On the numerical approximation of a viscoelastic problem with unilateral constraints

Adrien Petrov
Weierstraß-Institute für Angewandte Analysis und Stochastik

joint work with M. Schatzman

DFG Research Center MATHEON
Mathematics for key technologies

Lisboa, 8 September 2009
1. The viscoelastodynamic problem
2. Numerical scheme
3. Convergence result
4. Numerical results
5. Outlook
The viscoelastodynamic problem

1. Numerical scheme
2. Convergence result
3. Numerical results
4. Outlook
State variables \[ u : [0, T] \times [0, L] \to \mathbb{R} \text{ displacement} \]

Applied field \[ \ell : [0, T] \times [0, L] \to \mathbb{R} \text{ density forces} \]

We consider the motion of a bar between two obstacles
State variables \( u : [0, T] \times [0, L] \to \mathbb{R} \) displacement

Applied field \( \ell : [0, T] \times [0, L] \to \mathbb{R} \) density forces

The mathematical formulation is given by (DI):

\[
\begin{align*}
  u_{tt} - u_{xx} - \alpha u_{xxt} &= \ell, \quad \alpha > 0 \\
  0 \leq u(0, t) + a_0 \perp -(u_x + \alpha u_{xt})(0, t) &\geq 0 \\
  0 \leq u(L, t) + a_L \perp (u_x + \alpha u_{xt})(L, t) &\geq 0 \\
  u(\cdot, 0) = u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1
\end{align*}
\]

damped wave equation

unilateral bdry conditions

Cauchy initial data
State variables
\( u : [0, T] \times [0, L] \to \mathbb{R} \) displacement

Applied field
\( \ell : [0, T] \times [0, L] \to \mathbb{R} \) density forces

The mathematical formulation is given by \((\text{DI})\):

\[
\begin{align*}
\dddot{u} - \ddot{u} - \alpha \dot{u} &= \ell, \quad \alpha > 0 && \text{damped wave equation} \\
\ell &\in L^2(0, T; L^2(\Omega)) \\
0 &\leq u(0, t) + a_0 \perp -(u_x + \alpha u_{xt})(0, t) \geq 0 && \text{unilateral bdry conditions} \\
0 &\leq u(L, t) + a_L \perp (u_x + \alpha u_{xt})(L, t) \geq 0 \\
u(\cdot, 0) &= u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1 && \text{Cauchy initial data} \\
u_0 &\in H^2(\Omega) \\
u_1 &\in H^1(\Omega)
\end{align*}
\]
The Mathematical formulation

State variables
\( u : [0, T] \times [0, L] \rightarrow \mathbb{R} \) displacement

Applied field
\( \ell : [0, T] \times [0, L] \rightarrow \mathbb{R} \) density forces

The weak formulation (VI) is given by

\[
\text{Find } u \in K \text{ such that for all } v \in K \text{ and for all } \tau \in [0, T], \text{ we have}
\]
\[
\int_0^L (u_t(v-u))|_0^\tau \, dx - \int_0^\tau \int_0^L u_t(v_t-u_t) \, dx \, dt
\]
\[
+ \int_0^\tau \int_0^L (u_x+\alpha u_{xt})(v_x-u_x) \, dx \, dt \geq \int_0^\tau \int_0^L \ell(v-u) \, dx \, dt
\]

Notation:

\( K = \{ u \in H^1([0, L] \times (0, T)) : u_{xt} \in L^2([0, L] \times (0, T)), u(0, \cdot) \geq -a_0, u(L, \cdot) \geq -a_L \} \)
Penalized Problem (PP) (here $r^- = -\min(r, 0)$)

\[
\begin{align*}
    u^{\varepsilon}_{tt} - u^{\varepsilon}_{xx} - \alpha u^{\varepsilon}_{xxt} &= \ell, \quad \alpha > 0 & \text{damped wave equation} \\
    (u^{\varepsilon}_{x} + \alpha u^{\varepsilon}_{xt})(0, t) &= -(u^{\varepsilon}(0, t) + a_0)^{-}/\varepsilon & \text{normal compliance cond.}^1 \\
    (u^{\varepsilon}_{x} + \alpha u^{\varepsilon}_{xt})(L, t) &= (u^{\varepsilon}(L, t) + a_L)^{-}/\varepsilon \\
    u^{\varepsilon}(\cdot, 0) &= u_0 \quad \text{and} \quad u^{\varepsilon}_t(\cdot, 0) = u_1 & \text{Cauchy initial data}
\end{align*}
\]

