On the numerical approximation of a viscoelastic problem with unilateral constraints

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Mathematics for key technologies

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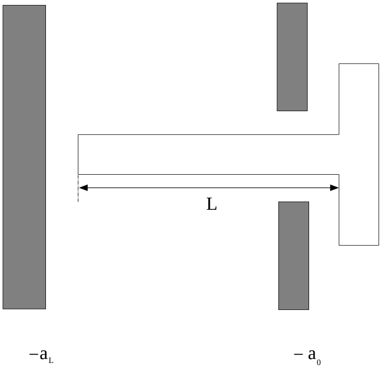
State variables

$u : [0, T] \times [0, L] \to \mathbb{R}$ displacement

Applied field

$\ell : [0, T] \times [0, L] \to \mathbb{R}$ density forces

We consider the motion of a bar between two obstacles



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The mathematical formulation is given by **(DI)**:

$u_{tt} - u_{xx} - \alpha u_{xxt} = \ell, \quad \alpha > 0$	damped wave equation
$0 \leq u(0,t) + a_0 \perp -(u_x + \alpha u_{xt})(0,t) \geq 0$	unilateral bdry conditions
$0 \leq u(L,t) + a_L \perp (u_x + \alpha u_{xt})(L,t) \geq 0$	
$u(\cdot,0) = u_0$ and $u_t(\cdot,0) = u_1$	Cauchy initial data

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$$u_{tt} - u_{xx} - \alpha u_{xxt} = \underbrace{\ell}_{\in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \quad , \quad \alpha > 0$$
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unilateral bdry conditions

$$u(\cdot,0) = \underbrace{u_0}_{\in \mathrm{H}^2(\Omega)} \quad \text{and} \quad u_t(\cdot,0) = \underbrace{u_1}_{\in \mathrm{H}^1(\Omega)}$$
Cauchy initial data

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The weak formulation **(VI)** is given by

Find $u \in K$ such that for all $v \in K$ and for all $\tau \in [0,T]$, we have

$$\int_0^L (u_t(v-u))\big|_0^\tau \mathrm{d}x - \int_0^\tau \int_0^L u_t(v_t-u_t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau \int_0^L (u_x+\alpha u_{xt})(v_x-u_x)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\tau \int_0^L \ell(v-u)\,\mathrm{d}x\,\mathrm{d}t$$

Notation:

$$K=\{u \in \mathrm{H}^1([0,L]\times(0,T)) : u_{xt} \in \mathrm{L}^2([0,L]\times(0,T)), u(0,\cdot) \geq -a_0, u(L,\cdot) \geq -a_L\}$$

Penalized Problem (PP) (here $r^- = -\min(r, 0)$)

$u^\epsilon_{tt} - u^\epsilon_{xx} - \alpha u^\epsilon_{xxt} = \ell, \quad \alpha > 0$	damped wave equation
$(u^\epsilon_x + \alpha u^\epsilon_{xt})(0,t) = -(u^\epsilon(0,t) + a_0)^- / \epsilon$	normal compliance cond.[1]
$(u^\epsilon_x + \alpha u^\epsilon_{xt})(L,t) = (u^\epsilon(L,t) + a_L)^- / \epsilon$	
$u^\epsilon(\cdot, 0) = u_0$ and $u^\epsilon_t(\cdot, 0) = u_1$	Cauchy initial data

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[2] Jarušek. *Czechoslovak Math. J.*, 1996.

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Theorem (Existence and uniqueness results[2])

There exists a unique weak solution $u^\epsilon \in \mathrm{H}^1([0,T]\times[0,L])$ *of the problem* **(PP)** *such that* $u^\epsilon_{xt} \in \mathrm{L}^2(0,T;\mathrm{L}^2(0,L))$.

Idea of the proof: Use Galerkin method.

[1] Martins, Oden. *Nonlinear Anal.*, 1988.

[2] Jarušek. *Czechoslovak Math. J.*, 1996.

Theorem (Existence result)

There exists a solution of **(VI)**; *this solution can be obtained as a limit of a subsequence of* **(PP)**.

