industrial fluids such as polymers, liquid crystals) especially when the characteristic dimension of the flow becomes small.

The **unknowns of the problem** are

- $u = (u_1, u_2, u_3)$ velocity field
- p pressure field
- $\omega = (\omega_1, \omega_2, \omega_3)$ micro-rotation field

and the **equilibrium of momentum, mass and moment of momentum** are given by

$$\begin{aligned}
&u_t - (\nu + \nu_r)\Delta u + (u \cdot \nabla)u + \nabla p = 2\nu_r \mathrm{rot}\omega + f \\
&\mathrm{div}(u) = 0 \\
&\omega_t - \alpha\Delta\omega - \beta\nabla\mathrm{div}\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r \mathrm{rot}u + g
\end{aligned}$$

ν is the usual Newtonian viscosity and ν_r is the micro-rotation viscosity.

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We consider **a lubrication problem**, i.e. a flow in a infinite journal bearing with small gap. After rectification the fluid domain is

$$\Omega^\varepsilon \times (-\infty, +\infty), \quad \Omega^\varepsilon = \big\{(z_1, z_2) \in \mathbb{R}^2;\ 0 < z_1 < L,\ 0 < z_2 < \varepsilon h^\varepsilon(z_1)\big\}$$

with h^ε L-periodic with respect to z_1.

We assume that the external excitation fields and the flow do not depend on z_3. Thus $u = \big(u_1(t,z), u_2(t,z), 0\big)$, $\omega = \big(0, 0, \omega_3(t,z)\big)$, $f = \big(f_1(t,z), f_2(t,z), 0\big)$, $g = \big(0, 0, g_3(t,z)\big)$ and

$$\begin{aligned}
&u^\varepsilon_t - (\nu + \nu_r)\Delta u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = 2\nu_r \mathrm{rot}\omega^\varepsilon + f^\varepsilon \\
&\mathrm{div}(u^\varepsilon) = 0 \\
&\omega^\varepsilon_t - \alpha\Delta\omega^\varepsilon + (u^\varepsilon \cdot \nabla)\omega^\varepsilon + 4\nu_r\omega^\varepsilon = 2\nu_r \mathrm{rot}u^\varepsilon + g^\varepsilon
\end{aligned}$$

with $u^\varepsilon = \big(u^\varepsilon_1(t,z), u^\varepsilon_2(t,z)\big)$, $\omega^\varepsilon = \omega^\varepsilon_3(t,z)$ and

$$\mathrm{rot}u^\varepsilon = \frac{\partial u^\varepsilon_2}{\partial z_1} - \frac{\partial u^\varepsilon_1}{\partial z_2}, \quad \mathrm{rot}\omega^\varepsilon = \left(\frac{\partial \omega^\varepsilon}{\partial z_2}, -\frac{\partial \omega^\varepsilon}{\partial z_1}\right).$$

The **boundary and initial conditions** are given by

$$u^\varepsilon,\ \omega^\varepsilon,\ p^\varepsilon \text{ are L-periodic with respect to } z_1$$

$$u^\varepsilon \cdot n^\varepsilon = 0,\ \tau^\varepsilon \cdot \sigma(u^\varepsilon, p^\varepsilon) \cdot n^\varepsilon = 0,\ \omega^\varepsilon = 0 \text{ on } (0,T) \times \Gamma_1^\varepsilon,\ \Gamma_1^\varepsilon = \big\{\big(z_1, \varepsilon h^\varepsilon(z_1)\big);\ 0 < z_1 < L\big\}$$

$$u^\varepsilon = \big(U_0(t), 0\big),\ \omega^\varepsilon = W_0(t) \text{ on } (0,T) \times \Gamma_0,\ \Gamma_0 = (0,L) \times \{0\}$$

$$u^\varepsilon(z,0) = u_0^\varepsilon(z),\ \omega^\varepsilon(z,0) = \omega_0^\varepsilon(z) \text{ for } z \in \Omega^\varepsilon$$

where τ^ε and n^ε are respectively the tangential and the normal components of the unit outward normal to the boundary and $\sigma(u^\varepsilon, p^\varepsilon) = (\sigma_{ij}(u^\varepsilon, p^\varepsilon))_{1 \le i,j \le 2}$ is the Cauchy stress tensor given by

$$\sigma_{ij}(u^\varepsilon, p^\varepsilon) = -p^\varepsilon \delta_{ij} + (\nu + \nu_r) \underbrace{\left(\frac{\partial u_i^\varepsilon}{\partial z_j} + \frac{\partial u_j^\varepsilon}{\partial z_i}\right)}_{=D_{ij}(u)}.$$

II - Existence and uniqueness of a solution for any $\varepsilon > 0$

As a first step we define an extension of U_0 and W_0 in order to get homogeneous boundary conditions.

Let $U^\varepsilon = U^\varepsilon(t, z_2)$ and $W^\varepsilon = W^\varepsilon(t, z_2)$ be smooth functions such that

$$U^\varepsilon(t,0) = U_0(t),\ U^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = 0,\ \frac{\partial U^\varepsilon}{\partial z_2}(t, \varepsilon h^\varepsilon(z_1)) = 0,\ (t, z_1) \in (0,T) \times (0,L),$$

$$W^\varepsilon(t,0) = W_0(t),\ W^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = 0,\ (t, z_1) \in (0,T) \times (0,L).$$

Then we define

$$v^\varepsilon(t,z) = u^\varepsilon(t,z) - U^\varepsilon(t, z_2)e_1,\ Z^\varepsilon(t,z) = \omega^\varepsilon(t,z) - W^\varepsilon(t, z_2) \text{ in } (0,T) \times \Omega^\varepsilon$$

and

$$v_0^\varepsilon(z) = u_0^\varepsilon(z) - U^\varepsilon(0, z_2)e_1,\ Z_0^\varepsilon(z) = \omega_0^\varepsilon(z) - W^\varepsilon(0, z_2) \text{ in } \Omega^\varepsilon.$$

We may observe that $\sigma(u^\varepsilon, p^\varepsilon) = \sigma(v^\varepsilon, p^\varepsilon)$ on $(0,T) \times \Gamma_1^\varepsilon$.

