On existence for viscoelastodynamic problems with unilateral boundary conditions

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joint work with M. Schatzman

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Mathematics for key technologies

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1. The damped wave equation
2. The evolution of a Kelvin-Voigt material
3. Outlook

State variables

$u : [0,T]\times\Omega \to \mathbb{R}$ displacement

Applied field

$\ell : [0,T]\times\Omega \to \mathbb{R}$ density forces

Notations:

- $\Omega = (-\infty, 0]\times\mathbb{R}^{d-1}$ and $\Sigma = \{0\}\times\mathbb{R}^{d-1}$
- $x = (x_1, x')$ with $x' = (x_2, \dots, x_d)$
- $K=\{v \in \mathrm{H}^1_{\mathrm{loc}}(\Omega\times[0,\infty)) : \nabla v_t \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{L}^2(\Omega)),\ v|_{\{0\}\times\mathbb{R}^{d-1}} \geq 0\}$

We consider the following problem **(DI)**:

$u_{tt} - \Delta u - \alpha\Delta u_t = \ell, \quad \alpha > 0$	damped wave equation
$0 \leq u(0,x',t) \perp (u_{x_1}+\alpha u_{x_1 t})(0,x',t) \geq 0$	unilateral bdry conditions
$u(\cdot,0) = u_0$ and $u_t(\cdot,0) = u_1$	Cauchy initial data

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$$\boxed{\begin{array}{ll} u_{tt} - \Delta u - \alpha\Delta u_t = \underbrace{\ell}_{\in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))}, \quad \alpha > 0 & \text{damped wave equation} \\ 0 \leq u(0,x',t) \perp (u_{x_1} + \alpha u_{x_1 t})(0,x',t) \geq 0 & \text{unilateral bdry conditions} \\ u(\cdot,0) = \underbrace{u_0}_{\in \mathrm{H}^2(\Omega)} \quad \text{and} \quad u_t(\cdot,0) = \underbrace{u_1}_{\in \mathrm{H}^1(\Omega)} & \text{Cauchy initial data} \end{array}}$$

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The weak formulation **(VI)** is given by

Find $u \in K$ such that for all $v \in K$ and for all $\tau \in (0,\infty)$, we have

$$\int_\Omega (u_t(v-u))\big|_0^\tau \mathrm{d}x - \int_0^\tau \int_\Omega u_t(v_t-u_t)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega (\nabla u + \alpha\nabla u_t)(\nabla v - \nabla u)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\tau \int_\Omega \ell(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

Penalized Problem (PP) (here $r^- = -\min(r, 0)$)

$u^\epsilon_{tt} - \Delta u^\epsilon - \alpha \Delta u^\epsilon_t = \ell, \quad \alpha > 0$	damped wave equation
$(u^\epsilon_{x_1} + \alpha u^\epsilon_{x_1 t})(0, x', t) = (u^\epsilon(0, x', t))^- / \epsilon$	normal compliance cond.[1]
$u^\epsilon(\cdot, 0) = u_0$ and $u^\epsilon_t(\cdot, 0) = u_1$	Cauchy initial data

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[2] Jarušek. *Czechoslovak Math. J.*, 1996.

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Theorem (Existence and uniqueness results[2])

There exists a unique weak solution $u^\epsilon \in \mathrm{H}^1_{\mathrm{loc}}([0,\infty)\times\Omega)$ *of the problem* **(PP)** *such that* $\nabla u^\epsilon_t \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{L}^2(\Omega))$.

Idea of the proof: Use Galerkin method.

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Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates give us

- $u^\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $u^\epsilon_t \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\nabla u^\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\nabla u^\epsilon_t \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\Delta u^\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$

Moreover $u_t \in \mathrm{C}^0([0,\infty);\mathrm{L}^2(\Omega))$ equipped with the weak topology.

Theorem (Existence result)

There exists a solution of **(VI)**; *this solution can be obtained as a limit of a subsequence of* **(PP)**.

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_\Omega u_t^\epsilon \varphi(v-u^\epsilon)\Big|_0^\tau \mathrm{d}x - \int_0^\tau \int_\Omega u_t^\epsilon (\varphi(v-u^\epsilon))_t \,\mathrm{d}x\,\mathrm{d}t$$
$$-\frac{1}{\epsilon}\int_0^\tau \int_\Sigma \left(\left((u^\epsilon)^-\right)^2 \varphi\right) \mathrm{d}x'\,\mathrm{d}t - \frac{1}{\epsilon}\int_0^\tau \int_\Sigma \left((u^\epsilon)^- \varphi v\right) \mathrm{d}x'\,\mathrm{d}t$$
$$+\int_0^\tau \int_\Omega (\nabla u^\epsilon + \alpha \nabla u_t^\epsilon)\nabla(\varphi(v-u^\epsilon))\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_\Omega \ell \varphi(v-u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

where $\varphi \in \mathrm{C}_0^\infty(\bar{\Omega} \times [0,\infty))$ taken its values in $[0,1]$.

