On the viscoelastodynamic problem with Signorini boundary conditions

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The dynamical evolution system for Kelvin-Voigt material is

$$\rho u_{tt} = Au + Bu_t + f, \ x \in \Omega, \ t > 0$$

The boundary conditions are given by

$$\sigma_T^A + (\sigma_T^B)_t = 0,$$
$$\sigma_N^A + (\sigma_N^B)_t \geq 0, \ u_N \geq 0, \ u_N\left(\sigma_N^A + (\sigma_N^B)_t\right) = 0$$

Notations: Let $C = A$ or $C = B$

- σ_N^C: the normal component of the stress vector at the boundary
- σ_T^C: the tangential component of the stress vector at the boundary

Assumptions: the material is homogeneous and isotropic

State variables

$u : [0,T]\times\Omega \to \mathbb{R}$ displacement

Applied field

$\ell : [0,T]\times\Omega \to \mathbb{R}$ density forces

Notations:

- $\Omega = (-\infty, 0)$
- $K = \{ v \in \mathrm{H}^1_{\mathrm{loc}}(\Omega\times[0,\infty)) : v_{xt} \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{L}^2(\Omega)),\ v(0,\cdot) \geq 0 \}$

We consider the following problem **(DI)**:

$u_{tt} - u_{xx} - \alpha u_{xt} = \ell, \quad \alpha > 0$	damped wave equation
$0 \leq u(0,t) \perp (u_x + \alpha u_{xt})(0,t) \geq 0$	unilateral bdry conditions
$u(\cdot,0) = u_0$ and $u_t(\cdot,0) = u_1$	Cauchy initial data

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$$u_{tt} - u_{xx} - \alpha u_{xxt} = \underbrace{\ell}_{\in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))}, \quad \alpha > 0$$ damped wave equation

$$0 \leq u(0,t) \perp (u_x+\alpha u_{xt})(0,t) \geq 0$$ unilateral bdry conditions

$$u(\cdot,0) = \underbrace{u_0}_{\in \mathrm{H}^2(\Omega)} \quad \text{and} \quad u_t(\cdot,0) = \underbrace{u_1}_{\in \mathrm{H}^1(\Omega)}$$ Cauchy initial data

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The weak formulation **(VI)** is given by

Find $u \in K$ such that for all $v \in K$ and for all $\tau \in (0,\infty)$, we have

$$\int_\Omega (u_t(v-u))\big|_0^\tau \mathrm{d}x - \int_0^\tau \int_\Omega u_t(v_t - u_t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega (u_x + \alpha u_{xt})(v_x - u_x)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\tau \int_\Omega \ell(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

Penalized Problem (PP) (here $r^- = -\min(r, 0)$)

$u^\epsilon_{tt} - u^\epsilon_{xx} - \alpha u^\epsilon_{xxt} = \ell, \quad \alpha > 0$	damped wave equation
$(u^\epsilon_x + \alpha u^\epsilon_{xt})(0,t) = \frac{1}{\epsilon}(u^\epsilon(0,t))^-$	normal compliance conditions[1]
$u^\epsilon(\cdot,0) = u_0$ and $u^\epsilon_t(\cdot,0) = u_1$	Cauchy initial data

[1] Martins, Oden. *Nonlinear Anal.*, 1988.

[2] Jarušek. *Czechoslovak Math. J.*, 1996.

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Theorem (Existence and uniqueness results[2])

There exists a unique weak solution $u^\epsilon \in \mathrm{H}^1_{\mathrm{loc}}([0, \infty) \times \Omega)$ *of the problem* **(PP)** *such that* $u^\epsilon_{xt} \in \mathrm{L}^2_{\mathrm{loc}}([0, \infty); \mathrm{L}^2(\Omega))$.

Idea of the proof: Use Galerkin method.

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Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)**.

Sketch of the proof: A-priori estimates give us

- $u^\epsilon \in L^\infty_{loc}([0,\infty); L^2(\Omega))$
- $u^\epsilon_t \in L^\infty_{loc}([0,\infty); L^2(\Omega))$
- $u^\epsilon_x \in L^\infty_{loc}([0,\infty); L^2(\Omega))$
- $u^\epsilon_{xt} \in L^2_{loc}([0,\infty); L^2(\Omega))$
- $u^\epsilon_{xx} \in L^\infty_{loc}([0,\infty); L^2(\Omega))$

Moreover $u_t \in C^0([0,\infty); L^2(\Omega))$ equipped with the weak topology.

Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_\Omega u_t^\epsilon \varphi(v-u^\epsilon)\Big|_0^T \mathrm{d}x - \int_0^T \int_\Omega u_t^\epsilon(\varphi(v-u^\epsilon))_t \,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{1}{\epsilon}\int_0^T \left(\left((u^\epsilon(0,\cdot))^-\right)^2\varphi\right)\mathrm{d}t - \frac{1}{\epsilon}\int_0^T \left((u^\epsilon(0,\cdot))^-\varphi v\right)\mathrm{d}t$$

$$+\int_0^T \int_\Omega (u_x^\epsilon + \alpha u_{xt}^\epsilon)(\varphi(v_x - u_x^\epsilon))\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \ell\varphi(v-u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

where $\varphi \in \mathrm{C}_0^\infty(\bar{\Omega}\times[0,\infty))$ taken its values in $[0,1]$.

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$$+\int_0^T \int_\Omega (\nabla u^\epsilon + \alpha u_{xt}^\epsilon)(\varphi(v_x - u_x^\epsilon))\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_\Omega \ell\varphi(v-u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

where $\varphi \in \mathrm{C}_0^\infty(\bar{\Omega} \times [0,\infty))$ taken its values in $[0,1]$.

Remark: Nothing is known about uniqueness.

AIM: Characterize the trace spaces

Recall that $u(0,t) \in \mathrm{H}^a_{\mathrm{loc}}([0,\infty))$ for $a \in \mathbb{R}$

$$\Updownarrow$$

$$|\omega|^a \widehat{u}(0,\omega) \in \mathrm{L}^2(\mathbb{R})$$

- ω : the dual variable of t
- $\widehat{u}(0,\omega)$: the Fourier transform of $u(0,t)$

Theorem (Regularity of the trace[3])

Let u^ϵ be the solution of **(PP)***. Then we may extract a subsequence, still denoted by u^ϵ, such that*

$$u^\epsilon(0,t) \rightharpoonup u(0,t) \quad \textit{weakly} \quad \textit{in} \quad \mathrm{H}^{5/4}_{\mathrm{loc}}([0,\infty)).$$

Moreover u is a strong solution of **(DI)***.*

[3] P., Schatzman. *C. R. Math. Acad. Sci.*, 2002.

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- Let $v^\epsilon = u^\epsilon - \bar{u}$ be a solution of

$$v^\epsilon_{tt} - (1+\alpha\partial_t)v^\epsilon_{xx} = 0, \quad x \in \Omega, \quad t > 0$$

$$(1+\alpha\partial_t)v^\epsilon_x(0,t) = \bar{g} + \frac{1}{\epsilon}(v^\epsilon(0,t))^- \quad \text{where} \quad \bar{g} = -(\bar{u}_x + \alpha\bar{u}_{xt})(0,t)$$

$$v^\epsilon(\cdot,0) = 0 \quad \text{and} \quad v^\epsilon_t(\cdot,0) = 0$$

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- Use the Fourier transform in t

$$\widehat{v}^\epsilon_{xx}(x,\omega) = \widehat{\lambda}^2\widehat{v}^\epsilon(x,\omega) \quad \text{where} \quad \widehat{\lambda} = \frac{i\omega}{\sqrt{1+i\alpha\omega}}$$

$$\rightsquigarrow \widehat{v}^\epsilon(x,\omega) = \widehat{a}^\epsilon e^{\widehat{\lambda}x}$$

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- Use the boundary conditions

$$((1+\alpha\partial_t)v^\epsilon_x\widehat{)}(0,\omega) = \widehat{\lambda}_1\widehat{v}^\epsilon(0,\omega) \quad \text{where} \quad \widehat{\lambda}_1 = (1+i\alpha\omega)\widehat{\lambda}$$

$$\rightsquigarrow \lambda_1 * v^\epsilon(0,t) = \bar{g} + \tfrac{1}{\epsilon}(v^\epsilon(0,t))^-$$

The energy balance is obtained by multiplying **(DI)** by u_t and integrating over $\Omega\times(0,\tau)$, i.e.

$$\frac{1}{2}\int_\Omega (|u_t|^2+|u_x|^2)\Big|_0^\tau \mathrm{d}x + \alpha\int_0^\tau\int_\Omega |u_{xt}|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau\int_\Omega \ell u_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau (u_x+\alpha u_{xt})(0,t)u_t(0,t)\,\mathrm{d}t$$

The energy losses are purely viscous iff $\int_0^\tau (u_x+\alpha u_{xt})(0,t)u_t(0,t)\,\mathrm{d}t = 0$.

