A time-stepping scheme for multibody dynamics with unilateral constraints

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January 20, 2010
**Description of the dynamics**

We consider a mechanical system with a finite number of degrees of freedom. The unconstrained dynamics is given by

\[ M(q)\ddot{q} = g(t, q, \dot{q}). \]

We assume that the system is submitted to unilateral constraints described by

\[ q(t) \in K = \{ q \in \mathbb{R}^d; \ f_\alpha(q) \geq 0 \ \forall \alpha \in \{1, \ldots, \nu\}\}, \ \nu \geq 1. \]

Adding the reaction force due to the constraints, we obtain

\[ M(q)\ddot{q} = g(t, q, \dot{q}) + R, \quad \text{Supp}(R) \subset \{ t; q(t) \in \partial K \}. \]
We assume moreover that the constraints are perfect, i.e.

- there is **no adhesion**

\[
\forall v \in T_K(q) : (R, v) \geq 0,
\]

- contact is **without friction**

\[
\forall v \in T_K(q) \cap (-T_K(q)) : (R, v) = 0,
\]

with

\[
T_K(q) = \{ w \in \mathbb{R}^d; \ (\nabla f_\alpha(q), w) \geq 0 \ \forall \alpha \in J(q) \},
\]

and

\[
J(q) = \{ \alpha \in \{1, \ldots, \nu\}; \ f_\alpha(q) \leq 0 \}.
\]
Using Farkas’ lemma we infer that

\[ R = \sum_{\alpha \in J(q)} \lambda_\alpha \nabla f_\alpha(q), \quad \lambda_\alpha \geq 0. \]

Moreover the velocity may be discontinuous whenever \( q(t) \in \partial K \) since

\[ \dot{q}^+(t) \in T_K(q(t)), \quad \dot{q}^-(t) \in T_K(q(t)). \]

So \( R \) is a measure and we get the following Measure Differential Inclusion

\[ M(q)\dot{q} - g(t, q, \dot{q}) \in -N_K(q). \]

It follows that

\[ M(q(t))(\dot{q}^+(t) - \dot{q}^-(t)) \in N_K(q(t)) \text{ if } q(t) \in \partial K. \]
If $J(q(t)) = \{\alpha\}$ we infer that there exists $e \geq 0$ such that

$$
\dot{q}^+(t) = \dot{q}^-(t) - (1 + e)\frac{(\nabla f_\alpha(q(t)), \dot{q}^-(t))}{(\nabla f_\alpha(q(t)), M^{-1}(q(t)) \nabla f_\alpha(q(t)))}
$$

which can be rewritten as

$$
\dot{q}^+(t) = -e\dot{q}^-(t) + (1 + e)\text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t))
= -e\text{Proj}_{M(q(t))}(M^{-1}(q(t)) N_K(q(t)), \dot{q}^-(t)) + \text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t)).
$$

The kinetic energy satisfies

$$
\mathcal{E}^+(t) = \frac{1}{2}\left|\dot{q}^+(t)\right|^2_{M(q(t))}
= \frac{1}{2}\left(|\text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t))|^2_{M(q(t))} + e^2|\text{Proj}_{M(q(t))}(M^{-1}(q(t)) N_K(q(t)), \dot{q}^-(t))|^2_{M(q(t))}\right)
$$

and $\mathcal{E}^+(t) \leq \mathcal{E}^-(t)$ if $e \in [0, 1]$ (mechanical consistency).
In the general case, the transmission of the velocity at impacts is modelled by the Newton’s law

\[ \dot{q}^+(t) = -e \text{Proj}_{M(q(t))}(N^*_K(q(t)), \dot{q}^-(t)) + \text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t)) \]

with a restitution coefficient \( e \in [0, 1] \).

