A time-stepping scheme for multibody dynamics with unilateral constraints

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January 20, 2010

Description of the dynamics

We consider a mechanical system with a finite number of degrees of freedom. The unconstrained dynamics is given by

$$M(q)\ddot{q} = g(t, q, \dot{q}).$$

We assume that the system is submitted to unilateral constraints described by

$$q(t) \in K = \Big\{ q \in \mathbb{R}^d;\ f_\alpha(q) \geq 0\ \forall \alpha \in \{1, \dots, \nu\} \Big\}, \quad \nu \geq 1.$$

Adding the reaction force due to the constraints, we obtain

$$M(q)\ddot{q} = g(t, q, \dot{q}) + R, \quad \mathrm{Supp}(R) \subset \Big\{ t; q(t) \in \partial K \Big\}.$$

We assume moreover that the constraints are perfect, i.e.

• there is no adhesion

$$\forall v \in T_K(q) : \ (R, v) \geq 0,$$

• contact is without friction

$$\forall v \in T_K(q) \cap \big(-T_K(q)\big) : \ (R, v) = 0,$$

with

$$T_K(q) = \big\{w \in \mathbb{R}^d ; \ \big(\nabla f_\alpha(q), w\big) \geq 0 \ \forall \alpha \in J(q)\big\},$$

and

$$J(q) = \big\{\alpha \in \{1, \dots, \nu\} ; \ f_\alpha(q) \leq 0\big\}.$$

Using Farkas' lemma we infer that

$$R = \sum_{\alpha \in J(q)} \lambda_\alpha \nabla f_\alpha(q), \quad \lambda_\alpha \geq 0.$$

Moreover the velocity may be discontinuous whenever $q(t) \in \partial K$ since

$$\dot{q}^+(t) \in T_K\big(q(t)\big), \quad \dot{q}^-(t) \in T_K\big(q(t)\big).$$

So R is a measure and we get the following Measure Differential Inclusion

$$M(q)\ddot{q} - g(t, q, \dot{q}) \in -N_K(q).$$

It follows that

$$M\big(q(t)\big)\big(\dot{q}^+(t) - \dot{q}^-(t)\big) \in N_K\big(q(t)\big) \text{ if } q(t) \in \partial K.$$

If $J\big(q(t)\big) = \{\alpha\}$ we infer that there exists $e \geq 0$ such that

$$\dot{q}^+(t) = \dot{q}^-(t) - (1+e)\frac{\big(\nabla f_\alpha\big(q(t)\big), \dot{q}^-(t)\big)}{\big(\nabla f_\alpha\big(q(t)\big), M^{-1}\big(q(t)\big)\nabla f_\alpha\big(q(t)\big)\big)}$$

which can be rewritten as

$$\begin{aligned}&\dot{q}^+(t) = -e\dot{q}^-(t) + (1+e)\mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big)\\ &= -e\mathrm{Proj}_{M(q(t))}\big(M^{-1}\big(q(t)\big)N_K\big(q(t)\big), \dot{q}^-(t)\big) + \mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big).\end{aligned}$$

The kinetic energy satisfies

$$\begin{aligned}&\mathcal{E}^+(t) = \frac{1}{2}\big|\dot{q}^+(t)\big|^2_{M(q(t))}\\ &= \frac{1}{2}\Big(\big|\mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big)\big|^2_{M(q(t))} + e^2\big|\mathrm{Proj}_{M(q(t))}\big(M^{-1}\big(q(t)\big)N_K\big(q(t)\big), \dot{q}^-(t)\big)\big|^2_{M(q(t))}\Big)\end{aligned}$$

and $\mathcal{E}^+(t) \leq \mathcal{E}^-(t)$ if $e \in [0,1]$ (mechanical consistency).

In the general case, the transmission of the velocity at impacts is modelled by the Newton's law

$$\dot{q}^+(t) = -e\mathrm{Proj}_{M(q(t))}\big(N_K^*\big(q(t)\big), \dot{q}^-(t)\big) + \mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big)$$

with a restitution coefficient $e \in [0,1]$.

Remarks:

- This model is mechanically consistent
- If $e = 1$, the kinetic energy is conserved at impacts (elastic shocks)
- If $e = 0$, we have

$$\dot{q}^+(t) = \mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big) = \mathrm{Argmin}_{u \in T_K(q(t))} \|u - \dot{q}^-(t)\|_{M(q(t))}$$

(standard inelastic shocks)

- If $\mathrm{Card}\big(J\big(q(t)\big)\big) > 1$ this is not the only mechanically consistent model.