---


Penalized Problem (PP) (here \( r^- = -\min(r, 0) \))

\[
\begin{align*}
  u_{tt}^\varepsilon - u_{xx}^\varepsilon - \alpha u_{xxt}^\varepsilon &= \ell, \quad \alpha > 0 \\
  (u_x^\varepsilon + \alpha u_{xt}^\varepsilon)(0, t) &= -(u^\varepsilon(0, t) + a_0)^- / \varepsilon \\
  (u_x^\varepsilon + \alpha u_{xt}^\varepsilon)(L, t) &= (u^\varepsilon(L, t) + a_L)^- / \varepsilon \\
  u^\varepsilon(\cdot, 0) &= u_0 \quad \text{and} \quad u_t^\varepsilon(\cdot, 0) = u_1
\end{align*}
\]

damped wave equation
normal compliance cond.\(^1\)

Theorem (Existence and uniqueness results\(^2\))

There exists a unique weak solution \( u^\varepsilon \in H^1([0, T] \times [0, L]) \) of the problem (PP) such that \( u_{xt}^\varepsilon \in L^2(0, T; L^2(0, L)) \).

Idea of the proof: Use Galerkin method.


Theorem (Existence result)

There exists a solution of (VI); this solution can be obtained as a limit of a subsequence of (PP).

Sketch of the proof: A-priori estimates give us

- $u^\varepsilon \in L^\infty(0, T; L^2(0, L))$
- $u_t^\varepsilon \in L^\infty(0, T; L^2(0, L))$
- $u_x^\varepsilon \in L^\infty(0, T; L^2(0, L))$
- $u_{xt}^\varepsilon \in L^2(0, T; L^2(0, L))$
- $u_{xx}^\varepsilon \in L^\infty(0, T; L^2(0, L))$

Moreover $u_t \in C^0([0, T]; L^2(0, L))$ equipped with the weak topology.
Theorem (Existence result)

There exists a solution of (VI); this solution can be obtained as a limit of a subsequence of (PP).

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

\[
\int_0^L u_t^\varepsilon (v - u^\varepsilon) |^\tau_0 \, dx - \int_0^\tau \int_0^L u_t^\varepsilon (v_t - u_t^\varepsilon) \, dx \, dt \\
- \frac{1}{\varepsilon} \int_0^\tau ((u^\varepsilon (0, \cdot) + a_0)^-(v - u^\varepsilon)(0, \cdot)) \, dt \\
- \frac{1}{\varepsilon} \int_0^\tau ((u^\varepsilon (L, \cdot) + a_L)^-(v - u^\varepsilon)(L, \cdot)) \, dt \\
+ \int_0^\tau \int_0^L (u_x^\varepsilon + \alpha u_{xt}^\varepsilon)(v_x - u_x^\varepsilon) \, dx \, dt = \int_0^\tau \int_0^L \ell(v - u^\varepsilon) \, dx \, dt
\]
Theorem (Existence result)

There exists a solution of (VI); this solution can be obtained as a limit of a subsequence of (PP).

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$
\int_0^L u_t^\epsilon (v - u^\epsilon) |^\tau_0 \, dx - \int_0^\tau \int_0^L u_t^\epsilon (v_t - u_t^\epsilon) \, dx \, dt \\
+ \int_0^\tau \int_0^L (u_x^\epsilon + \alpha u_{xt}^\epsilon)(v_x - u_x^\epsilon) \, dx \, dt \geq \int_0^\tau \int_0^L \ell (v - u^\epsilon) \, dx \, dt
$$

Remark: Nothing is known about uniqueness.
1. The viscoelastodynamic problem
2. Numerical scheme
3. Convergence result
4. Numerical results
5. Outlook
Finite-element spaces $H^h \subset L^2(0, L)$ and $V^h \subset H^1(0, L)$, $\Delta x = \frac{L}{J+1}$

We consider the following variational formulation $(\textbf{VI})_h$:

Find $u^{n+1}_h \in K_h$ such that for all $v \in K_h$ we have

\[
\langle \frac{u^{n+1}_h-2u^n_h+u^{n-1}_h}{\Delta t^2}, v-u^n_h \rangle + a\left(\frac{u^{n+1}_h+u^{n-1}_h}{2}, v-u^n_h \right) \geq \langle \ell^n_h, v-u^n_h \rangle
\]

Notations:

\(\triangleright\) $K_h = \{v \in V_h : v_{xt} \in H_h, v(0) \geq -a_0, v(L) \geq -a_L\}$