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where $\varphi \in \mathrm{C}_0^\infty(\bar{\Omega} \times [0,\infty))$ taken its values in $[0,1]$.

Remark: Nothing is known about uniqueness.

AIM: Characterize the trace spaces

Recall that $u(0,x',t) \in \mathrm{H}^{a,b}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty))$ for $(a,b)\in\mathbb{R}^2$

$$\Updownarrow$$

$|\xi|^a\widehat{u}(0,\xi,\omega)\in \mathrm{L}^2(\mathbb{R}^d)$ and $|\omega|^b\widehat{u}(0,\xi,\omega)\in \mathrm{L}^2(\mathbb{R}^d)$

- $\xi = (\xi_2,\dots,\xi_d)^{\mathsf{T}}$: the dual variable to $x' = (x_2,\dots,x_d)^{\mathsf{T}}$
- ω : the dual variable to t
- $\widehat{u}(0,\xi,\omega)$: the Fourier transform of $u(0,x',t)$

Lemma (Regularity of the trace[3])

Let u^ϵ be the solution of **(PP)**. *Then we may extract a subsequence, still denoted by u^ϵ such that*

$$u^\epsilon(0,x',t) \rightharpoonup u(0,x',t) \quad \textit{weakly} \quad \textit{in} \quad \mathrm{H}^{1/2,5/4}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty)).$$

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- Let $v^\epsilon = \mathrm{e}^{-\nu t}(u^\epsilon - \bar{u})$, $\nu > 0$, be a solution of

$$(\nu+\partial_t)^2 v^\epsilon - (1+\alpha(\nu+\partial_t))\Delta v^\epsilon = 0, \quad x \in \Omega, \quad t > 0$$
$$(1+\alpha(\nu+\partial_t))v^\epsilon_{x_1}(0,x',t) = \mathrm{e}^{-\nu t}\bar{g} - (v^\epsilon(0,x',t)+\mathrm{e}^{-\nu t}\bar{u}(0,x',t))^-/\epsilon$$
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$$v^\epsilon(\cdot,0) = 0 \quad \text{and} \quad v^\epsilon_t(\cdot,0) = 0$$

- Use the partial Fourier transform in x' and t

$$\widehat{v}^\epsilon_{x_1 x_1}(x_1,\xi,\omega) = \widehat{\lambda}^2 \widehat{v}^\epsilon(x_1,\xi,\omega) \quad \text{where} \quad \widehat{\lambda} = \sqrt{|\xi|^2 + \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}}$$

$$\rightsquigarrow \widehat{v}^\epsilon(x_1,\xi,\omega) = \widehat{a}^\epsilon \mathrm{e}^{\widehat{\lambda} x_1}$$

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- Use the boundary conditions

$$((1+\alpha(\nu+\partial_t))v^\epsilon_{x_1})\widehat{\ }(0,\xi,\omega) = \widehat{\lambda}_1 \widehat{v}^\epsilon(0,\xi,\omega) \text{ where } \widehat{\lambda}_1 = (1+\alpha(\nu+i\omega))\widehat{\lambda}$$
$$\rightsquigarrow \lambda_1 * v^\epsilon(0,x',t) = \mathrm{e}^{-\nu t}\bar{g} + (v^\epsilon(0,x',t) + \mathrm{e}^{-\nu t}\bar{u}(0,x',t))^-/\epsilon$$

State variables

$u : [0,T]\times\Omega \to \mathbb{R}^3$ displacement

Applied field

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Notations:

- $\varepsilon_{ij}(u) = (u_{j,x_i} + u_{i,x_j})/2$ and $a^n_{ijkl} = \lambda^n \delta_{ij}\delta_{kl} + 2\mu^n \delta_{ik}\delta_{jl}$ for $n = 0, 1$
- $K = \{v \in \mathbf{H}^1(\Omega \times (0,\tau)) : \nabla v_t \in \mathbf{L}^2(\Omega \times (0,\tau)), v(0,\cdot) \leq 0\}$

