# Some mathematical results for a model of thermally-induced phase transformation in shape-memory materials

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**Previous works:**

- The model for average of the transformation strain
  - Souza, Mamiya & Zouain'98, Auricchio & Petrini'04
  - Mielke'07, Mielke & P.'07, Auricchio, Mielke & Stefanelli'08
- The model for each pure phase independently
  - Mielke, Theil & Levitas'02, Govindjee, Mielke & Hall'02, Kružík, Mielke & Roubíček'05
- The energetic formulation theory of rate-independent systems
  - Mielke & Theil'04, Mielke'05

We consider a body with reference configuration $\Omega \subset \mathbb{R}^d$.

- $u : \Omega \to \mathbb{R}^d$: the phase transformation and deformations
- $z : \Omega \to Z := \mathrm{conv}\{e_1, \dots, e_N\}$: the internal variable

The potential energy has the following form:

$$\mathcal{E}(t, u, z) := \int_\Omega \Big( W(e(u), z, \theta) + \frac{\sigma}{2}|\nabla z|^2 \Big) \,\mathrm{d}x - \langle l(t), u \rangle,$$

- $W$: the stored energy density depends on $e(u)$, $z$, $\theta$
  - $e(u) := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$: the linearized deformation $\quad \theta$: the temperature
- $\sigma > 0$: measures some nonlocal interaction effect for $z$
- $l(t)$: the applied mechanical loading

The dissipation potential is defined by

$$\mathcal{D}(z_1, z_2) = \int_\Omega \psi(x, z_2 - z_1) \,\mathrm{d}x,$$

- $\psi(x, \cdot)$: convex, l.s.c., positively homogeneous of degree 1 for a.e. $x \in \Omega$

**Assumptions**

- We do not solve an associated heat equation, i.e. $\theta = \theta_{\mathrm{appl}}(t, x)$ is given
  - This approximation used in engineering models:
    - the changes of the loading are slow
    - the body is small in at least one direction

    $\Rightarrow$ excess heat can be transported very fast to the surface
  - Examples: the wires
- $u = u_{\mathrm{Dir}}(t)$ on $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$

**Notations**: the set $\mathcal{Q} := \mathcal{F} \times \mathcal{Z}$ where

- $\mathcal{F} := \{\widetilde{u} \in \mathrm{H}^1(\Omega; \mathbb{R}^d) \,|\, \widetilde{u}_{|\Gamma_{\mathrm{Dir}}} = 0\}$: the set of admissible displacements
- $\mathcal{Z} := \{z \in \mathrm{H}^1(\Omega; \mathbb{R}^N) \,|\, z(x) \in Z \text{ a.e. } x \in \Omega\}$: the set where lies the internal variable

**Assumption**: Initial data $(u(0), z(0)) = (u_0, z_0) \in \mathcal{F} \times \mathcal{Z}$ are given.

**Energetic formulation:** (Theil & Mielke'04, Mielke'05)
A function $q := (\widetilde{u} = u - u_{\mathrm{Dir}}(t), z) : [0, T] \to \mathcal{Q}$ is an *energetic solution* of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{D}$ if for all $(s, t) \in [0, T]^2$, the *global stability condition* (S) and the *global energy conservation* (E) are satisfied, i.e.

$$\text{(S)} \quad \forall \overline{q} = (\overline{u}, \overline{z}) \in \mathcal{Q} : \widetilde{\mathcal{E}}(t, q(t)) \leq \widetilde{\mathcal{E}}(t, \overline{q}) + \mathcal{D}(z(t), \overline{z}),$$

$$\text{(E)} \quad \widetilde{\mathcal{E}}(t, q(t)) + \mathrm{Var}_{\mathcal{D}}(z; [0, t]) = \widetilde{\mathcal{E}}(0, q(0)) + \int_0^t \partial_s \widetilde{\mathcal{E}}(s, q(s))\,\mathrm{d}s,$$

with $\widetilde{\mathcal{E}}(t, q(t)) := \mathcal{E}(t, \widetilde{u}(t) + u_{\mathrm{Dir}}(t), z(t), \theta_{\mathrm{appl}}(t))$,

$$\mathrm{Var}_{\mathcal{D}}(z; [r, s]) := \sup\Big\{ \sum_{j=1}^{p} \mathcal{D}\big(z(t_{j-1}), z(t_j)\big) \,\Big|\, p \in \mathbb{N},\ r \leq t_0 < \ldots < t_p \leq s \Big\}.$$