We consider the following **functional framework** for the velocities

$$\widetilde{V^\varepsilon} = \{v \in \mathcal{C}^\infty(\Omega^\varepsilon)^2 : \ v \text{ is L} - \text{periodic in } z_1, \ \ v_{|\Gamma_0} = 0, \ v \cdot n_{|\Gamma_1^\varepsilon} = 0\}$$

$$H^\varepsilon = \text{closure of } \widetilde{V^\varepsilon} \text{ in } L^2(\Omega^\varepsilon) \times L^2(\Omega^\varepsilon),$$

$$V^\varepsilon = \text{closure of } \widetilde{V^\varepsilon} \text{ in } H^1(\Omega^\varepsilon) \times H^1(\Omega^\varepsilon), \qquad V^\varepsilon_{div} = \{v \in V^\varepsilon \ : \ \operatorname{div} v = 0, \ in \ \Omega^\varepsilon\}$$

and for the micro-rotations and the pressure

$$\widetilde{H^1}^\varepsilon = \{Z \in \mathcal{C}^\infty(\Omega^\varepsilon) : \ \ Z \ \text{ is } \ \text{L} - \text{periodic } \text{ in } \ z_1, \ \ Z = 0 \ \text{ on } \ \Gamma_0 \cup \Gamma_1^\varepsilon\}$$

$$H^{0,\varepsilon} = \text{closure of } \widetilde{H^1}^\varepsilon \text{ in } L^2(\Omega^\varepsilon), \qquad H^{1,\varepsilon} = \text{closure of } \widetilde{H^1}^\varepsilon \text{ in } H^1(\Omega^\varepsilon),$$

$$L^2_0(\Omega^\varepsilon) = \left\{q \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} q(z)dz = 0\right\}.$$

For all $\bar{v} = (v, Z)$, $\bar{u} = (u, w)$ and $\Theta = (\varphi, \psi)$ in $V^\varepsilon \times H^{1,\varepsilon}$, we define

$$\begin{aligned} a(\bar{v}, \Theta) &= (\nu + \nu_r)\big(D(v), D(\varphi)\big) + \alpha(\nabla Z, \nabla \psi), \\ \mathcal{R}(\bar{v}, \Theta) &= -2\nu_r(\mathrm{rot}\, Z, \varphi) - 2\nu_r(\mathrm{rot}\, v, \psi) + 4\nu_r(Z, \psi), \\ B(\bar{v}, \bar{u}, \Theta) &= b(v, u, \varphi) + b_1(v, w, \psi) = \sum_{i,j=1}^{2} \int_{\Omega^\varepsilon} v_i \frac{\partial u_j}{\partial z_i} \varphi_j dz + \sum_{i=1}^{2} \int_{\Omega^\varepsilon} v_i \frac{\partial w}{\partial z_i} \psi dz \end{aligned}$$

and for all $\bar{v} = (v, Z)$ and $\Theta = (\varphi, \psi)$ in $H^\varepsilon \times H^{0,\varepsilon}$,

$$[\bar{v}, \Theta] = (v, \varphi) + (Z, \psi).$$

The **variational formulation of the problem** is given by

Problem (P^ε) Find $(v^\varepsilon, Z^\varepsilon, p^\varepsilon)$ such that

$$\bar{v^\varepsilon} = (v^\varepsilon, Z^\varepsilon) \in \mathcal{C}([0,T]; H^\varepsilon) \cap L^2(0,T; V^\varepsilon_{div}) \times \mathcal{C}([0,T]; H^{0,\varepsilon}) \cap L^2(0,T; H^{1,\varepsilon}),$$

$$p^\varepsilon \in H^{-1}(0,T; L^2_0(\Omega^\varepsilon)),$$

and

$$\begin{aligned}
&[\frac{\partial \bar{v^\varepsilon}}{\partial t}(t), \Theta] + a(\bar{v^\varepsilon}(t), \Theta) + B(\bar{v^\varepsilon}(t), \bar{v^\varepsilon}(t), \Theta) + \mathcal{R}(\bar{v^\varepsilon}(t), \Theta) \\
&= (p^\varepsilon(t)\,,\, \mathrm{div}\varphi) + (\mathcal{F}(\bar{v^\varepsilon}(t)), \Theta), \qquad \forall \Theta = (\varphi, \psi) \in V^\varepsilon \times H^{1,\varepsilon}
\end{aligned}$$

with the initial condition

$$\bar{v^\varepsilon}(z,0) = \bar{v^\varepsilon_0}(z) = (v^\varepsilon_0(z), Z^\varepsilon_0(z)),$$

where

$$\begin{aligned}
(\mathcal{F}(\bar{v^\varepsilon}(t)), \Theta) \;=\; & -a(\bar{\xi^\varepsilon}(t), \Theta) - B(\bar{\xi^\varepsilon}(t), \bar{v^\varepsilon}(t), \Theta) - B(\bar{v^\varepsilon}(t), \bar{\xi^\varepsilon}(t), \Theta) - \mathcal{R}(\bar{\xi^\varepsilon}(t), \Theta) \\
& -[\frac{\partial \bar{\xi^\varepsilon}}{\partial t}(t), \Theta] + [\bar{f^\varepsilon}(t), \Theta], \qquad \bar{\xi^\varepsilon} = (U^\varepsilon e_1, W^\varepsilon) \qquad \bar{f^\varepsilon} = (f^\varepsilon, g^\varepsilon).
\end{aligned}$$

Theorem Let $f^\varepsilon \in (L^2((0,T)\times\Omega^\varepsilon))^2$, $g^\varepsilon \in L^2((0,T)\times\Omega^\varepsilon)$, and $(v_0^\varepsilon\,,\, Z_0^\varepsilon) \in H^\varepsilon \times H^{0,\varepsilon}$. Then the problem (P^ε) admits an unique solution.

Sketch of the proof

Existence is proved by using the same ideas as in J.L.Lions (1978) i.e. the condition "$\mathrm{div}(v^\varepsilon) = 0$" is considered as a constraint. Hence we introduce the penalized problem

$$
\begin{aligned}
&[\frac{\partial \bar{v}_\delta^\varepsilon}{\partial t}, \Theta] + a(\bar{v}_\delta^\varepsilon, \Theta) + B(\bar{v}_\delta^\varepsilon, \bar{v}_\delta^\varepsilon, \Theta) - \frac{1}{2}\{(v_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\,\varphi\} + (Z_\delta^\varepsilon \mathrm{div}\, v_\delta^\varepsilon\,,\,\psi)\} \\
&= \big(-\frac{1}{\delta}\mathrm{div} v_\delta^\varepsilon\,,\, \mathrm{div}\varphi\big) + (\mathcal{F}(\bar{v}_\delta^\varepsilon), \Theta) - \mathcal{R}(\bar{v}_\delta^\varepsilon, \Theta) \qquad \forall \Theta = (\varphi, \psi) \in V^\varepsilon \times H^{1,\varepsilon},
\end{aligned}
$$

with the initial condition $\bar{v}_\delta^\varepsilon(0) = \bar{v}_0^\varepsilon$ and we pass to the limit as δ tends to 0.

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$$[\frac{\partial \bar{v}^\varepsilon_\delta}{\partial t},\Theta]+a(\bar{v}^\varepsilon_\delta,\Theta)+B(\bar{v}^\varepsilon_\delta,\bar{v}^\varepsilon_\delta,\Theta)-\frac{1}{2}\{(v^\varepsilon_\delta \mathrm{div}\, v^\varepsilon_\delta\,,\,\varphi)+(Z^\varepsilon_\delta \mathrm{div}\, v^\varepsilon_\delta\,,\,\psi)\}$$
$$=\big(\underbrace{-\frac{1}{\delta}\mathrm{div} v^\varepsilon_\delta}_{=p^\varepsilon_\delta}\,,\,\mathrm{div}\varphi\big)+(\mathcal{F}(\bar{v}^\varepsilon_\delta),\Theta)-\mathcal{R}(\bar{v}^\varepsilon_\delta,\Theta)\qquad \forall\Theta=(\varphi,\psi)\in V^\varepsilon\times H^{1,\varepsilon},$$

with the initial condition $\bar{v}^\varepsilon_\delta(0)=\bar{v}^\varepsilon_0$ and we pass to the limit as δ tends to 0.