**Remarks:**

- This model is mechanically consistent
- If \( e = 1 \), the kinetic energy is conserved at impacts (elastic shocks)
- If \( e = 0 \), we have

\[ \dot{q}^+(t) = \text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t)) = \text{Argmin}_{u \in T_K(q(t))} \| u - \dot{q}^-(t) \|_{M(q(t))} \]

(standard inelastic shocks)
- If \( \text{Card}(J(q(t))) > 1 \) this is not the only mechanically consistent model.
We consider the following Cauchy problem:

**Problem (P)** Let \((q_0, u_0) \in K \times T_K(q_0)\) be admissible initial data. Find a function \(q : [0, \tau] \to \mathbb{R}^d\), with \(\tau > 0\), s.t.

(P1) \(q \in C^0([0, \tau]; \mathbb{R}^d), \dot{q} \in BV(0, \tau; \mathbb{R}^d),\)

(P2) \(q(t) \in K\) for all \(t \in [0, \tau],\)

(P3) there exists a non negative measure \(\mu\) such that the Stieltjes measure \(d\dot{q} = \dot{q}\) and the Lebesgue’s measure \(dt\) admit densities relatively to \(d\mu\), denoted respectively \(u^\prime_\mu\) and \(t^\prime_\mu\), and

\[
M(q(t))u^\prime_\mu(t) - g(t, q(t), \dot{q}(t))t^\prime_\mu(t) \in -N_K(q(t)) \quad d\mu \text{ a.e.},
\]

(P4) \(q(0) = q_0, \dot{q}^+(0) = u_0,\)

(P5) \(\dot{q}^+(t) = -e\dot{q}^-(t) + (1 + e)\text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t))\) for all \(t \in (0, \tau).\)
Existence and approximation of solutions

A lot of results since M.Schatzman (78).

- Penalty method

M.Schatzman (78), M.Carriero - E.Pascali (80), G.Buttazzo - D.Percivale (81, 83),
LP - M.Schatzman (93), M.Schatzman (01)
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LP - M.Schatzman (93), LP - M.Schatzman (98), LP - M.Schatzman (02)

- Time-discretization at the velocity level

J.J.Moreau (83, 85 ...), M.Monteiro Marques (87, 93), M.Mabrouk (98),
R.Dzonou - M.Monteiro Marques - LP (06, 09).

Uniqueness is not true in general (A.Bressan 1959).
Existence and approximation of solutions: multi-constraint case ($\nu > 1$)

- Penalty method

M.Schatzman (78), LP (00),
Existence and approximation of solutions: multi-constraint case \((\nu > 1)\)

- Penalty method
  M.Schatzman (78), LP (00),

- Theoretical existence result
  P.Ballard (00)
Existence and approximation of solutions: multi-constraint case ($\nu > 1$)

- **Penalty method**

M.Schatzman (78), LP (00),

- **Theoretical existence result**

P.Ballard (00)

**New difficulty:** continuity on data does not hold in general but holds if the following "angle condition" on the active constraints is satisfied:

$$\begin{align*}
(\nabla f_\alpha (q), M(q)^{-1} \nabla f_\beta (q)) &\leq 0 \quad \text{if } e = 0 \\
(\nabla f_\alpha (q), M(q)^{-1} \nabla f_\beta (q)) &= 0 \quad \text{if } e \neq 0
\end{align*}$$

for all $(\alpha, \beta) \in J(q)^2$ such that $\alpha \neq \beta$, for all $q \in \partial K$.

**References:** P.Ballard (00) and LP (05).
A "simple" case: $M \equiv \text{Id}$ and $K$ convex

The MDI can be rewritten as

$$\ddot{q} + \partial\psi_K(q) \ni g(t, q, \dot{q})$$

and we propose the following implicit time-discretization

$$\frac{q^{n+1} - 2q^n + q^{n-1}}{h^2} + \partial\psi_K(q^{n+1}) \ni G^n$$

which is equivalent to

$$q^{n+1} = \text{Proj}(K, 2q^n - q^{n-1} + h^2G^n)$$

where $G^n$ is an approximate value of $g(t, q, \dot{q})$ at $t = t_n = nh$.