We consider the following Cauchy problem:

Problem (P) Let $(q_0, u_0) \in K \times T_K(q_0)$ be admissible initial data. Find a function $q : [0, \tau] \to \mathbb{R}^d$, with $\tau > 0$, s.t.

(P1) $q \in C^0\big([0, \tau]; \mathbb{R}^d\big)$, $\dot{q} \in BV(0, \tau; \mathbb{R}^d)$,

(P2) $q(t) \in K$ for all $t \in [0, \tau]$,

(P3) there exists a non negative measure μ such that the Stieltjes measure $d\dot{q} = \ddot{q}$ and the Lebesgue's measure dt admit densities relatively to $d\mu$, denoted respectively u'_μ and t'_μ, and

$$M\big(q(t)\big)u'_\mu(t) - g\big(t, q(t), \dot{q}(t)\big)t'_\mu(t) \in -N_K\big(q(t)\big) \quad d\mu \text{ a.e.},$$

(P4) $q(0) = q_0$, $\dot{q}^+(0) = u_0$,

(P5) $\dot{q}^+(t) = -e\dot{q}^-(t) + (1+e)\mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big)$ for all $t \in (0, \tau)$.

Existence and approximation of solutions

A lot of results since M.Schatzman (78).

- **Penalty method**

M.Schatzman (78), M.Carriero - E.Pascali (80), G.Buttazzo - D.Percivale (81, 83), LP - M.Schatzman (93), M.Schatzman (01)

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- **Time-discretization at the position level**

LP - M.Schatzman (93), LP - M.Schatzman (98), LP - M.Schatzman (02)

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• **Time-discretization at the position level**

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• **Time-discretization at the velocity level**

J.J.Moreau (83, 85 ...), M.Monteiro Marques (87, 93), M.Mabrouk (98), R.Dzonou - M.Monteiro Marques - LP (06, 09).

Uniqueness is not true in general (A.Bressan 1959)).

Existence and approximation of solutions: multi-constraint case ($\nu > 1$)

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New difficulty: continuity on data does not hold in general but holds if the following "angle condition" on the active constraints is satisfied:

$$\begin{array}{ll} (\nabla f_\alpha(q), M(q)^{-1}\nabla f_\beta(q)) \leq 0 & \text{if } e = 0 \\ (\nabla f_\alpha(q), M(q)^{-1}\nabla f_\beta(q)) = 0 & \text{if } e \neq 0 \end{array}$$

for all $(\alpha, \beta) \in J(q)^2$ such that $\alpha \neq \beta$, for all $q \in \partial K$.

References: P.Ballard (00) and LP (05).

A "simple" case: $M \equiv \mathrm{Id}$ and K convex

The MDI can be rewritten as

$$\ddot{q} + \partial\psi_K(q) \ni g(t, q, \dot{q})$$

and we propose the following implicit time-discretization

$$\frac{q^{n+1} - 2q^n + q^{n-1}}{h^2} + \partial\psi_K(q^{n+1}) \ni G^n$$

which is equivalent to

$$q^{n+1} = \mathrm{Proj}(K, 2q^n - q^{n-1} + h^2 G^n)$$

where G^n is an approximate value of $g(t, q, \dot{q})$ at $t = t_n = nh$.

We initialize the algorithm by defining

$$q^0 = q_0, \quad q^1 = \mathrm{Proj}\big(K, q_0 + hu_0 + hz(h)\big) \quad \text{with } \lim_{h \to 0} z(h) = 0.$$

Example (bouncing ball): $d = 1$, $K = \mathbb{R}^+$, $M(q) \equiv 1$, $g \equiv 0$, $q_0 = 1$, $u_0 = -1$.

The solution of problem (P) is

$$q(t) = 1 - t \quad \text{if } t \in [0, 1], \quad q(t) = 0 \quad \text{if } t \geq 1.$$

Assume that $h \in (0, 1/2)$. We obtain $q^0 = 1$, $q^1 = 1 - h$ and for all $n \geq 1$

$$q^{n+1} = \mathrm{Proj}(\mathbb{R}^+, 2q^n - q^{n-1}) = \max(2q^n - q^{n-1}, 0).$$

There exists $p \geq 1$ ($p = \lfloor 1/h \rfloor - 1$) such that

$$p = \max\left\{k \geq 0;\ 2q^n - q^{n-1} \geq 0\ \forall n \in \{0, \ldots, k\}\right\}$$

and $q^n = 1 - nh$ for all $n \in \{0, \ldots, p+1\}$. Then

$$q^{p+2} = 0, \quad 2q^{p+2} - q^{p+1} = -q^{p+1} \leq 0$$

and $q^k = 0$ for all $k \geq p + 2$.

