Some mathematical results for a model of thermally-induced phase transformation in shape-memory materials

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Mathematics for key technologies
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Outline

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Previous works:

- The model for average of the transformation strain
  - Souza, Mamiya & Zouain’98, Auricchio & Petrini’04
  - Mielke’07, Mielke & P.’07, Auricchio, Mielke & Stefanelli’08

- The model for each pure phase independently
  - Mielke, Theil & Levitas’02, Govindjee, Mielke & Hall’02, Kružík, Mielke & Roubíček’05

- The energetic formulation theory of rate-independent systems
  - Mielke & Theil’04, Mielke’05
1. Introduction
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We consider a body with reference configuration $\Omega \subset \mathbb{R}^d$.

- $u : \Omega \to \mathbb{R}^d$: the phase transformation and deformations
- $z : \Omega \to Z := \text{conv}\{e_1, \ldots, e_N\}$: the internal variable

The **potential energy** has the following form:

$$
\mathcal{E}(t, u, z) := \int_{\Omega} \left( \mathcal{W}(e(u), z, \theta) + \frac{\sigma}{2} |\nabla z|^2 \right) \mathrm{d}x - \langle l(t), u \rangle,
$$

- $\mathcal{W}$: the stored energy density depends on $e(u), z, \theta$
  - $e(u) := \frac{1}{2}(\nabla u + \nabla u^T)$: the linearized deformation
  - $\theta$: the temperature
- $\sigma > 0$: measures some nonlocal interaction effect for $z$
- $l(t)$: the applied mechanical loading

The **dissipation potential** is defined by

$$
\mathcal{D}(z_1, z_2) = \int_{\Omega} \psi(x, z_2 - z_1) \mathrm{d}x,
$$

- $\psi(x, \cdot)$: convex, l.s.c., positively homogeneous of degree 1 for a.e. $x \in \Omega$
Assumptions

- We do not solve an associated heat equation, i.e. $\theta = \theta_{\text{appl}}(t, x)$ is given

- This approximation used in engineering models:
  - the changes of the loading are slow
  - the body is small in at least one direction

  $\Rightarrow$ excess heat can be transported very fast to the surface

- Examples: the wires

- $u = u_{\text{Dir}}(t)$ on $\Gamma_{\text{Dir}} \subset \partial\Omega$

Notations: the set $Q := \mathcal{F} \times \mathcal{Z}$ where

- $\mathcal{F} := \{\tilde{u} \in H^1(\Omega; \mathbb{R}^d) | \tilde{u}|_{\Gamma_{\text{Dir}}} = 0\}$: the set of admissible displacements

- $\mathcal{Z} := \{z \in H^1(\Omega; \mathbb{R}^N) | z(x) \in Z \text{ a.e. } x \in \Omega\}$: the set where lies the internal variable
Assumption: Initial data \((u(0), z(0)) = (u_0, z_0) \in \mathcal{F} \times \mathcal{Z}\) are given.

Energetic formulation: (Theil & Mielke’04, Mielke’05)
A function \(q := (\bar{u} = u - u_{\text{Dir}}(t), z) : [0, T] \to \mathcal{Q}\) is an energetic solution of the rate-independent problem associated with \(E\) and \(\mathcal{D}\) if for all \((s, t) \in [0, T]^2\), the global stability condition (S) and the global energy conservation (E) are satisfied, i.e.

\[
\begin{align*}
\text{(S)} & \quad \forall \bar{q} = (\bar{u}, \bar{z}) \in \mathcal{Q} : \tilde{E}(t, q(t)) \leq \tilde{E}(t, \bar{q}) + \mathcal{D}(z(t), \bar{z}), \\
\text{(E)} & \quad \tilde{E}(t, q(t)) + \text{Var}_\mathcal{D}(z; [0, t]) = \tilde{E}(0, q(0)) + \int_0^t \partial_s \tilde{E}(s, q(s)) \, ds,
\end{align*}
\]