- ▷ $u_t(0,t) = 0$ a.e. on $\mathrm{supp}((u_x+\alpha u_{xt})(0,t))$
- ▷ $(u_x+\alpha u_{xt})(0,t)$ is only a measure

$\not\Rightarrow \int_0^\tau (u_x+\alpha u_{xt})(0,t)u_t(0,t)\,\mathrm{d}t = 0$

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The energy losses are purely viscous iff $\int_0^\tau (u_x^\epsilon+\alpha u_{xt}^\epsilon)(0,t)u_t^\epsilon(0,t)\,\mathrm{d}t = 0$.

- ▷ $u_t^\varepsilon(0,t) \in \mathrm{H}^{1/4}_{\mathrm{loc}}(0,\infty)$
- ▷ $(u_x^\varepsilon+\alpha u_{xt}^\varepsilon)(0,t) \in \mathrm{H}^{-1/4}_{\mathrm{loc}}(0,\infty)$

$\not\Rightarrow$ $\int_0^\tau (u_x^\epsilon+\alpha u_{xt}^\epsilon)(0,t)u_t^\epsilon(0,t)\,\mathrm{d}t = 0$

(DI) is reduced to pseudodifferential linear complementarity pb **(LCP)**

$$\lambda_1 * w(t) = \bar{g} + b, \quad 0 \leq w(t) \perp b \geq 0 \quad \text{with} \quad w(t) = v(0,t)$$

Assumptions & Notations: let $\sigma_j, \tau_j : \ 0 \leq \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \dots$

▷ $\operatorname{supp} w \subset \bigcup_j [\sigma_j, \tau_j]$ and $\operatorname{supp} b \subset \bigcup_j [\tau_j, \sigma_{j+1}]$

▷ $\bar{g} = \mu * \phi$, ϕ is a measure, $\mu = (\pi t)^{-1/2}$

▷ $\omega = \frac{H(t)}{\pi(t+1)\sqrt{t}}$ and $\nu = \frac{H(t)\mathrm{e}^{-t/\alpha}}{\alpha}$ with H: Heaviside function

▷ $\phi_0 = \phi$ and $\phi_{j+1} = 1_{[\sigma_{j+1},\infty)}(\phi_j + \int_{[\tau_j,\sigma_{j+1})} \mathrm{e}^{-(\cdot - s)/\alpha} \omega(\frac{\cdot - \sigma_{j+1}}{\sigma_{j+1} - s}) \frac{\psi_j(s)}{\sigma_{j+1} - s}$

with $\psi_j = \phi_j 1_{[\tau_j,\sigma_{j+1})} + \delta(\cdot - \tau_j)\mathrm{e}^{-\tau_j/\alpha} \int_{[\sigma_j,\tau_j)} \mathrm{e}^{s/\alpha} \phi_j$

Then w is given by

$$w 1_{[\sigma_j,\tau_j]} = \big(H * \nu * \phi_j\big) 1_{[\sigma_j,\tau_j]}$$

We do not know if for a given $\bar{g} = \mu * \phi$, there is a solution of **(LCP)** having a locally finite structure $\Rightarrow$ construction of w^n satisfying **(ALCP)**, i.e.

$$\lambda_1 * w^n(t) = \bar{g}^n + b^n, \quad 0 \le w^n(t) \perp b^n \ge 0$$

A-priori estimates **(ALCP)** imply that there exists a subsequence s.t.

- ▷ $w^n \to w$ uniformly on compact sets and $w_t^n \rightharpoonup w_t$ in $\mathrm{L}^\infty(0,T)$ weakly $*$
- ▷ $b^n \rightharpoonup b$ in $\mathcal{M}^1(0,T)$ weakly $*$
- ▷ w and b satisfy **(LCP)**

In particular, w_t is essentially bounded $\Rightarrow$ b has no atoms.

Theorem

Let N be the set of atoms of ϕ, $N_1 = \cup_j \{\sigma_j, \tau_j\}$, $U = \{t \in \mathbb{R} : w(t) > 0\}$. Then for all $t \notin N \cup N_1 \cup U$, w is differentiable at t with $w_t(t) = 0$.

$\Rightarrow \langle w_t, b\rangle = 0 \Rightarrow$ the energy losses are purely viscous.

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Thank you for your attention !

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