Let us assume

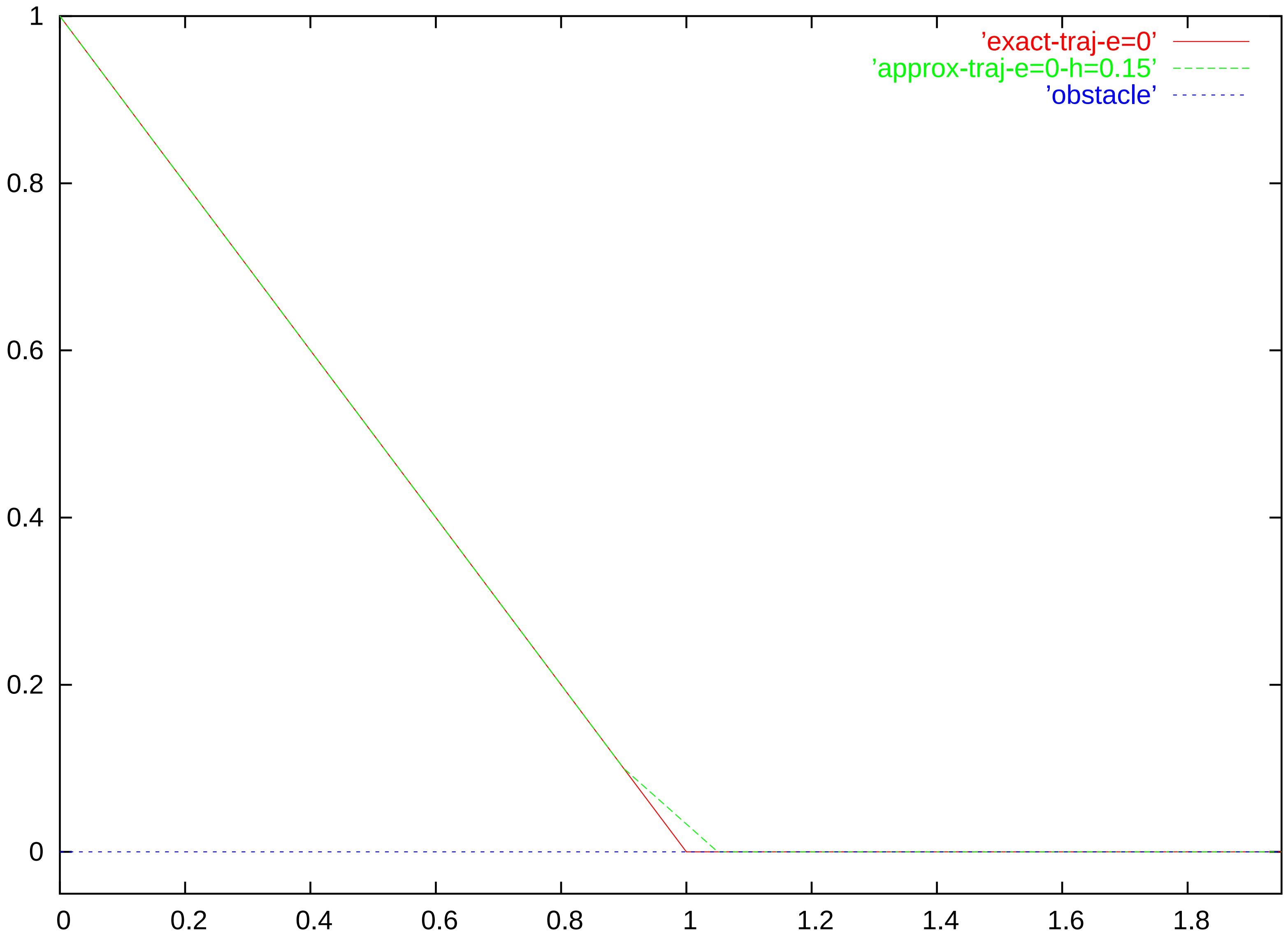
(H1) g is a continuous function from $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ $(T > 0)$ to $\mathbb{R}^d$,