**Remark**: (S) is equivalent to

$$q(t) \in \mathcal{S}(t) := \left\{ q \in \mathcal{Q} \,|\, \forall \overline{q} \in \mathcal{Q} : \widetilde{\mathcal{E}}(t,q) \le \widetilde{\mathcal{E}}(t,\overline{q}) + \mathcal{D}(z,\overline{z}) \right\}$$

for all $t \in [0, T]$.

**Notations**:

- ▷ $\mathcal{F}_h$, $V_h$: closed subspaces resp. of $\mathcal{F}$ and $V := \mathrm{H}^1(\Omega;\mathbb{R}^N)$
- ▷ $\mathcal{Q}_h := \mathcal{F}_h \times \mathcal{Z}_h$ and $\mathcal{Z}_h = \{z_h \in V_h \,|\, z_h(x) \in Z \text{ a.e. in } \Omega\} := \mathcal{Z} \cap V_h$
- ▷ $\Pi^\tau := \{0 = t_0^\tau < t_1^\tau < \ldots < t_{k^\tau}^\tau = T\}$: a partition with $\tau \in (0,T)$ and $t_k^\tau - t_{k-1}^\tau \leq \tau$ for $k = 1,\ldots,k^\tau$

**Assumptions**:

- ▷ $\forall q = (\widetilde{u}, z) \in \mathcal{Q} \quad \exists (q_h)_{h>0} : \; q_h = (\widetilde{u}_h, z_h) \in \mathcal{Q}_h$ and $q_h \to q$ strongly in $\mathcal{Q}$
- ▷ the initial condition $q_0$ is approximated by $[q_0]^h \in \mathcal{Q}_h$

One has to solve the following incremental problems:

$$(\mathrm{IP})^{\tau,h} \begin{cases} \text{for } k = 1,\ldots,k^\tau \text{ find} \\ q_k^{\tau,h} \in \operatorname{Argmin}\{\widetilde{\mathcal{E}}(t_k^\tau, \widehat{q}^h) + \mathcal{D}(z_{k-1}^{\tau,h}, \widehat{z}^h) \,|\, \widehat{q}^h \in \mathcal{Q}_h\}, \end{cases}$$

where $q_k^{\tau,h} := (\widetilde{u}_k^{\tau,h}, z_k^{\tau,h})$ and $\widehat{q}^h := (\widehat{u}^h, \widehat{z}^h)$.

**Assumptions**:

- $\theta_{\mathrm{appl}} \in \mathrm{C}^1([0,T];\mathrm{L}^\infty(\Omega;[\theta_{\min},\theta_{\max}]))$, $\quad l \in \mathrm{C}^1([0,T];(\mathrm{H}^1(\Omega;\mathbb{R}^d))')$
- $u_{\mathrm{Dir}} \in \mathrm{C}^1([0,T];\mathrm{H}^1(\Omega;\mathbb{R}^d))$
- $\forall v \in \mathbb{R}^N_0 \quad \exists c_\psi, C_\psi > 0 : \; c_\psi |v|_1 \leq \psi(x,v) \leq C_\psi |v|_1$ with $|v|_1 := \sum_{j=1}^N |v_j|$
- $W \in C^0(\mathbb{R}^{d\times d}_{\mathrm{sym}} \times Z \times [\theta_{\min},\theta_{\max}];\mathbb{R})$ is stricly convex $\forall (z,\theta) \in Z \times [\theta_{\min},\theta_{\max}]$
- $\forall (e,z,\theta) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times Z \times [\theta_{\min},\theta_{\max}] \quad \exists c, C > 0$:

$$c\left(|e|^2+|z|^2\right) - C \leq W(e,z,\theta) \leq c\left(|e|^2+|z|^2\right) + C$$