Sketch of the proof: A-priori estimates give us

- $u^\epsilon \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(0, L))$
- $u_t^\epsilon \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(0, L))$
- $u_x^\epsilon \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(0, L))$
- $u_{xt}^\epsilon \in \mathrm{L}^2(0, T; \mathrm{L}^2(0, L))$
- $u_{xx}^\epsilon \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(0, L))$

Moreover $u_t \in \mathrm{C}^0([0, T]; \mathrm{L}^2(0, L))$ equipped with the weak topology.

Theorem (Existence result)

There exists a solution of **(VI)***; this solution can be obtained as a limit of a subsequence of* **(PP)***.*

Sketch of the proof: A-priori estimates enable to pass to the limit in the variational formulation associated to the penalized problem

$$\int_0^L u_t^\epsilon (v - u^\epsilon)\Big|_0^T \mathrm{d}x - \int_0^T \int_0^L u_t^\epsilon (v_t - u_t^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{1}{\epsilon}\int_0^T \big((u^\epsilon(0,\cdot) + a_0)^- (v - u^\epsilon)(0,\cdot)\big)\,\mathrm{d}t$$

$$-\frac{1}{\epsilon}\int_0^T \big((u^\epsilon(L,\cdot) + a_L)^- (v - u^\epsilon)(L,\cdot)\big)\,\mathrm{d}t$$

$$+\int_0^T \int_0^L (u_x^\epsilon + \alpha u_{xt}^\epsilon)(v_x - u_x^\epsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_0^L \ell(v - u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

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$$\int_0^L u_t^\epsilon (v - u^\epsilon)\Big|_0^\mathcal{T} \,\mathrm{d}x - \int_0^\mathcal{T} \int_0^L u_t^\epsilon (v_t - u_t^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^\mathcal{T} \int_0^L (u_x^\epsilon + \alpha u_{xt}^\epsilon)(v_x - u_x^\epsilon)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\mathcal{T} \int_0^L \ell (v - u^\epsilon)\,\mathrm{d}x\,\mathrm{d}t$$

Remark: Nothing is known about uniqueness.

Finite-element spaces $H_h \subset \mathrm{L}^2(0, L)$ and $V_h \subset \mathrm{H}^1(0, L)$, $\Delta x = \frac{L}{J+1}$

We consider the following variational formulation $\mathbf{(VI)}_h$:

Find $u_h^{n+1} \in K_h$ such that for all $v \in K_h$ we have

$$\left\langle \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, v - u_h^{n+1} \right\rangle + a\left(\frac{u_h^{n+1} + u_h^{n-1}}{2}, v - u_h^{n+1}\right)$$

$$\alpha a\left(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, v - u_h^{n+1}\right) \geq \left\langle \ell_h^n, v - u_h^{n+1} \right\rangle$$

Notations:

- $K_h = \{v \in V_h : v_{xt} \in H_h,\ v(0) \geq -a_0,\ v(L) \geq -a_L\}$
- $a(u, v) = \int_0^L u_x v_x \,\mathrm{d}x$
- $u_h^n \overset{\text{def}}{=} \sum_{i=0}^{J+1} u_i^n w_i$

$(\mathbf{VI})_h$ can be written in the differential inclusion form:

$$\mathbf{M}\,\frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2}+\mathbf{R}\,\frac{u^{n+1}+u^{n-1}}{2}+\alpha\mathbf{R}\,\frac{u^{n+1}-u^{n-1}}{2\Delta t}+\partial\psi_{K_h}(u^{n+1})\ni F^n$$ [3]

where for all $i \in [1, J]$

$$\mathbf{M}_{ii}=\frac{2\Delta x}{3},\quad \mathbf{M}_{i,i+1}=\mathbf{M}_{i,i-1}=\frac{\Delta x}{3},\quad \mathbf{M}_{00}=\mathbf{M}_{J+1,J+1}=\frac{2\Delta x}{3},$$
$$\mathbf{R}_{ii}=\frac{2}{\Delta x},\quad \mathbf{R}_{i,i+1}=\mathbf{R}_{i,i-1}=-\frac{1}{\Delta x},\quad \mathbf{R}_{00}=\mathbf{R}_{J+1,J+1}=\frac{2}{\Delta x},$$

$$\partial\psi_{K_h}(u)=\begin{cases}\{0\} & \text{if}\quad u\in\text{int}(K_h),\\ \{w\in K_h : \langle w, v-u\rangle \le 0,\ \forall v\in K_h\} & \text{if}\quad u\in\partial K_h,\\ \emptyset & \text{otherwise.}\end{cases}$$

[3] Brézis. *Opérateurs maximaux monotones...*, 1973.