(H2) for all $\alpha \in \{1, \ldots, \nu\}$, the function f_α belongs to $C^1(\mathbb{R}^d; \mathbb{R})$, ∇f_α is Lispchitz continuous and does not vanish in a neighbourhood of $\big\{q \in \mathbb{R}^d \ : \ f_\alpha(q) = 0\big\}$,

(H3) the active constraints along are functionnally independent i.e., for all $q \in K$ the vectors $\big(\nabla f_\alpha(q)\big)_{\alpha \in J(q)}$ are linearly independent.

We define the approximate solution $(q_h)_{h>0}$ by a linear interpolation of the q^n's, i.e.

$$q_h(t) = q^n + (t - nh)\frac{q^{n+1} - q^n}{h} \quad \forall t \in \big[nh, (n+1)h\big] \cap [0, T]$$


'exact-traj-e=0'
'approx-traj-e=0-h=0.15'
'obstacle'
1
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0.2
0.4
0.6
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1
1.2
1.4
1.6
1.8

Theorem (LP 2005) Let $(q_0, u_0) \in K \times T_K(q_0)$ be admissible data. Then there exist $\tau \in (0, T]$ and a subsequence of $(q_h)_{h>0}$, still denoted $(q_h)_{h>0}$, such that

$$q_h \to q \quad \text{in } C^0\big([0,\tau]; \mathbb{R}^d\big)$$

and q satisfies the properties (P1)-(P2)-(P3)-(P4).

If we assume moreover that, for all $\widetilde{q} \in \partial K$, we have

$$\big(\nabla f_\alpha(\widetilde{q}), \nabla f_\beta(\widetilde{q})\big) \leq 0 \quad \text{for all } (\alpha, \beta) \in J(\widetilde{q})^2 \text{ such that } \alpha \neq \beta,$$

then the limit function q satisfies also (P5) with $e = 0$, i.e.

$$\dot{q}^+(t) = \operatorname{proj}\Big(T_K\big(q(t)\big), \dot{q}^-(t)\Big) \quad \forall t \in (0, \tau)$$

and q is a solution of the Cauchy problem.

Furthermore, if g is Lispchitz continuous in its last two arguments, uniformly with respect to the first one, the previous convergence holds on the whole time interval $[0, T]$.

Sketch of the proof

Step 1: We establish uniform estimates for the discrete velocities and accelerations.

Lemma 1: For all $n \geq 1$, let $V^n = \dfrac{q^{n+1} - q^n}{h}$. Then

$$V^{n-1} - V^n + hG^n \in N_K(q^{n+1}), \quad V^n \in -T_K(q^{n+1}).$$

As a consequence

$$\|V^n\| \leq \|V^{n-1}\| + h\|G^n\|.$$

Step 2: We define the approximate solutions $(q_h)_{h>0}$ by a linear interpolation of the q^n's. We pass to the limit by using Ascoli's and Helly's theorems. Hence the limit q satisfies the property (P1) and, by using again lemma 1, we prove that q takes its values in K and satisfies the MDI and the initial conditions.

Step 3: We assume now that the "angle condition" holds i.e. for all $\widetilde{q} \in \partial K$:

$$\big(\nabla f_\alpha(\widetilde{q}), \nabla f_\beta(\widetilde{q})\big) \leq 0 \quad \text{for all } (\alpha, \beta) \in J(\widetilde{q})^2 \text{ such that } \alpha \neq \beta.$$

We observe first that

$$\dot{q}^+(t) \in T_K\big(q(t)\big), \quad \dot{q}^-(t) \in -T_K\big(q(t)\big), \quad \dot{q}^+(t) - \dot{q}^-(t) \in -N_K\big(q(t)\big).$$

Hence the impact law reduces to $\big(\dot{q}^-(t) - \dot{q}^+(t), \dot{q}^+(t)\big) = 0$ and we have

$$\dot{q}^+(t) - \dot{q}^-(t) = \sum_{\alpha \in J(q(t))} -\mu_\alpha \nabla f_\alpha\big(q(t)\big),\ \mu_\alpha \leq 0,\ \big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t)\big) \geq 0 \quad \forall \alpha \in J\big(q(t)\big).$$