with \(\tilde{E}(t, q(t)) := E(t, \bar{u}(t) + u_{\text{Dir}}(t), z(t), \theta_{\text{appl}}(t))\),

\[
\text{Var}_\mathcal{D}(z; [r, s]) := \sup \left\{ \sum_{j=1}^p \mathcal{D}(z(t_{j-1}), z(t_j)) \ \big| \ p \in \mathbb{N}, \ r \leq t_0 < \ldots < t_p \leq s \right\}.
\]
**Remark:** $(S)$ is equivalent to

\[
q(t) \in S(t) := \{ q \in Q \mid \forall \overline{q} \in Q : \tilde{E}(t, q) \leq \tilde{E}(t, \overline{q}) + D(z, \overline{z}) \}
\]

for all \( t \in [0, T] \).
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Convergence of the space-time discretization

Notations:

- \( \mathcal{F}_h, \mathcal{V}_h \): closed subspaces resp. of \( \mathcal{F} \) and \( \mathcal{V} := H^1(\Omega; \mathbb{R}^N) \)
- \( Q_h := \mathcal{F}_h \times \mathcal{Z}_h \) and \( \mathcal{Z}_h = \{ z_h \in \mathcal{V}_h \mid z_h(x) \in \mathcal{Z} \text{ a.e. in } \Omega \} := \mathcal{Z} \cap \mathcal{V}_h \)
- \( \Pi^\tau := \{ 0 = t_0^\tau < t_1^\tau < \ldots < t_k^\tau = T \} \): a partition with \( \tau \in (0, T) \) and \( t_k^\tau - t_{k-1}^\tau \leq \tau \) for \( k = 1, \ldots, k^\tau \)

Assumptions:

- \( \forall q = (\tilde{u}, z) \in Q \quad \exists (q_h)_{h>0} : q_h = (\tilde{u}_h, z_h) \in Q_h \) and \( q_h \to q \) strongly in \( Q \)
- the initial condition \( q_0 \) is approximated by \( [q_0]^h \in Q_h \)

One has to solve the following incremental problems:

\[
(IP)^{\tau,h} \left\{ \begin{array}{l}
\text{for } k = 1, \ldots, k^\tau \text{ find }

q_k^{\tau,h} \in \operatorname{Argmin}\{ \tilde{\mathcal{E}}(t_k^\tau, \hat{q}^h) + \mathcal{D}(z_{k-1}^{\tau,h}, \hat{z}^h) \mid \hat{q}^h \in Q_h \},

\end{array} \right.
\]

where \( q_k^{\tau,h} := (\tilde{u}_k^{\tau,h}, z_k^{\tau,h}) \) and \( \hat{q}^h := (\hat{u}^h, \hat{z}^h) \).
Convergence of the space-time discretization

Assumptions:

- $\theta_{\text{appl}} \in C^1([0, T]; L^\infty(\Omega; [\theta_{\text{min}}, \theta_{\text{max}}]))$, $l \in C^1([0, T]; (H^1(\Omega; \mathbb{R}^d))')$
- $u_{\text{Dir}} \in C^1([0, T]; H^1(\Omega; \mathbb{R}^d))$
- $\forall v \in \mathbb{R}^N_0 \ \exists c_\psi, C_\psi > 0: c_\psi |v|_1 \leq \psi(x, v) \leq C_\psi |v|_1$ with $|v|_1 := \sum_{j=1}^{N} |v_j|$
- $W \in C^0(\mathbb{R}_\text{sym}^{d \times d} \times \mathbb{Z} \times [\theta_{\text{min}}, \theta_{\text{max}}]; \mathbb{R})$ is strictly convex $\forall (z, \theta) \in \mathbb{Z} \times [\theta_{\text{min}}, \theta_{\text{max}}]$
- $\forall (e, z, \theta) \in \mathbb{R}_\text{sym}^{d \times d} \times \mathbb{Z} \times [\theta_{\text{min}}, \theta_{\text{max}}] \ \exists c, C > 0:\n\quad c(|e|^2 + |z|^2) - C \leq W(e, z, \theta) \leq c(|e|^2 + |z|^2) + C$