Uniqueness and continuity with respect to time are obtained as in G.Lukaszewicz (2001)

References:

J.L.Lions, *Some problems connected with Navier-Stokes equations*, Lectures at the IVth latin-american school of mathematics, 1978.

G.Lukaszewicz, *Long time behaviour of 2D micropolar fluid flows*, Math. Comp. Modelling, 2001.

III - Rescaling and a priori estimates

We assume now some **roughness of the outer cylinder** i.e. $h^\varepsilon(z_1) = h\left(z_1, \dfrac{z_1}{\varepsilon}\right)$ with

$$h : (z_1, \eta_1) \mapsto h(z_1, \eta_1) \text{ is L-periodic in } z_1 \text{ and 1-periodic in } \eta_1,\ \frac{L}{\varepsilon} \in \mathbb{N}.$$

We assume also that $h \in \mathcal{C}^1\big([0, L] \times [0, 1]\big)$, $\dfrac{\partial h}{\partial \eta_1}$ is 1-periodic in η_1 and we define

$$0 < h_m = \min_{[0,L]\times[0,1]} h(z_1, \eta_1) < \max_{[0,L]\times[0,1]} h(z_1, \eta_1) = h_M.$$

We rescale the problem:

Let $x_1 = z_1$, $x_2 = \dfrac{z_2}{\varepsilon}$ and

$$\Omega_\varepsilon = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 :\ 0 < x_1 < L,\ 0 < x_2 < h^\varepsilon(x_1) = h(x_1, \frac{x_1}{\varepsilon}) \right\}.$$

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We rescale the problem **in two steps**:

- $x_1 = z_1$, $x_2 = \dfrac{z_2}{\varepsilon}$, $\Omega_\varepsilon = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 :\ 0 < x_1 < L,\ 0 < x_2 < h^\varepsilon(x_1) = h(x_1, \dfrac{x_1}{\varepsilon}) \right\}$,
- $y_1 = x_1 = z_1$, $y_2 = \dfrac{x_2}{h^\varepsilon(x_1)} = \dfrac{z_2}{\varepsilon h^\varepsilon(z_1)}$, $\Omega = (0, L) \times (0, 1)$.

We have

$$\begin{aligned} v^\varepsilon(t,z_1,z_2) &= v^\varepsilon(t,y_1,\varepsilon h^\varepsilon(y_1)y_2) \quad \forall(t,z)\in(0,T)\times\Omega^\varepsilon \\ &:= v^\varepsilon(t,y_1,y_2) \quad \forall(t,y)\in(0,T)\times\Omega, \end{aligned}$$

and similarly (with some lack of consistency in the notations)

$$Z^\varepsilon(t,z) := Z^\varepsilon(t,y), \quad p^\varepsilon(t,z) := p^\varepsilon(t,y).$$

We get

$$\begin{aligned} \frac{\partial}{\partial z_2} = \frac{1}{\varepsilon h}\frac{\partial}{\partial y_2}, \quad \frac{\partial}{\partial z_1} &= \frac{\partial}{\partial y_1}\frac{\partial y_1}{\partial z_1} + \frac{\partial}{\partial y_2}\frac{\partial y_2}{\partial z_1} = \frac{\partial}{\partial y_1} + \left(-\frac{y_2}{h^\varepsilon(y_1)}\frac{\partial h^\varepsilon}{\partial y_1}\right)\frac{\partial}{\partial y_2} \\ &= \left(1, -\frac{y_2}{h^\varepsilon(y_1)}\frac{\partial h^\varepsilon}{\partial y_1}\right)\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \\ \frac{\partial}{\partial y_2} \end{pmatrix} = b_\varepsilon \cdot \nabla \end{aligned}$$

We choose

$$U^\varepsilon(t, z_2) = U\left(\frac{z_2}{\varepsilon}\right) U_0(t) = U(x_2)U_0(t), \quad W^\varepsilon(t, z_2) = W\left(\frac{z_2}{\varepsilon}\right) W_0(t) = W(x_2)W_0(t)$$

with $U, W \in \mathcal{D}(-\infty, h_m)$ and we assume that

$$f^\varepsilon(t, z) = f(t, y_1), \quad g^\varepsilon(t, z) = g(t, y_1) \quad \forall (t, z) \in (0, T) \times \Omega^\varepsilon$$

with $(f, g) \in \big(L^2\big((0, T) \times \Omega\big)\big)^2 \times L^2\big((0, T) \times \Omega\big)$.

Proposition There exists a constant $C > 0$ which does not depends on ε, such that, for $0 \le i \le 2$, we have the following estimates

$$\|(\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon)\|_{L^2((0,T)\times\Omega)} \le C, \qquad \|(\varepsilon b_\varepsilon \cdot \nabla Z^\varepsilon)\|_{L^2((0,T)\times\Omega)} \le C,$$

$$\|\frac{\partial v_i^\varepsilon}{\partial y_2}\|_{L^2((0,T)\times\Omega)} \le C, \qquad \|\frac{\partial Z^\varepsilon}{\partial y_2}\|_{L^2((0,T)\times\Omega)} \le C,$$

$$\|\frac{\partial v_i^\varepsilon}{\partial y_1}\|_{L^2((0,T)\times\Omega)} \le \frac{C}{\varepsilon}, \qquad \|\frac{\partial Z^\varepsilon}{\partial y_1}\|_{L^2((0,T)\times\Omega)} \le \frac{C}{\varepsilon},$$

$$\|v_i^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C, \qquad \|Z^\varepsilon\|_{L^2((0,T)\times\Omega)} \le C, \qquad \varepsilon^2\|p^\varepsilon\|_{H^{-1}(0,T;L^2(\Omega))} \le C.$$

Sketch of the proof

With $\Theta = \bar{v}^\varepsilon(t)$ we get:

$$[\frac{\partial \bar{v}^\varepsilon}{\partial t}(t), \bar{v}^\varepsilon] + a(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon) + \underbrace{B(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon(t), \bar{v}^\varepsilon)}_{=0} + \mathcal{R}(\bar{v}^\varepsilon(t), \bar{v}^\varepsilon)$$

$$= \underbrace{(p^\varepsilon(t)\,,\,\mathrm{div} v^\varepsilon)}_{=0} - a(\bar{\xi}^\varepsilon(t), \bar{v}^\varepsilon) - \underbrace{B(\bar{\xi}^\varepsilon(t), \bar{v}^\varepsilon(t), \bar{v}^\varepsilon)}_{=0} - B(\bar{v}^\varepsilon(t), \bar{\xi}^\varepsilon(t), \bar{v}^\varepsilon)$$

$$-\mathcal{R}(\bar{\xi}^\varepsilon(t), \bar{v}^\varepsilon) - [\frac{\partial \bar{\xi}^\varepsilon}{\partial t}(t), \bar{v}^\varepsilon] + [\bar{f}^\varepsilon(t), \bar{v}^\varepsilon], \qquad \bar{\xi}^\varepsilon = (U^\varepsilon e_1, W^\varepsilon) \qquad \bar{f}^\varepsilon = (f^\varepsilon, g^\varepsilon)$$

We rewrite all these integrals by using the rescaled variables (y_1, y_2), we play with Young's inequality and we use Grönwall's lemma.