We initialize the algorithm by defining

$$q^0 = q_0, \quad q^1 = \text{Proj}(K, q_0 + hu_0 + hz(h)) \quad \text{with } \lim_{h \to 0} z(h) = 0.$$
**Example (bouncing ball):** \(d = 1, \ K = \mathbb{R}^+, \ M(q) \equiv 1, \ g \equiv 0, \ q_0 = 1, \ u_0 = -1.\)

The solution of problem \((P)\) is

\[
q(t) = 1 - t \quad \text{if} \ t \in [0, 1], \quad q(t) = 0 \quad \text{if} \ t \geq 1.
\]

Assume that \(h \in (0, 1/2).\) We obtain \(q^0 = 1, \ q^1 = 1 - h\) and for all \(n \geq 1\)

\[
q^{n+1} = \text{Proj}(\mathbb{R}^+, \ 2q^n - q^{n-1}) = \max(2q^n - q^{n-1}, 0).
\]

There exists \(p \geq 1 \ (p = \lfloor 1/h \rfloor - 1)\) such that

\[
p = \max\{k \geq 0; \ 2q^n - q^{n-1} \geq 0 \ \forall n \in \{0, \ldots, k\}\}
\]

and \(q^n = 1 - nh\) for all \(n \in \{0, \ldots, p + 1\}.\) Then

\[
q^{p+2} = 0, \quad 2q^{p+2} - q^{p+1} = -q^{p+1} \leq 0
\]

and \(q^k = 0\) for all \(k \geq p + 2.\)
Let us assume

(H1) $g$ is a continuous function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ($T > 0$) to $\mathbb{R}^d$,

(H2) for all $\alpha \in \{1, \ldots, \nu\}$, the function $f_\alpha$ belongs to $C^1(\mathbb{R}^d; \mathbb{R})$, $\nabla f_\alpha$ is Lispchitz continuous and does not vanish in a neighbourhood of $\{q \in \mathbb{R}^d : f_\alpha(q) = 0\}$.

(H3) the active constraints along are functionnally independent i.e., for all $q \in K$ the vectors $\left(\nabla f_\alpha(q)\right)_{\alpha \in J(q)}$ are linearly independent.

We define the approximate solution $(q_h)_{h > 0}$ by a linear interpolation of the $q^n$'s, i.e.

$$q_h(t) = q^n + (t - nh) \frac{q^{n+1} - q^n}{h} \quad \forall t \in [nh, (n + 1)h] \cap [0, T]$$
**Theorem** (LP 2005) Let \((q_0, u_0) \in K \times T_K(q_0)\) be admissible data. Then there exist \(\tau \in (0, T]\) and a subsequence of \((q_h)_{h>0}\), still denoted \((q_h)_{h>0}\), such that

\[
q_h \to q \quad \text{in } C^0([0, \tau]; \mathbb{R}^d)
\]

and \(q\) satisfies the properties (P1)-(P2)-(P3)-(P4).

If we assume moreover that, for all \(\tilde{q} \in \partial K\), we have

\[
(\nabla f_\alpha(\tilde{q}), \nabla f_\beta(\tilde{q})) \leq 0 \quad \text{for all } (\alpha, \beta) \in J(\tilde{q})^2 \quad \text{such that } \alpha \neq \beta,
\]

then the limit function \(q\) satisfies also (P5) with \(e = 0\), i.e.

\[
\dot{q}^+(t) = \text{proj}(T_K(q(t)), \dot{q}^-(t)) \quad \forall t \in (0, \tau)
\]

and \(q\) is a solution of the Cauchy problem.

Furthermore, if \(g\) is Lipschitz continuous in its last two arguments, uniformly with respect to the first one, the previous convergence holds on the whole time interval \([0, T]\).
Sketch of the proof

Step 1: We establish uniform estimates for the discrete velocities and accelerations.