**Proposition 1.** The following properties hold:

(i) $\mathcal{D}$ is continuous for the weak topology of $\mathrm{H}^1(\Omega)$

(ii) for all $t \in [0,T]$, $\widetilde{\mathcal{E}}(t,\cdot)$ has weakly compact sublevels

(iii) $\forall (t,q) \in [0,T]\times\mathcal{Q} \quad \exists C, c > 0$: $\; C\|q\|^2_{\mathcal{Q}} - c \leq \widetilde{\mathcal{E}}(t,q) \leq C\|q\|^2_{\mathcal{Q}} + c$

The incremental problems $(\mathrm{IP})^h_\Pi$ admit a solution $(q_k^{\tau,h})_{1\le k\le k^\tau}$, we have

$$\forall \overline{q}^{\tau,h} \in \mathcal{Q}_h : \ \widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) \le \widetilde{\mathcal{E}}(t_k^\tau, \overline{q}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, \overline{z}^{\tau,h}) - \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}) \\ \le \widetilde{\mathcal{E}}(t_k^\tau, \overline{q}^{\tau,h}) + \mathcal{D}(z_k^{\tau,h}, \overline{z}^{\tau,h}),$$

i.e. $q_k^{\tau,h} \in \mathcal{S}_h(t_k^\tau) = \big\{ q^h \in \mathcal{Q}_h \,|\, \overline{q}^h \in \mathcal{Q}_h : \widetilde{\mathcal{E}}(t, q^h) \le \widetilde{\mathcal{E}}(t, \overline{q}^h) + \mathcal{D}(z^h, \overline{z}^h) \big\}$.

**Assumptions**:

- ▷ $\partial_\theta W \in C^0(\mathbb{R}^{d\times d}_{\mathrm{sym}} \times Z \times [\theta_{\min}, \theta_{\max}]; \mathbb{R})$
- ▷ $\partial_e W \in C^0(\mathbb{R}^{d\times d}_{\mathrm{sym}} \times Z \times [\theta_{\min}, \theta_{\max}]; \mathbb{R}^{d\times d}_{\mathrm{sym}})$

Then for all $1 \le k \le k^\tau$ (discrete upper energy inequality)

$$\widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) - \widetilde{\mathcal{E}}(t_{k-1}^\tau, q_{k-1}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}) \le \int_{t_{k-1}^\tau}^{t_k^\tau} \partial_t \widetilde{\mathcal{E}}(t, q_{k-1}^{\tau,h})\, \mathrm{d}t,$$

and for all $2 \le k \le k^\tau$ (discrete lower energy inequality)

$$\widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) - \widetilde{\mathcal{E}}(t_{k-1}^\tau, q_{k-1}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}) \ge \int_{t_{k-1}^\tau}^{t_k^\tau} \partial_t \widetilde{\mathcal{E}}(t, q_k^{\tau,h})\, \mathrm{d}t.$$

We define now the approximate solution $\overline{q}^{\tau,h} : [0,T] \to \mathcal{Q}$ as the right-continuous piecewise constant approximation, namely

$$\overline{q}^{\tau,h}(t) := \begin{cases} q_{k-1}^{\tau,h} & \text{for } \; t_{k-1}^{\tau} \le t < t_k^{\tau},\; k = 1, \dots, k^{\tau}, \\ q_{k^{\tau}}^{\tau,h} & \text{for } t = T. \end{cases}$$