IV - Two scale convergence properties

We introduce the following notations: $Y = [0,1]^2$,

$$\mathcal{C}^\infty_\sharp(Y) = \Big\{\varphi \in \mathcal{C}^\infty(\mathbb{R}^2) \text{ s.t. } \varphi \text{ is } Y\text{-periodic}\Big\},$$

$$L^2_\sharp(Y) = \text{closure of } \mathcal{C}^\infty_\sharp(Y) \text{ in } L^2(Y), \quad H^1_\sharp(Y) = \text{closure of } \mathcal{C}^\infty_\sharp(Y) \text{ in } H^1(Y).$$

Definition A sequence $(w^\varepsilon)_{\varepsilon>0}$ of $L^2\big((0,T)\times\Omega\big)$ (resp. $H^{-1}\big(0,T;L^2(\Omega)\big)$) **two-scale converges** to $w^0 \in L^2\big(0,T;L^2(\Omega\times Y)\big)$ (resp. $w^0 \in H^{-1}\big(0,T;L^2(\Omega\times Y)\big)$) if and only if

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega w^\varepsilon(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt = \int_0^T\int_{\Omega\times Y} w^0(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt$$

for all $\theta \in \mathcal{D}(0,T)$, for all $\varphi \in \mathcal{D}\big(\Omega;\mathcal{C}^\infty_\sharp(Y)\big)$. In such a case we will denote $w^\varepsilon \longrightarrow\!\!\!\!\!\!\longrightarrow w^0$.

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Theorem Let $(w^\varepsilon)_{\varepsilon>0}$ be a bounded sequence of $L^2\big((0,T)\times\Omega\big)$ (resp. of $H^{-1}\big((0,T)\times\Omega\big)$). There exists $w^0 \in L^2\big(0,T;L^2(\Omega\times Y)\big)$ (resp. $w^0 \in H^{-1}\big(0,T;L^2(\Omega\times Y)\big)$) such that, possibly extracting a subsequence still denoted $(w^\varepsilon)_{\varepsilon>0}$, we have

$$w^\varepsilon \longrightarrow\!\!\!\!\!\rightarrow w^0.$$

Sketch of the proof

We adapt the proof of G.Allaire (1992) to our time-dependent setting.

Reference:

G.Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. (1992).

Theorem Let $(w^\varepsilon)_{\varepsilon>0}$ be a bounded sequence of $L^2\big((0,T)\times\Omega\big)$ $\big($resp. of $H^{-1}\big(0,T;L^2(\Omega)\big)\big)$. There exists $w^0 \in L^2\big(0,T;L^2(\Omega\times Y)\big)$ $\big($resp. $w^0 \in H^{-1}\big(0,T;L^2(\Omega\times Y)\big)\big)$ such that, possibly extracting a subsequence still denoted $(w^\varepsilon)_{\varepsilon>0}$, we have

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Proposition (Two-scale limit of the velocity) There exist $v^0 \in \left(L^2(0,T;L^2(\Omega;H^1_\sharp(Y)))\right)^2$ such that $\dfrac{\partial v^0}{\partial y_2} \in \left(L^2(0,T;L^2(\Omega\times Y))\right)^2$ and $v^1 \in \left(L^2(0,T;L^2(\Omega\times(0,1);H^1_\sharp(0,1)_{/\mathbb{R}}))\right)^2$ such that, for $i=1,2$:

$$v_i^\varepsilon \longrightarrow\!\!\!\!\!\!\longrightarrow v_i^0, \quad \frac{\partial v_i^\varepsilon}{\partial y_2} \longrightarrow\!\!\!\!\!\!\longrightarrow \frac{\partial v_i^0}{\partial y_2} + \frac{\partial v_i^1}{\partial \eta_2},$$

and

$$\varepsilon\frac{\partial v_i^\varepsilon}{\partial y_1} \longrightarrow\!\!\!\!\!\!\longrightarrow \frac{\partial v_i^0}{\partial \eta_1}.$$

Furthermore v^0 does not depend on η_2, v^0 is divergence free in the following sense

$$h(y_1,\eta_1)\frac{\partial v_1^0}{\partial \eta_1} - y_2\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\frac{\partial v_1^0}{\partial y_2} + \frac{\partial v_2^0}{\partial y_2} = 0 \quad \text{in } \Omega\times(0,1)\times(0,T),$$

and

$$v^0 = 0 \quad \text{on } \Gamma_0\times(0,1)\times(0,T),\ \Gamma_0 = (0,L)\times\{0\},$$
$$-v_1^0\frac{\partial h}{\partial \eta_1}(y_1,\eta_1) + v_2^0 = 0 \quad \text{on } \Gamma_1\times(0,1)\times(0,T),\ \Gamma_1 = (0,L)\times\{1\}.$$

Sketch of the proof

Let v_i^0, ξ_i^0 and ξ_i^1 be the two-scale limits of $(v_i^\varepsilon)_{\varepsilon>0}$, $\left(\dfrac{\partial v_i^\varepsilon}{\partial y_2}\right)_{\varepsilon>0}$ and $\left(\varepsilon\dfrac{\partial v_i^\varepsilon}{\partial y_1}\right)_{\varepsilon>0}$ respectively. We have

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varepsilon\frac{\partial v_i^\varepsilon}{\partial y_1}(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt = \int_0^T\int_{\Omega\times Y}\xi_i^1(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt$$
$$= -\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\left(\varepsilon\frac{\partial\varphi}{\partial y_1}\left(y,\frac{y}{\varepsilon}\right)+\frac{\partial\varphi}{\partial\eta_1}\left(y,\frac{y}{\varepsilon}\right)\right)\theta(t)\,dydt$$
$$= -\int_0^T\int_{\Omega\times Y} v_i^0(t,y,\eta)\frac{\partial\varphi}{\partial\eta_1}(y,\eta)\theta(t)\,d\eta dydt = \int_0^T\int_{\Omega\times Y}\frac{\partial v_i^0}{\partial\eta_1}(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt$$

for all $\theta\in\mathcal{D}(0,T)$, $\varphi\in\mathcal{D}\big(\Omega;\mathcal{C}_\sharp^\infty(Y)\big)$. Hence

$$\xi_i^1 = \frac{\partial v_i^0}{\partial\eta_1}\in L^2\big(0,T;L^2(\Omega\times Y)\big).$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\partial v_i^\varepsilon}{\partial y_2}(t,y)\varphi\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt = \int_0^T\int_{\Omega\times Y}\xi_i^0(t,y,\eta)\varphi(y,\eta)\theta(t)\,d\eta dydt$$
$$= -\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\left(\frac{\partial\varphi}{\partial y_2}\left(y,\frac{y}{\varepsilon}\right)+\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial\eta_2}\left(y,\frac{y}{\varepsilon}\right)\right)\theta(t)\,dydt.$$