Lemma 1: For all $n \geq 1$, let $V^n = \frac{q^{n+1} - q^n}{h}$. Then

$$V^{n-1} - V^n + hG^n \in N_K(q^{n+1}), \quad V^n \in -T_K(q^{n+1}).$$

As a consequence

$$\|V^n\| \leq \|V^{n-1}\| + h\|G^n\|.$$

Step 2: We define the approximate solutions $(q_h)_{h>0}$ by a linear interpolation of the $q^n$'s. We pass to the limit by using Ascoli’s and Helly’s theorems. Hence the limit $q$ satisfies the property (P1) and, by using again lemma 1, we prove that $q$ takes its values in $K$ and satisfies the MDI and the initial conditions.
**Step 3:** We assume now that the "angle condition" holds i.e. for all $\tilde{q} \in \partial K$:

$$\left(\nabla f_\alpha(\tilde{q}), \nabla f_\beta(\tilde{q})\right) \leq 0 \quad \text{for all } (\alpha, \beta) \in J(\tilde{q})^2 \text{ such that } \alpha \neq \beta.$$  

We observe first that

$$\dot{q}^+(t) \in T_K(q(t)), \quad \dot{q}^-(t) \in -T_K(q(t)), \quad \dot{q}^+(t) - \dot{q}^-(t) \in -N_K(q(t)).$$

Hence the impact law reduces to $(\dot{q}^-(t) - \dot{q}^+(t), \dot{q}^+(t)) = 0$ and we have

$$\dot{q}^+(t) - \dot{q}^-(t) = \sum_{\alpha \in J(q(t))} -\mu_\alpha \nabla f_\alpha(q(t)), \quad \mu_\alpha \leq 0, \quad (\nabla f_\alpha(q(t)), \dot{q}^+(t)) \geq 0 \quad \forall \alpha \in J(q(t)).$$

Consequently we only need to prove the following **complementarity condition**

$$\mu_\alpha(\nabla f_\alpha(q(t)), \dot{q}^+(t)) = 0 \quad \text{for all } \alpha \in J(q(t)).$$
\[
(\nabla f_\alpha(q(t)), \dot{q}^+(t)) \leq 0 \quad \text{for all } \alpha \in J(q(t)) \text{ such that } \mu_\alpha \neq 0.
\]

Using again lemma 1, we get
\[
V^{n-1} - V^n + hG^n = \sum_{\beta \in J(q^{n+1})} \mu^n_\beta \nabla f_\beta(q^{n+1}), \quad \mu^n_\beta \leq 0.
\]

We infer that, if \( \mu_\alpha \neq 0 \), in any neighbourhood \( \mathcal{V} \) of the impact instant \( t \), there exists at least one discrete impact i.e. there exists at least a discrete instant \( t_{n_i} \) such that \( f_\alpha(q^{n_i+1}) \leq 0 \).

It follows that \( \alpha \in J(q^{n_i+1}) \) and \( V^{n_i} \in -T_K(q^{n_i+1}) \) thus
\[
(\nabla f_\alpha(q^{n_i+1}), V^{n_i}) \leq 0.
\]

Finally, by considering the last discrete impact \( t_{n_i} \in \mathcal{V} \) and using the "angle condition" we obtain \( (\nabla f_\alpha(q(t)), \dot{q}^+(t)) \leq 0 \).
Let us consider now $e \in [0, 1]$ but still $M \equiv \text{Id}$ and $K$ convex. The vibro-impact problem is described by the MDI

$$\ddot{q} + \partial\psi_K(q) \ni g(t, q, \dot{q})$$

and the impact law

$$\dot{q}^+ = -e\dot{q}^- + (1 + e)\text{Proj}(T_K(q), \dot{q}^-) = -e\text{Proj}(N_K(q), \dot{q}^-) + \text{Proj}(T_K(q), \dot{q}^-).$$