Consequently we only need to prove the following complementarity condition

$$\mu_\alpha \big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t)\big) = 0 \quad \text{for all } \alpha \in J\big(q(t)\big)$$

i.e

$$\big(\nabla f_\alpha\big(q(t)\big), \dot q^+(t)\big) \le 0 \quad \text{for all } \alpha \in J\big(q(t)\big) \text{ such that } \mu_\alpha \ne 0.$$

Using again lemma 1, we get

$$V^{n-1} - V^n + hG^n = \sum_{\beta \in J(q^{n+1})} \mu_\beta^n \nabla f_\beta(q^{n+1}), \quad \mu_\beta^n \le 0.$$

We infer that, if $\mu_\alpha \ne 0$, in any neighbourhood $\mathcal{V}$ of the impact instant t, there exists at least one discrete impact i.e. there exists at least a discrete instant t_{n_i} such that $f_\alpha(q^{n_i+1}) \le 0$.

It follows that $\alpha \in J(q^{n_i+1})$ and $V^{n_i} \in -T_K(q^{n_i+1})$ thus

$$\big(\nabla f_\alpha(q^{n_i+1}), V^{n_i}\big) \le 0.$$

Finally, by considering the last discrete impact $t_{n_i} \in \mathcal{V}$ and using the "angle condition" we obtain $\big(\nabla f_\alpha\big(q(t)\big), \dot q^+(t)\big) \le 0$.

Let us consider now $e \in [0,1]$ but still $M \equiv \mathrm{Id}$ and K convex. The vibro-impact problem is described by the MDI

$$\ddot{q} + \partial\psi_K(q) \ni g(t, q, \dot{q})$$

and the impact law

$$\dot{q}^+ = -e\dot{q}^- + (1+e)\mathrm{Proj}\big(T_K(q), \dot{q}^-\big) = -e\mathrm{Proj}\big(N_K(q), \dot{q}^-\big) + \mathrm{Proj}\big(T_K(q), \dot{q}^-\big).$$

Starting from the model problem of the bouncing ball, propose the following algorithm:

$$\frac{q^{n+1} - 2q^n + q^{n-1}}{h^2} + \partial\psi_K\left(\frac{q^{n+1} + eq^{n-1}}{1+e}\right) \ni G^n$$

which can be rewritten as

$$q^{n+1} = -eq^{n-1} + (1+e)\mathrm{Proj}\left(K, \frac{2q^n - (1-e)q^{n-1} + h^2G^n}{1+e}\right).$$

Example (bouncing ball): $d = 1$, $K = \mathbb{R}^+$, $M(q) \equiv 1$, $f \equiv 0$, $q_0 = 1$, $u_0 = -1$.

The solution of problem (P) is

$$\begin{cases} q(t) = 1 - t & \text{if } t \in [0, 1], \\ q(t) = e(t - 1) & \text{if } t \geq 1. \end{cases}$$

Assume that $h \in (0, 1/2)$. We obtain $q^0 = 1$, $q^1 = 1 - h$ and

$$q^{n+1} = -eq^{n-1} + (1+e)\mathrm{Proj}\left(\mathbb{R}^+, \frac{2q^n - (1-e)q^{n-1}}{1+e}\right) = -eq^{n-1} + \max(2q^n - (1-e)q^{n-1}, 0).$$

There exists $p \geq 1$ such that

$$p = \max\Big\{k \geq 0;\ 2q^n - (1-e)q^{n-1} \geq 0\ \forall n \in \{0, \ldots, k\}\Big\}$$

and $q^n = 1 - nh$ for all $n \in \{0, \ldots, p+1\}$. Then $q^{p+2} = -eq^p$.

But

$$2q^{p+2} - (1-e)q^{p+1} = -2eq^{p} - (1-e)(2q^{p} - q^{p-1}) = -\big(2q^{p} - (1-e)q^{p-1}\big) \leq 0$$

so $q^{p+3} = -eq^{p+1}$ and

$$q^{k+p+2} = q^{p+2} + ekh \quad \forall k \geq 0.$$

General case: $e \in [0,1]$, $M \not\equiv \mathrm{Id}$ and /or K not convex

• We extend first this definition to the case of a non trivial inertia operator but still a convex set K by considering the projection on K relatively to the kinetic metric at q^n instead of the projection relatively to the Euclidean metric.

• In the case of a non convex set K we extend once again the definition of the algorithm by replacing the projection on K by the Argmin of the distance.