Proposition 1. The following properties hold:

(i) $D$ is continuous for the weak topology of $H^1(\Omega)$

(ii) for all $t \in [0, T]$, $\tilde{E}(t, \cdot)$ has weakly compact sublevels

(iii) $\forall (t, q) \in [0, T] \times \mathcal{Q} \ \exists C, c > 0: C\|q\|_{\mathcal{Q}}^2 - c \leq \tilde{E}(t, q) \leq C\|q\|_{\mathcal{Q}}^2 + c$
Convergence of the space-time discretization

The incremental problems $(\text{IP})^h_\Pi$ admit a solution $(q^{\tau,h}_k)_{1 \leq k \leq k^\tau}$, we have

$$\forall q^{\tau,h} \in Q_h: \ 	ilde{\mathcal{E}}(t_k^{\tau}, q^{\tau,h}_k) \leq \tilde{\mathcal{E}}(t_{k-1}^{\tau}, q^{\tau,h}_{k-1}) + \mathcal{D}(z^{\tau,h}_{k-1}, z^{\tau,h}_k) - \mathcal{D}(z^{\tau,h}_{k-1}, z^{\tau,h}_k)$$

$$\leq \tilde{\mathcal{E}}(t_k^{\tau}, q^{\tau,h}_k) + \mathcal{D}(z^{\tau,h}_k, z^{\tau,h}_k),$$

i.e. $q^{\tau,h}_k \in S_h(t_k^{\tau}) = \{ q^h \in Q_h | \bar{q}^h \in Q_h : \tilde{\mathcal{E}}(t, q^h) \leq \tilde{\mathcal{E}}(t, \bar{q}^h) + \mathcal{D}(z^h, \bar{z}^h) \}$.

**Assumptions:**

- $\partial_\theta \mathcal{W} \in C^0(\mathbb{R}^{d \times d}_{\text{sym}} \times Z \times [\theta_{\text{min}}, \theta_{\text{max}}]; \mathbb{R})$
- $\partial_e \mathcal{W} \in C^0(\mathbb{R}^{d \times d}_{\text{sym}} \times Z \times [\theta_{\text{min}}, \theta_{\text{max}}]; \mathbb{R}^{d \times d})$

Then for all $1 \leq k \leq k^\tau$ (discrete upper energy inequality)

$$\tilde{\mathcal{E}}(t_k^{\tau}, q^{\tau,h}_k) - \tilde{\mathcal{E}}(t_{k-1}^{\tau}, q^{\tau,h}_{k-1}) + \mathcal{D}(z^{\tau,h}_{k-1}, z^{\tau,h}_k) \leq \int_{t_{k-1}^{\tau}}^{t_k^{\tau}} \partial_t \tilde{\mathcal{E}}(t, q^{\tau,h}_{k-1}) \, dt,$$

and for all $2 \leq k \leq k^\tau$ (discrete lower energy inequality)

$$\tilde{\mathcal{E}}(t_k^{\tau}, q^{\tau,h}_k) - \tilde{\mathcal{E}}(t_{k-1}^{\tau}, q^{\tau,h}_{k-1}) + \mathcal{D}(z^{\tau,h}_{k-1}, z^{\tau,h}_k) \geq \int_{t_{k-1}^{\tau}}^{t_k^{\tau}} \partial_t \tilde{\mathcal{E}}(t, q^{\tau,h}_k) \, dt.$$
We define now the approximate solution $\bar{q}^{\tau,h} : [0,T] \rightarrow \mathcal{Q}$ as the right-continuous piecewise constant approximation, namely

$$
\bar{q}^{\tau,h}(t) := \begin{cases} 
q_{k-1}^{\tau,h} & \text{for } t^{\tau}_{k-1} \leq t < t^{\tau}_k, \ k = 1, \ldots, k^{\tau}, \\
q_{k^{\tau}}^{\tau,h} & \text{for } t = T.
\end{cases}
$$