We multiply by ε:

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega v_i^\varepsilon(t,y)\frac{\partial\varphi}{\partial\eta_2}\left(y,\frac{y}{\varepsilon}\right)\theta(t)\,dydt = 0 = \int_0^T\int_{\Omega\times Y} v_i^0(t,y,\eta)\frac{\partial\varphi}{\partial\eta_2}(y,\eta)\theta(t)\,d\eta dydt$$

and thus v_i^0 does not depend on η_2. Now we go back to the previous computation and we choose $\varphi\in\mathcal{D}(\Omega)$:

$$\int_0^T\int_{\Omega\times(0,1)}\frac{\partial v_i^0}{\partial\eta_1}(t,y,\eta_1)\varphi(y)\theta(t)\,d\eta_1 dydt = 0 = \int_0^T\int_\Omega\Big(v_i^0(t,y,1)-v_i^0(t,y,0)\Big)\varphi(y)\theta(t)\,dydt$$

and $v_i^0\in L^2\big(0,T;L^2(\Omega;H^1_\sharp(Y))\big)$.

Next by choosing $\varphi \in \mathcal{D}\big(\Omega; \mathcal{C}^\infty_\sharp(0,1)\big)$:

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega v_i^\varepsilon(t,y) \frac{\partial \varphi}{\partial y_2}\left(y, \frac{y_1}{\varepsilon}\right) \theta(t)\, dydt = \int_0^T \int_{\Omega\times Y} v_i^0(t,y,\eta_1) \frac{\partial \varphi}{\partial y_2}(y,\eta_1)\theta(t)\, d\eta dydt$$
$$= -\int_0^T \int_{\Omega\times Y} \xi_i^0(t,y,\eta)\varphi(y,\eta_1)\theta(t)\, d\eta dydt.$$

It follows that there exists $v_i^1 \in L^2\big(0,T; L^2\big(\Omega\times(0,1); H^1_\sharp(0,1)_{|\mathbb{R}}\big)\big)$ such that

$$\frac{\partial v_i^\varepsilon}{\partial y_2} \longrightarrow\!\!\!\!\!\longrightarrow \xi_i^0 = \frac{\partial v_i^0}{\partial y_2} + \frac{\partial v_i^1}{\partial \eta_2}.$$

Proposition (Two-scale limit of the micro-rotation) There exist $Z^0 \in L^2\big(0,T; L^2(\Omega; H^1_\sharp(Y$ such that $\dfrac{\partial Z^0}{\partial y_2} \in L^2\big(0,T; L^2(\Omega \times Y)\big)$ and $Z^1 \in L^2\big(0,T; L^2\big(\Omega \times (0,1); H^1_\sharp(0,1)_{/\mathbb{R}}\big)\big)$ such that

$$Z^\varepsilon \longrightarrow\!\!\!\!\to Z^0, \quad \frac{\partial Z^\varepsilon}{\partial y_2} \longrightarrow\!\!\!\!\to \frac{\partial Z^0}{\partial y_2} + \frac{\partial Z^1}{\partial \eta_2},$$

and

$$\varepsilon \frac{\partial Z^\varepsilon}{\partial y_1} \longrightarrow\!\!\!\!\to \frac{\partial Z^0}{\partial \eta_1}.$$

Furthermore Z^0 does not depend on η_2, and $Z^0 \equiv 0$ on $(\Gamma_0 \cup \Gamma 1) \times (0,1) \times (0,T)$.

Proposition (Two-scale limit of the pressure) There exists $p^0 \in H^{-1}\big(0,T; L^2(\Omega \times Y)\big)$ such that, possibly extracting a subsequence still denoted $(p^\varepsilon)_{\varepsilon>0}$, we have

$$\varepsilon^2 p^\varepsilon \longrightarrow\!\!\!\!\to p^0.$$

Moreover p^0 depends only on t and y_1 and $\displaystyle\int_0^L p^0(t, y_1) \left(\int_0^1 h(y_1, \eta_1)\, d\eta_1 \right) dy_1 = 0$ almost everywhere in $(0,T)$.

V - The limit problem

It is convenient to define

$$\widetilde{V} = \Big\{ \varphi \in \big(\mathcal{C}^\infty(\overline{\Omega}; \mathcal{C}^\infty_\sharp(0,1))\big)^2;\ \varphi \text{ is } L\text{-periodic in } y_1,\ \varphi = 0 \text{ on } \Gamma_0 \times (0,1),$$
$$-\varphi_1 \frac{\partial h}{\partial \eta_1} + \varphi_2 = 0 \text{ on } \Gamma_1 \times (0,1) \Big\}$$

$$\widetilde{V}_{div} = \Big\{ \varphi \in \widetilde{V};\ h\frac{\partial \varphi_1}{\partial \eta_1} - y_2 \frac{\partial h}{\partial \eta_1}\frac{\partial \varphi_1}{\partial y_2} + \frac{\partial \varphi_2}{\partial y_2} = 0 \text{ in } \Omega \times (0,1) \Big\}$$

$$\widetilde{H}^1 = \Big\{ \psi \in \mathcal{C}^\infty(\overline{\Omega}; \mathcal{C}^\infty_\sharp(0,1));\ \psi \text{ is } L\text{-periodic in } y_1,\ \psi = 0 \text{ on } (\Gamma_0 \cup \Gamma_1) \times (0,1) \Big\},$$

$$V_{div} = \text{closure of } \widetilde{V}_{div} \text{ in } \Big(L^2_\sharp\big([0,L]; H^1\big(0,1; H^1_\sharp(0,1)\big)\big) \Big)^2,$$
$$H^1_{0\sharp} = \text{closure of } \widetilde{H}^1 \text{ in } \Big(L^2_\sharp\big([0,L]; H^1(\Omega; H^1_\sharp(0,1))\big) \Big)^2.$$

Let us recall that

$$b_\varepsilon \cdot \nabla = \left(1, -\frac{y_2}{h^\varepsilon(y_1)}\frac{\partial h^\varepsilon}{\partial y_1}(y_1)\right)\begin{pmatrix} \dfrac{\partial}{\partial y_1} \\ \\ \dfrac{\partial}{\partial y_2} \end{pmatrix}$$