More precisely we propose now the following time-stepping scheme

$$q^0 = q_0, \quad q^1 \in \mathrm{Argmin}_{Z \in K} \left\| q_0 + hu_0 + hz(h) - Z \right\|_{M(q_0)}, \quad \lim_{h \to 0} z(h) = 0$$

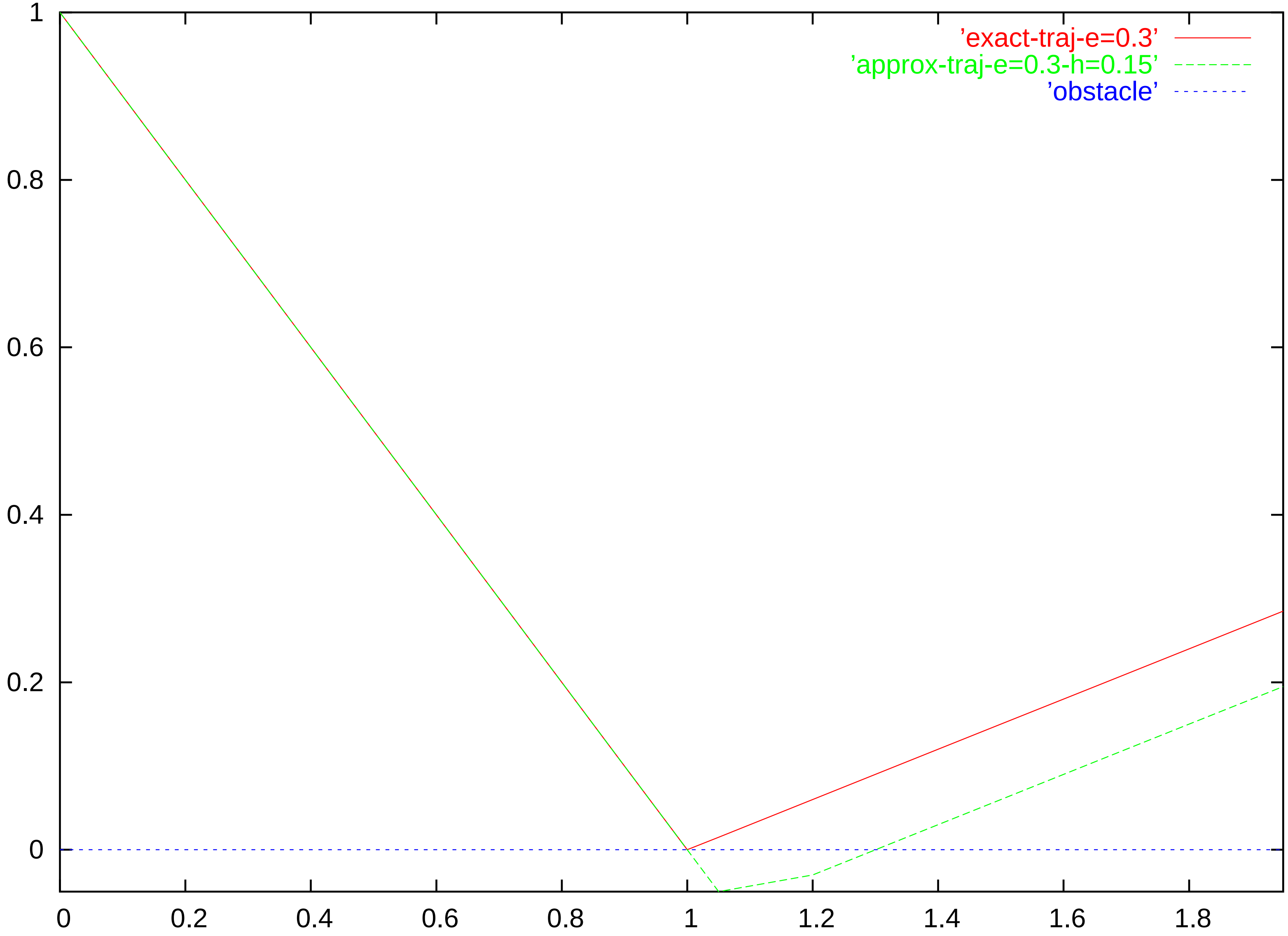
and for all $n \geq 1$

$$q^{n+1} = -eq^{n-1} + (1+e)Z^n$$

with

$$Z^n \in \mathrm{Argmin}_{Z \in K} \left\| \frac{2q^n - (1-e)q^{n-1} + h^2 G^n}{1+e} - Z \right\|_{M(q^n)}$$

and G^n is an approximate value of $M^{-1}(q)g(t,q,\dot{q})$ at $t = t_n = nh$.