**Goal**: investigate the asymptotics as $h \to 0$ and $\tau \to 0$.

**Assumptions on $W$**: There exist positive constants $C_0^W, C_1^W, C^{\theta}, C_0^{\theta}, C_1^{\theta}, C^e, C_0^e, C_1^e$ and a nondecreasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{\tau \to 0^+} \omega(\tau) = 0$ such that for all $e, e_1, e_2 \in \mathbb{R}_d^{d \times d}, z, z_1, z_2 \in \mathbb{Z}$ and $\theta, \theta_1, \theta_2 \in [\theta_{\text{min}}, \theta_{\text{max}}]$, we have

$$
|\partial_e W(e, z, \theta)|^2 + |\partial_\theta W(e, z, \theta)| \leq C_1^W (W(e, z, \theta) + C_0^W),
$$

$$
|\partial_\theta W(e, z, \theta_1) - \partial_\theta W(e, z, \theta_2)| \leq C_1^{\theta} (W(e, z, \theta_1) + C_0^{\theta}) \omega(|\theta_1 - \theta_2|),
$$

$$
|\partial_e W(e, z, \theta_1) - \partial_e W(e, z, \theta_2)|^2 \leq C_1^e (W(e, z, \theta_1) + C_0^e) \omega(|\theta_1 - \theta_2|),
$$

$$
|\partial_\theta W(e_1, z_1, \theta) - \partial_\theta W(e_2, z_2, \theta)| \leq C^{\theta} (|e_1 - e_2| + |z_1 - z_2|)(1 + |e_1 + e_2| + |z_1 + z_2|),
$$

$$
|\partial_e W(e_1, z_1, \theta) - \partial_e W(e_2, z_2, \theta)| \leq C^e (|e_1 - e_2| + |z_1 - z_2|).$$
Proposition 2. We have

(P1) Let $q = (\tilde{u}, z) \in Q$. Then $\tilde{E}(\cdot, q)$ lies in $C^1([0, T])$ and

$$
\partial_t \tilde{E}(t, q) = \int_{\Omega} \partial_e W(e(u + u_{\text{Dir}}(t)), z, \theta_{\text{appl}}(t)) e(u_{\text{Dir}}(t)) \, dx \\
+ \int_{\Omega} \partial_\theta W(e(u + u_{\text{Dir}}(t)), z, \theta_{\text{appl}}(t)) \dot{\theta}_{\text{appl}}(t) \, dx - \langle \dot{l}(t), u \rangle.
$$

(P2) There exist $C_0^\varepsilon, C_1^\varepsilon > 0$ such that

$$
\forall (t, q) \in [0, T] \times Q : \quad |\partial_t \tilde{E}(t, q)| \leq C_1^\varepsilon (\tilde{E}(t, q) + C_0^\varepsilon).
$$

(P3) For each $\varepsilon > 0$ and $E \in \mathbb{R}$ there exists $\delta > 0$ such that for all $(s, t, q) \in [0, T]^2 \times Q$ with $\tilde{E}(0, q) \leq E$ and $|s - t| < \delta$ we have

$$
|\partial_t \tilde{E}(s, q) - \partial_t \tilde{E}(t, q)| \leq \varepsilon.
$$
Notations: For all \( k \in \{0, \ldots, k^\tau\} \) and for all \( t \in [0, T] \):

\begin{itemize}
\item \( \bar{\eta}_{k}^{\tau,h}(t) := \mathcal{E}(t, \bar{q}_{k}^{\tau,h}(t)), \quad \eta_{k}^{h} := \mathcal{E}(0, [q_{0}]^{h}) \) and \( \bar{\delta}_{k}^{\tau,h}(t) := \text{Var}_{\mathcal{D}}(\bar{z}_{k}^{\tau,h}; [0, t]) \),
\item \( \bar{\eta}_{k}^{\tau,h}(t) := \mathcal{E}(t_{k}^{\tau}, q_{k}^{\tau,h}) \) and \( \delta_{k}^{\tau,h} := \mathcal{D}(z_{k-1}^{\tau,h}, z_{k}^{\tau,h}) \).
\end{itemize}