Starting from the model problem of the bouncing ball, propose the following algorithm:

$$\frac{q^{n+1} - 2q^n + q^{n-1}}{h^2} + \partial\psi_K\left(\frac{q^{n+1} + eq^{n-1}}{1 + e}\right) \ni G^n$$

which can be rewritten as

$$q^{n+1} = -eq^{n-1} + (1 + e)\text{Proj}\left(K, \frac{2q^n - (1 - e)q^{n-1} + h^2G^n}{1 + e}\right).$$
Example (bouncing ball): $d = 1$, $K = \mathbb{R}^+$, $M(q) \equiv 1$, $f \equiv 0$, $q_0 = 1$, $u_0 = -1$.

The solution of problem (P) is

\[
\begin{aligned}
q(t) &= 1 - t \quad \text{if } t \in [0, 1], \\
q(t) &= e(t - 1) \quad \text{if } t \geq 1.
\end{aligned}
\]

Assume that $h \in (0, 1/2)$. We obtain $q^0 = 1$, $q^1 = 1 - h$ and

\[
q^{n+1} = -eq^{n-1} + (1+e)\text{Proj}\left(\mathbb{R}^+, \frac{2q^n - (1-e)q^{n-1}}{1+e}\right) = -eq^{n-1} + \max(2q^n - (1-e)q^{n-1}, 0).
\]

There exists $p \geq 1$ such that

\[
p = \max\{k \geq 0; \ 2q^n - (1-e)q^{n-1} \geq 0 \ \forall n \in \{0, \ldots, k\}\}
\]

and $q^n = 1 - nh$ for all $n \in \{0, \ldots, p + 1\}$. Then $q^{p+2} = -eq^p.$
But

\[ 2q^{p+2} - (1 - e)q^{p+1} = -2eq^p - (1 - e)(2q^p - q^{p-1}) = -(2q^p - (1 - e)q^{p-1}) \leq 0 \]

so \( q^{p+3} = -eq^{p+1} \) and

\[ q^{k+p+2} = q^{p+2} + ekh \quad \forall k \geq 0. \]
General case: \( e \in [0, 1], \ M \neq \text{Id} \) and/or \( K \) not convex

- We extend first this definition to the case of a non trivial inertia operator but still a convex set \( K \) by considering the projection on \( K \) relatively to the kinetic metric at \( q^n \) instead of the projection relatively to the Euclidean metric.

- In the case of a non convex set \( K \) we extend once again the definition of the algorithm by replacing the projection on \( K \) by the Argmin of the distance.

More precisely we propose now the following time-stepping scheme

\[
q^0 = q_0, \quad q^1 \in \text{Argmin}_{Z \in K} \left\| q_0 + hu_0 + h z(h) - Z \right\|_{M(q_0)}, \quad \lim_{h \to 0} z(h) = 0
\]

and for all \( n \geq 1 \)

\[
q^{n+1} = -eq^{n-1} + (1 + e)Z^n
\]

with

\[
Z^n \in \text{Argmin}_{Z \in K} \left\| \frac{2q^n - (1 - e)q^{n-1} + h^2 G^n}{1 + e} - Z \right\|_{M(q^n)}
\]

and \( G^n \) is an approximate value of \( M^{-1}(q)g(t, q, \dot{q}) \) at \( t = t_n = nh \).
Let us assume

\textbf{(H1)} $g$ is a continuous function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ($T > 0$) to $\mathbb{R}^d$,

\textbf{(H2)} for all $\alpha \in \{1, \ldots, \nu\}$, the function $f_\alpha$ belongs to $C^1(\mathbb{R}^d; \mathbb{R})$, $\nabla f_\alpha$ is Lipschitz continuous and does not vanish in a neighbourhood of $\{q \in \mathbb{R}^d : f_\alpha(q) = 0\}$,

\textbf{(H3)} the active constraints along are functionally independent i.e., for all $q \in K$ the vectors $(\nabla f_\alpha(q))_{\alpha \in J(q)}$ are linearly independent.