\(\triangleright\) $a(u, v) = \int_0^L u_x v_x \, dx$

\(\triangleright\) $u^n_h \overset{\text{def}}{=} \sum_{i=0}^{J+1} u^n_i w_i$
(VI)$_h$ can be written in the differential inclusion form:

$$
M \frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2} + R \frac{u^{n+1}+u^{n-1}}{2} + \alpha R \frac{u^{n+1}-u^{n-1}}{2\Delta t} + \partial \psi_{K_h}(u^{n+1}) \ni F^n
$$

where for all $i \in [1, J]$

$$
M_{ii} = \frac{2\Delta x}{3}, \quad M_{i,i+1} = M_{i,i-1} = \frac{\Delta x}{3}, \quad M_{00} = M_{J+1,J+1} = \frac{2\Delta x}{3},
$$

$$
R_{ii} = \frac{2}{\Delta x}, \quad R_{i,i+1} = R_{i,i-1} = -\frac{1}{\Delta x}, \quad R_{00} = R_{J+1,J+1} = \frac{2}{\Delta x},
$$

$$
\partial \psi_{K_h}(u) = \begin{cases} 
\{0\} & \text{if } u \in \text{int}(K_h), \\
\{w \in K_h : \langle w, v-u \rangle \leq 0, \forall v \in K_h\} & \text{if } u \in \partial K_h, \\
\emptyset & \text{otherwise}.
\end{cases}
$$

---

(VI)$_h$ can be written in the differential inclusion form:

\[
\begin{align*}
M \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + R \frac{u^{n+1} + u^{n-1}}{2} + \alpha R \frac{u^{n+1} - u^{n-1}}{2\Delta t} + \partial \psi_{K_h}(u^{n+1}) \supseteq F^n \tag{3}
\end{align*}
\]

which is equivalent to minimize $J(u^{n+1})$ on $K_h$, 

\[
J(u^{n+1}) = \frac{1}{2} (u^{n+1})^T C u^{n+1} - (\tilde{F}_n)^T u^{n+1}
\]

**Notations:**

\[
\begin{align*}
\&\text{ C } = \frac{1}{\Delta t^2} M + \frac{1}{2} R + \frac{\alpha}{2\Delta t} R \\
\&\text{ } \tilde{F}_n = F^n + \frac{2}{\Delta t^2} M u^n - \left( \frac{1}{\Delta t^2} M + \frac{1}{2} R - \frac{\alpha}{2\Delta t} R \right) u^{n-1}
\end{align*}
\]

---

1. The viscoelastodynamic problem
2. Numerical scheme
3. Convergence result
4. Numerical results
5. Outlook
Theorem (Convergence result$^4$)

*The numerical scheme* $(\mathbf{VI})_h$ *converges to a solution of* $(\mathbf{DI})$ *when* $\Delta x$ *and* $\Delta t$ *tend to* $0$.

**Idea of the proof:** Stability and consistancy

---

**Figure:** Numerical experiments with Signorini conditions at the both ends and with $\ell(x, t) = 0$ (left) and $\ell(x, t) = \sin(t\sqrt{2})\cos(2x)$ (right).
Figure: Numerical experiments with Signorini condition at one end and Neumann condition at other end with $\ell(x, t) = 0$ (left) and $\ell(x, t) = \sin(t\sqrt{2})\cos(2x)$ (right).
Figure: Numerical experiments with distributed Signorini conditions with $\ell(x, t) = 0$ (left) and $\ell(x, t) = \sin(t\sqrt{2})\cos(2x)$ (right).
1 The viscoelastodynamic problem
2 Numerical scheme
3 Convergence result
4 Numerical results
5 Outlook
Understand the limit when $\alpha \to 0$
▷ Understand the limit when $\alpha \to 0$

▷ Study the same problems with Signorini boundary conditions distributed over the surface
- Understand the limit when $\alpha \to 0$
- Study the same problems with Signorini boundary conditions distributed over the surface
- Include the friction like a Coulomb friction law
- Understand the limit when $\alpha \to 0$
- Study the same problems with Signorini boundary conditions distributed over the surface
- Include the friction like a Coulomb friction law
- Include a heat equation
Understand the limit when $\alpha \to 0$

Study the same problems with Signorini boundary conditions distributed over the surface

Include the friction like a Coulomb friction law

Include a heat equation

Thank you for your attention!

Papers on line: http://www.wias-berlin.de/people/petrov