Step 1: A priori estimates

**Lemma 1.** \( \| \bar{q}_{k}^{\tau,h}(t) \|_{Q}, \| \bar{\eta}_{k}^{\tau,h}(t) \|, \| \partial_{t} \bar{\mathcal{E}}(t, \bar{q}_{k}^{\tau,h}(t)) \|, \text{Var}(\bar{\eta}_{k}^{\tau,h}; [0, t]) \) and \( \| \bar{\delta}_{k}^{\tau,h}(t) \| \) are bounded independently of \( \tau, h \) and \( t \).
Convergence of the space-time discretization

Step 2. Selection of subsequences

Proposition 3. (Helly’s selection principle) Assume that

\[ \forall (z, \tilde{z}) \in \mathcal{Z}^2, \forall (z^h, \tilde{z}^h) \in \mathcal{Z}^2_h : \]
\[ z = \lim_{h \to 0} z^h \text{ and } \tilde{z} = \lim_{h \to 0} \tilde{z}^h \implies \mathcal{D}(z, \tilde{z}) \leq \liminf_{h \to 0} \mathcal{D}(z^h, \tilde{z}^h), \]

\[ \forall z \in \mathcal{Z}, \forall K \subset \mathcal{Z} \text{ sequentially compact }, \forall (z_n)_{n \in \mathbb{N}} \in K^\mathbb{N} : \]
\[ \min(\mathcal{D}(z_n, z), \mathcal{D}(z, z_n)) \to 0 \text{ for } n \to \infty \implies z = \lim_{n \to \infty} z_n. \]

Let \((z_n)_{n \in \mathbb{N}}\) be a sequence such that

\[ \exists C > 0 \forall n \in \mathbb{N} : \text{Var}_\mathcal{D}(z_n; [0, T]) \leq C, \]
\[ \exists K \subset \mathcal{Z} \text{ sequentially compact } \forall n \in \mathbb{N} \forall t \in [0, T] : z_n(t) \in K. \]

Then, there exists \((z_{n_j})_{j \in \mathbb{N}},\) a nondecreasing function \(\delta : [0, T] \to \mathbb{R},\) and \(z : [0, T] \to \mathcal{Z}\) such that for all \((s, t) \in [0, T]^2\) with \(s \leq t\)

\[ z(t) = \lim_{j \to \infty} z_{n_j}(t), \delta(t) = \lim_{j \to \infty} \text{Var}_\mathcal{D}(z_{n_j}; [0, t]), \text{Var}_\mathcal{D}(z; [s, t]) \leq \delta(t) - \delta(s). \]
Convergence of the space-time discretization

Assumptions on \(\psi \Rightarrow \mathcal{D}\) satisfies the properties given in Proposition 1 with Lemma 1 \(\Rightarrow\) there exists a subsequence \((\tau_n, h_n)_{n \in \mathbb{N}}\) such that for all \((s, t) \in [0, T]^2, s \leq t:\)

\[
\overline{\eta}^{\tau_n, h_n}(t) \to \eta(t), \quad \overline{\delta}^{\tau_n, h_n}(t) \to \delta(t), \quad \overline{z}^{\tau_n, h_n}(t) \to z(t) \text{ in } \mathcal{Z},
\]

\[
\text{Var}_\mathcal{D}(z; [s, t]) \leq \delta(t) - \delta(s), \quad \partial_t \mathcal{E}(\cdot, \overline{q}^{\tau_n, h_n}) \rightharpoonup \xi_* \text{ weakly } * \text{ in } L^\infty([0, T]).
\]

with \(\eta \in \text{BV}([0, T]; \mathbb{R}), \delta : [0, T] \to \mathbb{R}\) a non decreasing function, \(z : [0, T] \to \mathcal{Z}\).