\textbf{(H4)} $M$ is a mapping of class $C^1$ from $\mathbb{R}^d$ to the set of symmetric positive definite $d \times d$ matrices.

Without further assumptions on $M$ and $g$, we can not expect a global existence result for problem (P) on $[0, T]$. 
Indeed, for any solution $q$ defined on $[0, \tau]$ (with $\tau \in (0, T]$), we have

$$E_k^+(t) \leq E_k^+(0) + \int_0^t (g(s, q(s), \dot{q}(s)), \dot{q}(s)) \, ds$$
$$+ \frac{1}{2} \int_0^t (\dot{q}(s), (dM(q(s))\dot{q}(s))\dot{q}(s)) \, ds \quad \forall t \in [0, \tau)$$

and finite time explosion may occur. Nevertheless, we can establish that

**Proposition (energy estimate):** Let $C' > \|u_0\|_{M(q_0)}$. Then, there exists $\tau(C') \in (0, T]$ s.t., for any solution $q$ of problem (P) defined on $[0, \tau]$, we have

$$\|q(t) - q_0\| \leq C' \quad \forall t \in [0, \min(\tau(C'), \tau)],$$
$$\|\dot{q}(t)\|_{M(q(t))} \leq C' \quad dt \text{ a.e. on } [0, \min(\tau(C'), \tau)].$$

We define once again the approximate solutions $(q_h)_{h>0}$ by a linear interpolation of the $q^n$'s and we establish the convergence to a solution of the Cauchy problem on $[0, \tau(C')]$ for any $C' > \|u_0\|_{M(q_0)}$. 
First we observe that

**Lemma 2:** For all $n \geq 1$

$$M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(Z^n).$$
First we observe that

**Lemma 2:** For all $n \geq 1$

\[
M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(Z^n).
\]

Next we prove

**Lemma 3:** For all $n \geq 2$ and for all $\alpha \in J(Z^n)$

\[
(\nabla f_\alpha(Z^n), V^n + eV^{n-2}) \leq O(h)\|V^n + eV^{n-2}\|^2.
\]
Proof: If $\alpha \in J(Z^n)$, we have $f_\alpha(Z^n) = 0$ and $f_\alpha(Z^{n-1}) \geq 0$, thus

$$0 \leq f_\alpha(Z^{n-1}) - f_\alpha(Z^n) = \int_0^1 (\nabla f_\alpha(Z^n + s(Z^{n-1} - Z^n), Z^{n-1} - Z^n) \, ds.$$ 

Observing that $Z^n - Z^{n-1} = \frac{h}{1 + e(V^n + eV^{n-2})}$, we get

$$\langle \nabla f_\alpha(Z^n), V^n + eV^{n-2} \rangle \leq -\int_0^1 (\nabla f_\alpha(Z^n + s(Z^{n-1} - Z^n)) - \nabla f_\alpha(Z^n), V^n + eV^{n-2}) \, ds$$

$$\leq O(h)\|V^n + eV^{n-2}\|^2.$$
**Case 1:** $e = 0$ With lemma 2 and lemma 3 we infer that

$$Z^n = q^{n+1} \in K, \quad M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(q^{n+1})$$

and

$$\left( \nabla f_\alpha(q^{n+1}), V^n \right) \leq O(h)\|V^n\|^2 \quad \forall \alpha \in J(q^{n+1}).$$

Hence $-V^n$ does not belong necessarily to $T_K(q^{n+1})$. We can still reproduce the convergence proof as in the "simple" case but now we have to deal with some $O(h)$ perturbing terms coming from the variation of the kinetic metric and the lack of convexity of $K$.