With the previous convergence results, we get

$$\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon = \varepsilon \frac{\partial v_i^\varepsilon}{\partial y_1}(y) - \frac{y_2}{h\left(y_1, \frac{y_1}{\varepsilon}\right)}\left(\varepsilon \frac{\partial h}{\partial y_1}\left(y_1, \frac{y_1}{\varepsilon}\right) + \frac{\partial h}{\partial \eta_1}\left(y_1, \frac{y_1}{\varepsilon}\right)\right)\frac{\partial v_i^\varepsilon}{\partial y_2}(y)$$

$$\longrightarrow\!\!\!\!\longrightarrow \frac{\partial v_i^0}{\partial \eta_1}(y, \eta_1) - \frac{y_2}{h(y_1, \eta_1)}\frac{\partial h}{\partial \eta_1}(y_1, \eta_1)\left(\frac{\partial v_i^0}{\partial y_2}(y, \eta_1) + \frac{\partial v_i^1}{\partial \eta_2}(y, \eta)\right) = \bar{b} \cdot \nabla v_i^0 - \frac{y_2}{h}\frac{\partial h}{\partial \eta_1}\frac{\partial v_i^1}{\partial \eta_2}$$

for $i = 1, 2$ and

$$b_\varepsilon \cdot \nabla Z^\varepsilon \longrightarrow\!\!\!\!\longrightarrow \bar{b} \cdot \nabla Z^0 - \frac{y_2}{h}\frac{\partial h}{\partial \eta_1}\frac{\partial Z^1}{\partial \eta_2}$$

where $\bar{b}\cdot\nabla$ is the differential operator defined by

$$\bar{b}\cdot\nabla = \left(1, -\frac{y_2}{h(y_1,\eta_1)}\frac{\partial h}{\partial \eta_1}(y_1,\eta_1)\right)\begin{pmatrix}\dfrac{\partial}{\partial \eta_1}\\[2ex] \dfrac{\partial}{\partial y_2}\end{pmatrix}$$

Similarly, let $\phi \in \mathcal{C}^\infty\left(\overline{\Omega}; \mathcal{C}^\infty_\sharp(0,1)\right)$ and $\phi^\varepsilon(y_1,y_2) = \phi\left(y_1,y_2,\dfrac{y_1}{\varepsilon}\right)$ for all $(y_1,y_2)\in\Omega$. We have

$$\begin{aligned} b_\varepsilon\cdot\nabla\phi^\varepsilon &= \frac{\partial \phi^\varepsilon}{\partial y_1}(y) - \frac{y_2}{h\left(y_1,\frac{y_1}{\varepsilon}\right)}\left(\frac{\partial h}{\partial y_1}\left(y_1,\frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta_1}\left(y_1,\frac{y_1}{\varepsilon}\right)\right)\frac{\partial \phi^\varepsilon}{\partial y_2}(y)\\ &= \frac{\partial \phi}{\partial y_1}\left(y,\frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial \phi}{\partial \eta_1}\left(y,\frac{y_1}{\varepsilon}\right) - \frac{y_2}{h\left(y_1,\frac{y_1}{\varepsilon}\right)}\left(\frac{\partial h}{\partial y_1}\left(y_1,\frac{y_1}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial h}{\partial \eta_1}\left(y_1,\frac{y_1}{\varepsilon}\right)\right)\frac{\partial \phi}{\partial y_2}\left(y,\frac{y_1}{\varepsilon}\right). \end{aligned}$$

Now let $\theta \in \mathcal{D}(0,T)$, $\Theta = (\varphi,\psi) \in \widetilde{V}_{div}\times\widetilde{H}^1$ and let $\Theta^\varepsilon = (\varphi^\varepsilon,\psi^\varepsilon)$ with $\varphi^\varepsilon(z) = \varphi\left(z_1,\dfrac{z_2}{\varepsilon h^\varepsilon(z_1)},\dfrac{z_1}{\varepsilon}\right)$ and $\psi^\varepsilon(z) = \psi\left(z_1,\dfrac{z_2}{\varepsilon h^\varepsilon(z_1)},\dfrac{z_1}{\varepsilon}\right)$ for all $(z_1,z_2)\in\Omega^\varepsilon$.

We have $\Theta^\varepsilon \in \widetilde{V}^\varepsilon \times \widetilde{H}^{1,\varepsilon}$ and we introduce this test-function in (P^ε). We rewrite the integrals in terms of (y_1,y_2), we multiply by ε and we pass to the limit as $\varepsilon\longrightarrow 0^+$.

Theorem The functions v^0, Z^0 and p^0 satisfy the following **limit problem**:

$$(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}\sum_{i=1}^{2}\left(h(\bar b\cdot\nabla v_i^0)(\bar b\cdot\nabla\varphi_i)+\frac{1}{h}\frac{\partial v_i^0}{\partial y_2}\frac{\partial\varphi_i}{\partial y_2}\right)\theta\,d\eta_1 dydt$$
$$+\alpha\int_0^T\int_{\Omega\times(0,1)}\left(h(\bar b\cdot\nabla Z^0)(\bar b\cdot\nabla\psi)+\frac{1}{h}\frac{\partial Z^0}{\partial y_2}\frac{\partial\psi}{\partial y_2}\right)\theta\,d\eta_1 dydt$$
$$-\int_0^T\int_{\Omega\times(0,1)}\frac{\partial p^0}{\partial y_1}h\varphi_1\theta\,d\eta_1 dydt$$
$$=-(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}U_0\left(h(\bar b\cdot\nabla\bar U)(\bar b\cdot\nabla\varphi_1)+\frac{1}{h}\frac{\partial\bar U}{\partial y_2}\frac{\partial\varphi_1}{\partial y_2}\right)\theta\,d\eta_1 dydt$$
$$-\alpha\int_0^T\int_{\Omega\times(0,1)}W_0\left(h(\bar b\cdot\nabla\bar W)(\bar b\cdot\nabla\psi)+\frac{1}{h}\frac{\partial\bar W}{\partial y_2}\frac{\partial\psi}{\partial y_2}\right)\theta\,d\eta_1 dydt$$

for all $\Theta=(\varphi,\psi)\in V_{div}\times H^1_{0\sharp}$ and $\theta\in\mathcal{D}(0,T)$ where

$$\bar U(y_1,y_2,\eta_1)=U\big(h(y_1,\eta_1)y_2\big),\quad \bar W(y_1,y_2,\eta_1)=W\big(h(y_1,\eta_1)y_2\big)$$

for all $(y_1,y_2,\eta_1)\in\Omega\times(0,1)$.

Theorem The functions v^0, Z^0 and p^0 satisfy the following **limit problem**:

$$
\begin{aligned}
&(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}\sum_{i=1}^{2}\left(h(\bar b\cdot\nabla v_i^0)(\bar b\cdot\nabla\varphi_i)+\frac{1}{h}\frac{\partial v_i^0}{\partial y_2}\frac{\partial\varphi_i}{\partial y_2}\right)\theta\,d\eta_1 dydt\\
&+\alpha\int_0^T\int_{\Omega\times(0,1)}\left(h(\bar b\cdot\nabla Z^0)(\bar b\cdot\nabla\psi)+\frac{1}{h}\frac{\partial Z^0}{\partial y_2}\frac{\partial\psi}{\partial y_2}\right)\theta\,d\eta_1 dydt\\
&-\int_0^T\int_{\Omega\times(0,1)}\frac{\partial p^0}{\partial y_1}h\varphi_1\theta\,d\eta_1 dydt\\
&=-(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}U_0\left(h(\bar b\cdot\nabla\bar U)(\bar b\cdot\nabla\varphi_1)+\frac{1}{h}\frac{\partial\bar U}{\partial y_2}\frac{\partial\varphi_1}{\partial y_2}\right)\theta\,d\eta_1 dydt\\
&-\alpha\int_0^T\int_{\Omega\times(0,1)}W_0\left(h(\bar b\cdot\nabla\bar W)(\bar b\cdot\nabla\psi)+\frac{1}{h}\frac{\partial\bar W}{\partial y_2}\frac{\partial\psi}{\partial y_2}\right)\theta\,d\eta_1 dydt
\end{aligned}
$$

for all $\Theta=(\varphi,\psi)\in V_{div}\times H^1_{0\sharp}$ and $\theta\in\mathcal{D}(0,T)$ where

$$
\bar U(y_1,y_2,\eta_1)=U\big(h(y_1,\eta_1)y_2\big),\quad \bar W(y_1,y_2,\eta_1)=W\big(h(y_1,\eta_1)y_2\big)
$$

for all $(y_1,y_2,\eta_1)\in\Omega\times(0,1)$.