**Step 3: Stability of the limit process.**

Since \(\|\overline{q}^{\tau, h}(t)\|_\mathcal{Q}\) is bounded independently of \(\tau\) and \(h\), there exists a subsequence \((n_j^t)_{j \in \mathbb{N}}\) (depending on \(t\)) such that

\[
\overline{q}^{n_j^t, h_j^t}(t) \rightharpoonup q(t) \text{ weakly in } \mathcal{Q},
\]

and thus \(q(t) = (\tilde{u}(t), z(t))\).

**Lemma 2.** We have

\[
q(t) \in S(t) = \{ q \in \mathcal{Q} \mid \forall \overline{q} \in \mathcal{Q} : \tilde{\mathcal{E}}(t, q) \leq \tilde{\mathcal{E}}(t, \overline{q}) + \mathcal{D}(z, \overline{z}) \}.
\]
Convergence of the space-time discretization

**Sketch of the proof.** Let \( \overline{q} \in Q \) and define

\[
t_j = \max \left\{ t_k^{\tau_{n_j}^f, h_{n_j}^f} \leq t, k = 0, \ldots, k_{\tau_{n_j}^f} \right\}.
\]

We have \( \lim_{j \to \infty} t_j = t \) and \( \overline{q}^{\tau_{n_j}^f, h_{n_j}^f}(t) \in S_{h_{n_j}^f}(t_j) \).

Using (P1), we have

\[
\tilde{E}(t, \overline{q}^{\tau_{n_j}^f, h_{n_j}^f}(t)) \leq \exp(C_1^E |t-t_j|)\tilde{E}(t_j, \overline{q}^{\tau_{n_j}^f, h_{n_j}^f}(t)) + C_0^E (\exp(C_1^E |t-t_j|)-1)
\]

\[
\leq \exp(C_1^E |t-t_j|)(\tilde{E}(t_j, q^{h_{n_j}^f}) + D(\overline{Z}^{\tau_{n_j}^f, h_{n_j}^f}(t), z^{h_{n_j}^f})) + C_0^E (\exp(C_1^E |t-t_j|)-1)
\]

for all \( q^{h_{n_j}^f} \in Q_{h_{n_j}^f} \). Then, we choose \( (q^{h_{n_j}^f})_{j \in \mathbb{N}} \) such that \( q^{h_{n_j}^f} \to \overline{q} \) in \( Q \) and we pass to the limit. As a consequence \( \tilde{u}(t) \in \text{Argmin}\{ \tilde{E}(t, \tilde{u}, z(t)) \mid \tilde{u} \in \mathcal{F} \} \).

Since \( \tilde{E}(t, \cdot, z(t)) \) is strictly convex, the whole sequence \( (\overline{u}^{\tau_{n}^f, h_n}(t))_{n \in \mathbb{N}} \) converges weakly in \( Q \).

So we have defined a limit process \( q \in L^\infty([0, T]; Q) \) which satisfies the global stability property.
Convergence of the space-time discretization

Step 4: Upper energy estimate

With the discrete upper energy estimate and step 1, there exists $C > 0$ such that

$$\overline{\eta}^{\tau,h}(t) + \overline{\delta}^{\tau,h}(t) \leq \eta_0^h + \int_0^t \partial_t \widehat{E}(s, \overline{q}^{\tau,h}(s)) \, ds + C(\exp(C_1^\varepsilon \tau) - 1)$$

for all $t \in [0, T]$, for all $\tau$ and $h$, which yields at the limit

$$\forall t \in [0, T] : \eta(t) + \text{Var}_D(q; [0, t]) \leq \eta(t) + \delta(t) \leq \eta(0) + \int_0^t \xi_*(s) \, ds.$$ 