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which is equivalent to minimize $J(u^{n+1})$ on K_h,

$$J(u^{n+1})=\frac{1}{2}(u^{n+1})^{\mathsf{T}}\mathbf{C}\,u^{n+1}-(\widetilde{F}^n)^{\mathsf{T}}u^{n+1}$$

Notations:

- $\mathbf{C}=\frac{1}{\Delta t^2}\,\mathbf{M}+\frac{1}{2}\,\mathbf{R}+\frac{\alpha}{2\Delta t}\,\mathbf{R}$
- $\widetilde{F}^n=F^n+\frac{2}{\Delta t^2}\,\mathbf{M}u^n-\left(\frac{1}{\Delta t^2}\,\mathbf{M}+\frac{1}{2}\,\mathbf{R}-\frac{\alpha}{2\Delta t}\,\mathbf{R}\right)u^{n-1}$

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Theorem (Convergence result[4])

The numerical scheme $(\mathbf{VI})_h$ *converges to a solution of* $(\mathbf{DI})$ *when* Δx *and* Δt *tend to* 0.

Idea of the proof: Stability and consistancy

[4] Schatzman, Bercovier. *Math. Comp.*, 1989.

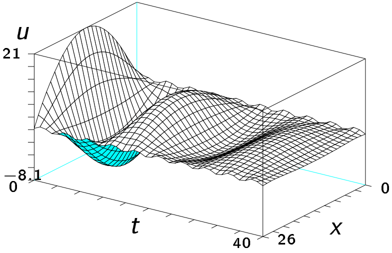

Figure: Numerical experiments with Signorini conditions at the both ends and with $\ell(x,t) = 0$ (left) and $\ell(x,t) = \sin(t\sqrt{2})\cos(2x)$ (right).

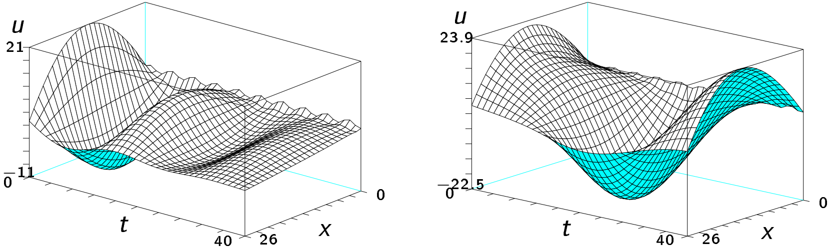

Figure: Numerical experiments with Signorini condition at one end and Neumann condition at other end with $\ell(x,t)=0$ (left) and $\ell(x,t)=\sin(t\sqrt{2})\cos(2x)$ (right).

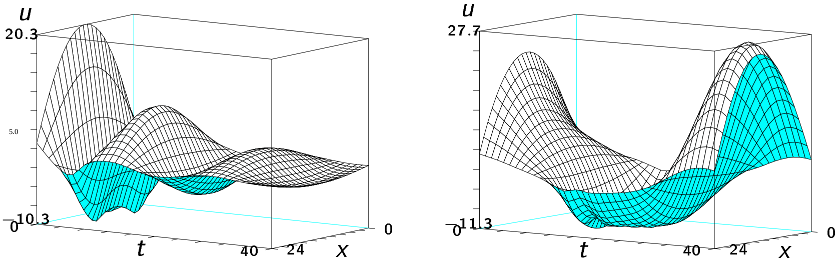

Figure: Numerical experiments with distributed Signorini conditions with $\ell(x,t) = 0$ (left) and $\ell(x,t) = \sin(t\sqrt{2})\cos(2x)$ (right).

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Thank you for your attention !

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