**Case 2:** $e \neq 0$ With lemma 2 and lemma 3 we infer that

$$Z^n = \frac{q^{n+1} + eq^{n-1}}{1 + e} \in K, \quad M(q^n)(V^{n-1} - V^n + hG^n) \in N_K(Z^n)$$

and

$$\left( \nabla f_\alpha(Z^n), V^n + eV^{n-2} \right) \leq O(h)\|V^n + eV^{n-2}\|^2 \quad \forall \alpha \in J(Z^n).$$
Consequently it is much more difficult to prove a priori uniform estimates for the discrete velocities and accelerations. Indeed, let us assume again that $M(q) \equiv \text{Id}$ and $K$ convex, we get

$$V^n + eV^{n-2} \in -TK(Z^n)$$

and

$$(V^{n-1} - V^n + hG^n, V^n + eV^{n-2}) \geq 0$$

which yields

$$(1 - e)\|V^n\|^2 \leq (1 + e)\|V^{n-1}\|^2 + 2e\|V^{n-2}\|^2 + O(h).$$

If $K$ is not convex and/or $M(q) \neq \text{Id}$, we obtain the same kind of estimate up to $O(h)$ perturbing terms.

Nevertheless, with more technicalities, we can still prove that the sequence $(q_h)_{h>0}$ is uniformly Lipschitz continuous on a non trivial time interval $[0, \tau], \tau \in (0, T]$, and the sequence $(\dot{q}_h)_{h>0}$ is bounded in $BV(0, \tau; \mathbb{R}^d)$. 
Furthermore it is also more difficult to establish that the limit $q$ satisfies the impact law

$$\dot{q}^+(t) = -e\dot{q}^-(t) + (1 + e)\text{Proj}_{M(q(t))}(T_K(q(t)), \dot{q}^-(t)).$$

(1)

With (P1), (P2) and (P3) we know that

$$\dot{q}^+(t) \in T_K(q(t)), \quad \dot{q}^-(t) \in -T_K(q(t)), \quad M(q(t))(\dot{q}^+(t) - \dot{q}^-(t)) \in -N_K(q(t)).$$

(2)

It follows that (1) holds if $q(t) \in \text{Int}(K)$. Otherwise $J(q(t)) \neq \emptyset$ and

$$M(q(t))(\dot{q}^+(t) - \dot{q}^-(t)) = -\sum_{\alpha \in J(q(t))} \mu_\alpha \nabla f_\alpha(q(t)), \quad \mu_\alpha \leq 0.$$

Hence (1) reduces to

$$\mu_\alpha(\nabla f_\alpha(q(t)), \dot{q}^+(t) + e\dot{q}^-(t)) = 0, \quad (\nabla f_\alpha(q(t)), \dot{q}^+(t) + e\dot{q}^-(t)) \geq 0$$

for all $\alpha \in J(q(t))$. 

Recalling the "angle condition" i.e. for all \( \tilde{q} \in \partial K \)

\[
(\nabla f_\alpha(\tilde{q}), M^{-1}(\tilde{q}) \nabla f_\beta(\tilde{q})) = 0 \quad \text{for all } (\alpha, \beta) \in J(\tilde{q})^2 \text{ such that } \alpha \neq \beta
\]

and using (2), we get

\[
(\nabla f_\alpha(q(t)), \dot{q}^+(t)) \geq 0, \quad (\nabla f_\alpha(q(t)), \dot{q}^-(t)) \leq 0
\]

and

\[
(\nabla f_\alpha(q(t)), \dot{q}^+(t)) = (\nabla f_\alpha(q(t)), \dot{q}^-(t)) - \mu_\alpha \left\| \nabla f_\alpha(q(t)) \right\|_{M^{-1}(q(t))}^2.
\]

If \( \mu_\alpha = 0 \) the conclusion follows immediately. Otherwise, we have to prove that

\[
(\nabla f_\alpha(q(t)), \dot{q}^+(t) + e \dot{q}^-(t)) = 0.
\]

By using lemma 2, we can prove as in the "simple case" the existence of discrete impacts and using lemma 3, we finally get the conclusion.
Implementation: some references

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Thank you for your attention