Lemma 3. For all $t \in [0, T]$:

$$\lim_{n \to \infty} \widehat{E}(t, \overline{q}^{\tau_n,h_n}(t)) = \eta(t) = \widehat{E}(t, q(t)),$$

$$\lim_{n \to \infty} \partial_t \widehat{E}(t, \overline{q}^{\tau_n,h_n}(t)) = \partial_t \widehat{E}(t, q(t)).$$
Convergence of the space-time discretization

**Sketch of the proof.** Since $\tilde{E}(t, \cdot)$ is l.s.c., we have

$$\tilde{E}(t, q(t)) \leq \liminf_{n \to \infty} \tilde{E}(t, q^{T_n \cdot h_n}(t)) = \liminf_{n \to \infty} \tilde{\eta}^{T_n \cdot h_n}(t) = \eta(t).$$

But $q^{T_n \cdot h_n}(t) \in S_{h_n}(t_j)$ and with the same computations as in step 2 with $q^{T_n \cdot h_n} \to q(t)$ in $Q$, we get

$$\eta(t) = \limsup_{n \to \infty} \tilde{E}(t, q^{T_n \cdot h_n}(t)) \leq \tilde{E}(t, q(t)) + D(z(t), z(t)) = \tilde{E}(t, q(t)),$$

and thus $\lim_{n \to \infty} \tilde{E}(t, q^{T_n \cdot h_n}(t)) = \eta(t) = \tilde{E}(t, q(t))$. Then, using (P2), we find

$$\lim_{n \to \infty} \partial_t \tilde{E}(t, q^{T_n \cdot h_n}(t)) = \partial_t \tilde{E}(t, q(t)).$$
Step 5: Lower energy estimate.
Notations:

- $\Pi^n := \{0 = t^n_0 < t^n_1 < \ldots < t^n_{N_n} = t\}$: a sequence of partitions of $[0, t]$, such that $\lim_{n \to \infty} \Delta(\Pi^n) = 0$ and

$$\int_0^t \partial_t \widetilde{E}(\sigma, q(\sigma)) \, d\sigma = \lim_{n \to \infty} \sum_{j=1}^{N_n} \partial_t \widetilde{E}(t^n_j, q(t^n_j))(t^n_j - t^n_{j-1}).$$

- $\mu^n_j := \int_{t^n_{j-1}}^{t^n_j} (\partial_t \widetilde{E}(s, q(t^n_j)) - \partial_t \widetilde{E}(t^n_j, q(t^n_j))) \, ds$

We have $q(t^n_{j-1}) \in S(t^n_{j-1})$. Choosing $\bar{q} = (\tilde{u}(t^n_j), z(t^n_j))$, we obtain

$$\widetilde{E}(t^n_{j-1}, q(t^n_j)) - \widetilde{E}(t^n_{j-1}, q(t^n_{j-1})) + \mathcal{D}(q(t^n_{j-1}), q(t^n_j)) \geq \int_{t^n_{j-1}}^{t^n_j} \partial_t \widetilde{E}(s, q(t^n_j)) \, ds$$

and after summation over $j$

$$\widetilde{E}(t, q(t)) - \widetilde{E}(0, q(0)) + \text{Var}_{\mathcal{D}}(q; [0, t]) \geq \sum_{j=1}^{N_n} \partial_t \widetilde{E}(t^n_j, q(t^n_j))(t^n_j - t^n_{j-1}) + \sum_{j=1}^{N_n} \mu^n_j,$$
Notice that
\[ |\mu_j^n| \leq (t_j^n - t_{j-1}^n) \omega_E(\Delta(\Pi^n)) \]
with \( \lim_{\rho \to 0} \omega_E(\rho) = 0 \).