We consider the following problem **(DI)**:

$$\boxed{\begin{array}{l}
\rho u_{tt} - a^n_{ijkl}\partial_j \varepsilon_{kl}(u) - a^n_{ijkl}\partial_j \varepsilon_{kl}(u_t) = \ell, \quad x \in \Omega, \quad t > 0 \\
0 \geq u_1 \perp a^0_{11kl}\varepsilon_{kl}(u) + a^1_{11kl}\varepsilon_{kl}(u_t) \leq 0 \quad \text{on} \quad \Sigma \times [0,\infty) \\
a^0_{12kl}\varepsilon_{kl}(u) + a^1_{12kl}\varepsilon_{kl}(u_t) = 0 \quad \text{and} \quad a^0_{13kl}\varepsilon_{kl}(u) + a^1_{13kl}\varepsilon_{kl}(u_t) = 0 \\
u(\cdot,0) = v_0 \quad \text{and} \quad u_t(\cdot,0) = v_1
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The weak formulation **(VI)** is given by

Find $u\in K$ such that for all $v\in K$ and for all $\tau\in(0,\infty)$, we have

$$\int_0^\tau\int_\Omega\rho u_{tt}\cdot(v-u)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega a^0_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega a^1_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v-u)\,\mathrm{d}x\,\mathrm{d}t\geq\int_0^\tau\int_\Omega\ell\cdot(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

Penalized Problem (PP) (here $r^+ = \max(r, 0)$)

$$\rho u^\epsilon_{tt} - a^n_{ijkl}\partial_j \varepsilon_{kl}(u^\epsilon) - a^n_{ijkl}\partial_j \varepsilon_{kl}(u^\epsilon_t) = \ell, \quad x \in \Omega, \quad t > 0$$

$$a^0_{11kl}\varepsilon_{kl}(u^\epsilon) + a^1_{11kl}\varepsilon_{kl}(u^\epsilon_t) = -(u^\epsilon_1)^+/\epsilon \quad \text{on} \quad \Sigma \times [0, \infty)$$

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$$u^\epsilon(\cdot, 0) = v_0 \quad \text{and} \quad u^\epsilon_t(\cdot, 0) = v_1$$

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Theorem (Existence and uniqueness results[4])

There exists a unique weak solution $u^\epsilon \in \mathbf{H}^1_{\mathrm{loc}}([0, \infty) \times \Omega)$ *of the problem* **(PP)** *such that* $\nabla u^\epsilon_t \in \mathbf{L}^2_{\mathrm{loc}}([0, \infty); \mathrm{L}^2(\Omega))$.

Idea of the proof: Use Galerkin method.

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Theorem (Existence result)

There exists a solution of **(VI)**; *this solution can be obtained as a limit of a subsequence of* **(PP)**.

Sketch of the proof: A-priori estimates and Korn's inequality lead to

- $u^\epsilon \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $u^\epsilon_t \in \mathbf{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $\nabla u^\epsilon \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$
- $a^n_{ijkl}\partial_j\varepsilon_{kl}(u) \in \mathbf{L}^\infty_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$ for $n = 0, 1$
- $\nabla u^\epsilon_t \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$

Moreover $u^\epsilon \longrightarrow u$ in $\mathbf{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$.

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Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_\Omega \rho u_t^\epsilon \cdot (\varphi(v-u^\epsilon))|_0^\tau \,\mathrm{d}x - \int_0^\tau \int_\Omega \rho u_t^\epsilon \cdot (\varphi(v-u^\epsilon))_t \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega (a_{ijkl}^0 \varepsilon_{kl}(u^\epsilon)\varepsilon_{ij}(u^\epsilon) + a_{ijkl}^1 \varepsilon_{kl}(u_t^\epsilon)\varepsilon_{ij}(u^\epsilon))(\varphi(v_i - u_i^\epsilon))\,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{1}{\epsilon}\int_0^\tau \int_\Sigma ((u_1^\epsilon)^+)^2 \varphi \,\mathrm{d}x'\,\mathrm{d}t + \frac{1}{\epsilon}\int_0^\tau \int_\Sigma (u_1^\epsilon)^+ \varphi v_1 \,\mathrm{d}x'\,\mathrm{d}t$$

$$= \int_{Q_\tau} \ell \cdot (\varphi(v-u^\epsilon))\,\mathrm{d}x\,\mathrm{d}t$$

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Remark: Nothing is known about uniqueness.

Lemma (Regularity of the trace[5])

Let $u^\epsilon = (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)^\mathsf{T}$ *be the solution of* **(PP)**. *Then we may extract a subsequence, still denoted by* u_1^ϵ *such that*

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- Introduce $\bar{u}$ solution of **(DI)** with the Dirichlet boundary conditions
- Use the Fourier transform in the tangential variable (x', t) and a Laplace transform in x_1

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Remark: We were unable to prove that the energy loss is purely viscous[6].

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Thank you for your attention !

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