Theorem The functions v^0, Z^0 and p^0 satisfy the following **limit problem**:

$$
\begin{aligned}
&(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}\sum_{i=1}^{2}\left(h(\bar b\cdot\nabla v_i^0)(\bar b\cdot\nabla\varphi_i)+\frac{1}{h}\frac{\partial v_i^0}{\partial y_2}\frac{\partial\varphi_i}{\partial y_2}\right)\theta\,d\eta_1 dy dt\\
&+\alpha\int_0^T\int_{\Omega\times(0,1)}\left(h(\bar b\cdot\nabla Z^0)(\bar b\cdot\nabla\psi)+\frac{1}{h}\frac{\partial Z^0}{\partial y_2}\frac{\partial\psi}{\partial y_2}\right)\theta\,d\eta_1 dy dt\\
&-\int_0^T\int_{\Omega\times(0,1)}\frac{\partial p^0}{\partial y_1}h\varphi_1\theta\,d\eta_1 dy dt\\
&=-(\nu+\nu_r)\int_0^T\int_{\Omega\times(0,1)}U_0\left(h(\bar b\cdot\nabla\bar U)(\bar b\cdot\nabla\varphi_1)+\frac{1}{h}\frac{\partial\bar U}{\partial y_2}\frac{\partial\varphi_1}{\partial y_2}\right)\theta\,d\eta_1 dy dt\\
&-\alpha\int_0^T\int_{\Omega\times(0,1)}W_0\left(h(\bar b\cdot\nabla\bar W)(\bar b\cdot\nabla\psi)+\frac{1}{h}\frac{\partial\bar W}{\partial y_2}\frac{\partial\psi}{\partial y_2}\right)\theta\,d\eta_1 dy dt
\end{aligned}
$$

for all $\Theta=(\varphi,\psi)\in V_{div}\times H^1_{0\sharp}$ and $\theta\in\mathcal{D}(0,T)$ where

$$
\bar U(y_1,y_2,\eta_1)=U\big(h(y_1,\eta_1)y_2\big),\quad \bar W(y_1,y_2,\eta_1)=W\big(h(y_1,\eta_1)y_2\big)
$$

for all $(y_1,y_2,\eta_1)\in\Omega\times(0,1)$.

For all $y_1 \in [0, L]$, we define

$$a_{y_1}(w,\psi) = \int_Y \Big(\quad h(y_1,\eta_1)(\bar{b}\cdot\nabla w)(y_2,\eta_1)(\bar{b}\cdot\nabla\psi)(y_2,\eta_1) \\ + \frac{1}{h(y_1,\eta_1)}\frac{\partial w}{\partial y_2}(y_2,\eta_1)\frac{\partial \psi}{\partial y_2}(y_2,\eta_1)\Big)\, d\eta_1 dy_2$$

for all $(w,\psi) \in H^1\big(0,1;H^1(0,1)\big)^2$.

The **limit micro-rotation field** is solution of the following **problem** (P_{Z^0}):

$$\text{Find } Z^0 \in L^2(0,T;H^1_{0,\sharp}) \text{ such that} \\ \int_0^L a_{y_1}(Z^0,\psi)\, dy_1 = -W_0(t)\int_0^L a_{y_1}\big(\bar{W},\psi\big)\, dy_1 \quad \forall \psi \in H^1_{0,\sharp},\ \text{a.e. } t \in [0,T]$$

and we obtain

Proposition The limit micro-rotation field Z^0 is uniquely given by

$$Z^0(t, y_1, y_2, \eta_1) = W_0(t) Z^0_{y_1}(y_2, \eta_1) \quad \text{a.e. in } (0,T) \times \Omega \times (0,1).$$

where $Z^0_{y_1} \in H^1_0\big(0,1; H^1_\sharp(0,1)\big)$ is such that

$$a_{y_1}(Z^0_{y_1}, \psi) = -a_{y_1}\big(\bar{W}(y_1, \cdot), \psi\big) \quad \forall \psi \in H^1_0\big(0,1; H^1_\sharp(0,1)\big).$$

Sketch of the proof

The mapping $y_1 \mapsto Z^0_{y_1}$ is continuous on $[0, L]$ with values in $H^1_0\big(0,1; H^1_\sharp(0,1)\big)$ and L-periodic in y_1. Hence $Z^0 \in L^2(0,T; H^1_{0,\sharp}\big)$ and solves the problem (P_{Z_0}). Uniqueness is a consequence of the uniform coercivity of a_{y_1} with respect to y_1.

Now we consider the **limit velocity and pressure**: they solve the following problem

$$\text{Find } v^0 \in L^2(0,T;V_{div}) \text{ and } p^0 \in L^2(0,T;L^2(0,L)) \text{ such that}$$
$$\int_0^L p^0(t,y_1)\left(\int_0^1 h(y_1,\eta_1)\,d\eta_1\right)\,dy_1 = 0 \text{ a.e. } t\in[0,T] \text{ and}$$
$$(\nu+\nu_r)\int_0^L \sum_{i=1}^2 a_{y_1}(v_i^0,\varphi_i)\,dy_1 - \int_0^L \frac{\partial p^0}{\partial y_1}\left(\int_0^1 h(y_1,\cdot)\varphi_1\,d\eta_1\right)\,dy_1$$
$$= -(\nu+\nu_r)U_0(t)\int_0^L a_{y_1}\big(\bar{U},\varphi_1\big)\,dy_1 \quad \forall\varphi\in V_{div},\ \text{a.e. } t\in[0,T]$$

For all $y_1 \in [0,L]$, let

$$\widetilde{V}_{y_1} = \left\{\varphi \in \left(\mathcal{C}^\infty\big([0,1];\mathcal{C}^\infty_\sharp(0,1)\big)\right)^2;\ \varphi(0,\cdot)=0,\ -\varphi_1(1,\cdot)\frac{\partial h}{\partial y_1}(y_1,\cdot)+\varphi_2(1,\cdot)=0 \text{ on } (0,1)\right\},$$

$$\widetilde{V}_{y_1,div} = \left\{\varphi\in\widetilde{V}_{y_1};\ h(y_1,\cdot)\frac{\partial\varphi_1}{\partial\eta_1} - y_2\frac{\partial h}{\partial\eta_1}(y_1,\cdot)\frac{\partial\varphi_1}{\partial y_2} + \frac{\partial\varphi_2}{\partial y_2} = 0 \text{ in } Y\right\}$$

$$V_{y_1,div} = \text{closure of } \widetilde{V}_{y_1,div} \text{ in } \left(H^1(0,1;H^1_\sharp(0,1))\right)^2.$$