Then passing to the limit as \( \Delta(\Pi^n) \) tends to zero, we get
\[
\tilde{E}(t, q(t)) - \tilde{E}(0, q(0)) + \text{Var}_D(q; [0, t]) \geq \int_0^t \partial_t \tilde{E}(s, q(s)) \, ds.
\]

Finally we have
\[
\tilde{E}(0, q(0)) + \int_0^t \partial_t \tilde{E}(s, q(s)) \, ds
\leq \tilde{E}(t, q(t)) + \text{Var}_D(q; [0, t]) \leq \eta(t) + \delta(t) \leq \tilde{E}(0, q(0)) + \int_0^t \xi_*(s) \, ds,
\]
and \( \xi_* = \partial_t \tilde{E}(\cdot, q) \) a.e. in \( [0, T] \). Hence all the inequalities are in fact equalities, thus
\[ \forall t \in [0, T] : \, \delta(t) = \text{Var}_D(q; [0, t]). \]

and
\[
\partial_t \tilde{E}(\cdot, \overline{q}^{tn} h^n) \to \partial_t \tilde{E}(\cdot, q) \text{ strongly in } L^1([0, T]).
\]
**Assumption:** Let $[q_0]^h \in \mathcal{Q}_h$ be such that $[q_0]^h \to q_0$ in $\mathcal{Q}$.

**Theorem 1 (Convergence of the approximate solutions).** There exist a subsequence $(\tau_n, h_n)_{n \in \mathbb{N}}$ tending to $(0, 0)$ and an energetic solution $q = (\tilde{u}, z) : [0, T] \to \mathcal{Q}$ of $(\mathcal{S})$ and $(\mathcal{E})$ such that $q(0) = q_0$,

$$
\tilde{u} \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^d)),
$$

$$
z \in L^\infty([0, T]; H^1(\Omega; Z)) \cap BV([0, T]; L^1(\Omega; Z)),
$$

and, for all $t \in [0, T]$, the following convergences hold:

$$
\overline{q}^{\tau_n, h_n}(t) \to q(t) \text{ strongly in } \mathcal{Q},
$$

$$
\tilde{\mathcal{E}}(t, \overline{q}^{\tau_n, h_n}(t)) \to \tilde{\mathcal{E}}(t, q(t)),
$$

$$
\text{Var}_D(\overline{z}^{\tau_n, h_n}; [0, t]) \to \text{Var}_D(z; [0, t]),
$$

$$
\partial_t \tilde{\mathcal{E}}(\cdot, \overline{q}^{\tau_n, h_n}) \to \partial_t \tilde{\mathcal{E}}(\cdot, q) \text{ strongly in } L^1([0, T]).
$$
1 Introduction
2 Mathematical formulation
3 Convergence of the space-time discretization
4 Temporal regularity via uniform convexity
Assumption:

- \( W \) is \( \alpha_W \)-uniformly convex in its first two arguments.

Proposition 3. \( q \) is Lipschitz continuous.

Sketch of the proof. The uniform convexity of \( W \) implies that there exists \( \kappa > 0 \) such that \((t, q_1, q_2) \in [0, T] \times (\mathcal{F} \times V)^2:\)

\[
\frac{\kappa}{2} \| q_2 - q_1 \|^2_Q \leq \tilde{E}(t, q_2) - \tilde{E}(t, q_1) - \langle D_q \tilde{E}(t, q_1), q_2 - q_1 \rangle.
\]

On the other hand, (S) implies

\( (S)_{\text{loc}} \forall s \in [0, T], \forall \nu = (\tilde{u}, z) \in Q : \langle D_q \tilde{E}(s), \nu - q(s) \rangle + D(z(s), z) \geq 0. \)

So, with \( q_1 = q(s), q_2 = q(t), 0 \leq s \leq t \leq T \)

\[
\frac{\kappa}{2} \| q(t) - q(s) \|^2_Q \leq \tilde{E}(s, q(t)) - \tilde{E}(s, q(s)) + \text{Var}_D(z; s, t)
\]

\[
= - \int_s^t \partial_r \tilde{E}(r, q(t)) \, dr + \int_s^t \partial_r \tilde{E}(r, q(r)) \, dr \leq C \int_s^t \| q(r) - q(t) \|_Q \, dr.
\]

Then we infer that \( \| q(t) - q(s) \|_Q \leq \frac{2C}{\kappa} (t - s) \)