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1.2
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Let us assume

(H1) g is a continuous function from $[0,T] \times \mathrm{I\!R}^d \times \mathrm{I\!R}^d$ $(T > 0)$ to $\mathrm{I\!R}^d$,

(H2) for all $\alpha \in \{1,\ldots,\nu\}$, the function f_α belongs to $C^1(\mathrm{I\!R}^d;\mathrm{I\!R})$, ∇f_α is Lispchitz continuous and does not vanish in a neighbourhood of $\{q \in \mathrm{I\!R}^d \ : \ f_\alpha(q) = 0\}$,

(H3) the active constraints along are functionnally independent i.e., for all $q \in K$ the vectors $\big(\nabla f_\alpha(q)\big)_{\alpha \in J(q)}$ are linearly independent.

(H4) M is a mapping of class C^1 from $\mathrm{I\!R}^d$ to the set of symmetric positive definite $d \times d$ matrices.

Without further assumptions on M and g, we can not expect a global existence result for problem (P) on $[0,T]$.

Indeed, for any solution q defined on $[0,\tau]$ (with $\tau \in (0,T]$), we have

$$E_k^+(t) \leq E_k^+(0) + \int_0^t \Big(g\big(s,q(s),\dot{q}(s)\big),\dot{q}(s)\Big)\,ds$$
$$+\frac{1}{2}\int_0^t \Big(\dot{q}(s),\big(dM\big(q(s)\big)\dot{q}(s)\big)\dot{q}(s)\Big)\,ds \quad \forall t \in [0,\tau)$$

and finite time explosion may occur. Nevertheless, we can establish that

Proposition (energy estimate): Let $C > \|u_0\|_{M(q_0)}$. Then, there exists $\tau(C) \in (0,T]$ s.t., for any solution q of problem (P) defined on $[0,\tau]$, we have

$$\|q(t)-q_0\| \leq C \quad \forall t \in \big[0,\min\big(\tau(C),\tau\big)\big],$$
$$\|\dot{q}(t)\|_{M(q(t))} \leq C \quad dt \text{ a.e. on } \big[0,\min\big(\tau(C),\tau\big)\big].$$

We define once again the approximate solutions $(q_h)_{h>0}$ by a linear interpolation of the q^n's and we establish the convergence to a solution of the Cauchy problem on $\big[0,\tau(C)\big]$ for any $C > \|u_0\|_{M(q_0)}$.

First we observe that

Lemma 2: For all $n \geq 1$

$$M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(Z^n).$$

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Next we prove

Lemma 3: For all $n \geq 2$ and for all $\alpha \in J(Z^n)$

$$\big(\nabla f_\alpha(Z^n), V^n + eV^{n-2}\big) \leq \mathcal{O}(h)\|V^n + eV^{n-2}\|^2.$$

Proof: If $\alpha \in J(Z^n)$, we have $f_\alpha(Z^n) = 0$ and $f_\alpha(Z^{n-1}) \geq 0$, thus

$$0 \leq f_\alpha(Z^{n-1}) - f_\alpha(Z^n) = \int_0^1 \big(\nabla f_\alpha(Z^n + s(Z^{n-1} - Z^n)), Z^{n-1} - Z^n\big)\, ds.$$

Observing that $Z^n - Z^{n-1} = \dfrac{h}{1+e}(V^n + eV^{n-2})$, we get

$$\big(\nabla f_\alpha(Z^n), V^n + eV^{n-2}\big) \leq -\int_0^1 \big(\nabla f_\alpha(Z^n + s(Z^{n-1} - Z^n)) - \nabla f_\alpha(Z^n), V^n + eV^{n-2}\big)\, ds$$
$$\leq \mathcal{O}(h)\|V^n + eV^{n-2}\|^2.$$

Case 1: $e = 0$ With lemma 2 and lemma 3 we infer that

$$Z^n = q^{n+1} \in K, \quad M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(q^{n+1})$$

and

$$\big(\nabla f_\alpha(q^{n+1}), V^n\big) \leq \mathcal{O}(h)\|V^n\|^2 \quad \forall \alpha \in J(q^{n+1}).$$

Hence $-V^n$ does not belong necessarily to $T_K(q^{n+1})$. We can still reproduce the convergence proof as in the "simple" case but now we have to deal with some $\mathcal{O}(h)$ perturbating terms coming from the variation of the kinetic metric and the lack of convexity of K.