Let

$$\bar{a}_{y_1}(w,\varphi) = (\nu+\nu_r)\sum_{i=1}^{2} a_{y_1}(w_i,\varphi_i) \quad \forall (w,\varphi)\in V^2_{y_1,div}$$

and $w^1_{y_1}, w^2_{y_1} \in V_{y_1,div}$ be the unique solutions of

$$\bar{a}_{y_1}(w^1_{y_1},\varphi) = -\int_Y h(y_1,\cdot)\varphi_1\, d\eta_1 dy_2 \quad \forall \varphi \in V_{y_1,div}$$

and

$$\bar{a}_{y_1}(w^2_{y_1},\varphi) = -(\nu+\nu_r)a_{y_1}\big(\bar{U}(y_1,\cdot),\varphi_1\big) \quad \forall \varphi\in V_{y_1,div}.$$

Proposition The limit velocity v^0 is uniquely given by

$$v^0(t,y_1,y_2,\eta_1) = \frac{\partial p^0}{\partial y_1}(t,y_1)w^1_{y_1}(y_2,\eta_1) + U_0(t)w^2_{y_1}(y_2,\eta_1) \quad \text{a.e. in } (0,T)\times\Omega\times(0,1).$$

Let $\theta \in \mathcal{D}(0,T)$, $\psi \in \mathcal{C}^\infty\big([0,L]\big)$ and $\psi^\varepsilon(z) = \psi(z_1)$ for all $z = (z_1, z_2) \in \Omega^\varepsilon$. Recalling that $\operatorname{div}_z v^\varepsilon = 0$ in Ω^ε

$$
\begin{aligned}
0 &= \frac{1}{\varepsilon}\int_0^T \int_{\Omega^\varepsilon} \left(\frac{\partial v_1^\varepsilon}{\partial z_1}(t,z) + \frac{\partial v_2^\varepsilon}{\partial z_2}(t,z)\right) \psi^\varepsilon(z)\theta(t)\, dz dt \\
&= -\frac{1}{\varepsilon}\int_0^T \int_{\Omega^\varepsilon} v_1^\varepsilon(t,z)\frac{\partial \psi^\varepsilon}{\partial z_1}(z)\theta(t)\, dz dt = -\int_0^T \int_\Omega v_1^\varepsilon(t,y)(b_\varepsilon \cdot \nabla \psi^\varepsilon)(y)\theta(t)\, dy dt \\
&= -\int_0^T \int_\Omega v_1^\varepsilon(t,y)\frac{\partial \psi}{\partial y_1}(y_1) h\left(y_1, \frac{y_1}{\varepsilon}\right) h^\varepsilon(y)\theta(t)\, dy dt.
\end{aligned}
$$

By passing to the limit as ε tends to zero we get

$$
0 = \int_0^T \int_{\Omega\times(0,1)} v_1^0(t,y,\eta_1)\frac{\partial \psi}{\partial y_1}(y_1) h(y_1,\eta_1)\theta(t)\, d\eta_1 dy dt.
$$

It follows that

$$
\int_0^L \frac{\partial p^0}{\partial y_1}\frac{\partial \psi}{\partial y_1} \underbrace{\left(\int_Y w^1_{y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2\right)}_{=-\bar{a}_{y_1}(w^1_{y_1}, w^1_{y_1})} dy_1 + \int_0^L U_0(t)\frac{\partial \psi}{\partial y_1} \underbrace{\left(\int_Y w^2_{y_1,1} h(y_1,\cdot)\, d\eta_1 dy_2\right)}_{=-\bar{a}_{y_1}(w^1_{y_1}, w^2_{y_1})} dy_1 = 0.
$$

Proposition We have $p^0(t,\cdot) = U_0(t)\bar{p}^0(\cdot)$ where $\bar{p}^0$ is the unique solution in $H^1(0,L)_{|\mathbb{R}}$ of the **stationnary Reynolds problem**

$$\int_0^L \frac{\partial \bar{p}^0}{\partial y_1}\frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1}, w^1_{y_1})\,dy_1 = -\int_0^L \frac{\partial \psi}{\partial y_1}\bar{a}\big(w^1_{y_1}, w^2_{y_1}\big)\,dy_1 \quad \forall \psi \in H^1(0,L)$$

satisfying $\displaystyle\int_0^L \bar{p}^0 \left(\int_0^1 h(\cdot,\eta_1)\,d\eta_1\right)\,dy_1 = 0.$

As a consequence of the uniqueness of p^0, we can state the next result:

Theorem The whole sequences $(\varepsilon^2 p^\varepsilon)_{\varepsilon>0}$, $(v^\varepsilon)_{\varepsilon>0}$ and $(Z^\varepsilon)_{\varepsilon>0}$ satisfy the following convergence:

$$\varepsilon p^\varepsilon \longrightarrow\!\!\!\!\!\!\longrightarrow p^0$$
$$v^\varepsilon \longrightarrow\!\!\!\!\!\!\longrightarrow v^0$$
$$Z^\varepsilon \longrightarrow\!\!\!\!\!\!\longrightarrow Z^0.$$

Proposition We have $p^0(t,\cdot) = U_0(t)\bar{p}^0(\cdot)$ where $\bar{p}^0$ is the unique solution in $H^1(0,L)_{|\mathbb{R}}$ of the **stationnary Reynolds problem**

$$\int_0^L \frac{\partial \bar{p}^0}{\partial y_1}\frac{\partial \psi}{\partial y_1}\bar{a}(w^1_{y_1}, w^1_{y_1})\, dy_1 = -\int_0^L \frac{\partial \psi}{\partial y_1}\bar{a}\big(w^1_{y_1}, w^2_{y_1}\big)\, dy_1 \quad \forall \psi \in H^1(0,L)$$

satisfying $\displaystyle\int_0^L \bar{p}^0 \left(\int_0^1 h(\cdot,\eta_1)\, d\eta_1\right)\, dy_1 = 0.$

As a consequence of the uniqueness of p^0, we can state the next result:

Theorem The whole sequences $(\varepsilon^2 p^\varepsilon)_{\varepsilon>0}$, $(v^\varepsilon)_{\varepsilon>0}$ and $(Z^\varepsilon)_{\varepsilon>0}$ satisfy the following convergence:

$$\begin{aligned} &\varepsilon p^\varepsilon \longrightarrow\!\!\!\!\!\longrightarrow p^0 \\ &v^\varepsilon \longrightarrow\!\!\!\!\!\longrightarrow v^0 \\ &Z^\varepsilon \longrightarrow\!\!\!\!\!\longrightarrow Z^0. \end{aligned}$$

Thank you for your attention