Case 2: $e \neq 0$ With lemma 2 and lemma 3 we infer that

$$Z^n = \frac{q^{n+1} + eq^{n-1}}{1+e} \in K, \quad M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(Z^n)$$

and

$$\big(\nabla f_\alpha(Z^n), V^n + eV^{n-2}\big) \leq \mathcal{O}(h)\|V^n + eV^{n-2}\|^2 \quad \forall \alpha \in J(Z^n).$$

Consequently it is much more difficult to prove a priori uniform estimates for the discrete velocities and accelerations.
Indeed, let us assume again that $M(q) \equiv \mathrm{Id}$ and K convex, we get

$$V^n + eV^{n-2} \in -T_K(Z^n)$$

and

$$(V^{n-1} - V^n + hG^n, V^n + eV^{n-2}) \geq 0$$

which yields

$$(1-e)\|V^n\|^2 \leq (1+e)\|V^{n-1}\|^2 + 2e\|V^{n-2}\|^2 + \mathcal{O}(h).$$

If K is not convex and/or $M(q) \not\equiv \mathrm{Id}$, we obtain the same kind of estimate up to $\mathcal{O}(h)$ perturbating terms.

Nevertheless, with more technicalities, we can still prove that the sequence $(q_h)_{h>0}$ is uniformly Lipschitz continuous on a non trivial time interval $[0,\tau]$, $\tau \in (0,T]$, and the sequence $(\dot{q}_h)_{h>0}$ is bounded in $BV(0,\tau;\mathbb{R}^d)$.

Furthermore it is also more difficult to establish that the limit q satisfies the impact law

$$\dot{q}^+(t) = -e\dot{q}^-(t) + (1+e)\mathrm{Proj}_{M(q(t))}\big(T_K\big(q(t)\big), \dot{q}^-(t)\big). \tag{1}$$

With (P1), (P2) and (P3) we know that

$$\dot{q}^+(t) \in T_K\big(q(t)\big), \quad \dot{q}^-(t) \in -T_K\big(q(t)\big), \quad M\big(q(t)\big)\big(\dot{q}^+(t) - \dot{q}^-(t)\big) \in -N_K\big(q(t)\big). \tag{2}$$

It follows that (1) holds if $q(t) \in \mathrm{Int}(K)$. Otherwise $J\big(q(t)\big) \neq \emptyset$ and

$$M\big(q(t)\big)\big(\dot{q}^+(t) - \dot{q}^-(t)\big) = -\sum_{\alpha \in J(q(t))} \mu_\alpha \nabla f_\alpha\big(q(t)\big),\ \mu_\alpha \leq 0.$$

Hence (1) reduces to

$$\mu_\alpha\big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t) + e\dot{q}^-(t)\big) = 0,\ \big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t) + e\dot{q}^-(t)\big) \geq 0$$

for all $\alpha \in J\big(q(t)\big)$.

Recalling the "angle condition" i.e. for all $\widetilde{q} \in \partial K$

$$\big(\nabla f_\alpha(\widetilde{q}), M^{-1}(\widetilde{q})\nabla f_\beta(\widetilde{q})\big) = 0 \quad \text{for all } (\alpha, \beta) \in J(\widetilde{q})^2 \text{ such that } \alpha \neq \beta$$

and using (2), we get

$$\big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t)\big) \geq 0, \quad \big(\nabla f_\alpha\big(q(t)\big), \dot{q}^-(t)\big) \leq 0$$

and

$$\big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t)\big) = \big(\nabla f_\alpha\big(q(t)\big), \dot{q}^-(t)\big) - \mu_\alpha \big\|\nabla f_\alpha\big(q(t)\big)\big\|^2_{M^{-1}(q(t))}.$$

If $\mu_\alpha = 0$ the conclusion follows immediately. Otherwise, we have to prove that

$$\big(\nabla f_\alpha\big(q(t)\big), \dot{q}^+(t) + e\dot{q}^-(t)\big) = 0.$$

By using lemma 2, we can prove as in the "simple case" the existence of discrete impacts and using lemma 3, we finally get the conclusion.

Implementation: some references

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Thank you for your attention