**Goal**: investigate the asymptotics as $h \to 0$ and $\tau \to 0$.

**Assumptions on** $W$: There exist positive constants $C_0^W$, $C_1^W$, $C^\theta$, $C_0^\theta$, $C_1^\theta$, $C^e$, $C_0^e$, $C_1^e$ and a nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ with $\lim_{\tau \to 0^+} \omega(\tau) = 0$ such that for all $e, e_1, e_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $z, z_1, z_2 \in Z$ and $\theta, \theta_1, \theta_2 \in [\theta_{\min}, \theta_{\max}]$, we have

$$|\partial_e W(e,z,\theta)|^2 + |\partial_\theta W(e,z,\theta)| \le C_1^W \big(W(e,z,\theta) + C_0^W\big),$$

$$\big|\partial_\theta W(e,z,\theta_1) - \partial_\theta W(e,z,\theta_2)\big| \le C_1^\theta \big(W(e,z,\theta_1) + C_0^\theta\big)\, \omega(|\theta_1 - \theta_2|),$$

$$\big|\partial_e W(e,z,\theta_1) - \partial_e W(e,z,\theta_2)\big|^2 \le C_1^e \big(W(e,z,\theta_1) + C_0^e\big)\, \omega(|\theta_1 - \theta_2|),$$

$$\big|\partial_\theta W(e_1,z_1,\theta) - \partial_\theta W(e_2,z_2,\theta)\big| \le C^\theta(|e_1 - e_2| + |z_1 - z_2|)(1 + |e_1 + e_2| + |z_1 + z_2|),$$

$$\big|\partial_e W(e_1,z_1,\theta) - \partial_e W(e_2,z_2,\theta)\big| \le C^e(|e_1 - e_2| + |z_1 - z_2|).$$

**Proposition 2.** We have

(P1) Let $q = (\widetilde{u}, z) \in \mathcal{Q}$. Then $\widetilde{\mathcal{E}}(\cdot, q)$ lies in $\mathrm{C}^1([0,T])$ and

$$\partial_t \widetilde{\mathcal{E}}(t,q) = \int_\Omega \partial_e W(e(u{+}u_{\mathrm{Dir}}(t)), z, \theta_{\mathrm{appl}}(t))\, e(\dot{u}_{\mathrm{Dir}}(t))\, \mathrm{d}x + \int_\Omega \partial_\theta W(e(u{+}u_{\mathrm{Dir}}(t)), z, \theta_{\mathrm{appl}}(t))\, \dot{\theta}_{\mathrm{appl}}(t)\, \mathrm{d}x - \langle \dot{l}(t), u\rangle.$$

(P2) There exist $C_0^{\mathcal{E}}, C_1^{\mathcal{E}} > 0$ such that

$$\forall (t,q) \in [0,T] \times \mathcal{Q}: \ |\partial_t \widetilde{\mathcal{E}}(t,q)| \leq C_1^{\mathcal{E}} \big(\widetilde{\mathcal{E}}(t,q) + C_0^{\mathcal{E}}\big).$$

(P3) For each $\varepsilon > 0$ and $E \in \mathbb{R}$ there exists $\delta > 0$ such that for all $(s,t,q) \in [0,T]^2 \times \mathcal{Q}$ with $\widetilde{\mathcal{E}}(0,q) \leq E$ and $|s{-}t| < \delta$ we have

$$|\partial_t \widetilde{\mathcal{E}}(s,q) - \partial_t \widetilde{\mathcal{E}}(t,q)| \leq \varepsilon.$$

**Notations**: For all $k \in \{0, \dots, k^\tau\}$ and for all $t \in [0, T]$:

▷ $\overline{\eta}^{\tau,h}(t) := \mathcal{E}(t, \overline{q}^{\tau,h}(t))$, $\eta_0^h := \mathcal{E}(0, [q_0]^h)$ and $\overline{\delta}^{\tau,h}(t) := \mathrm{Var}_{\mathcal{D}}(\overline{z}^{\tau,h}; [0, t])$,

▷ $\eta_k^{\tau,h} := \mathcal{E}(t_k^\tau, q_k^{\tau,h})$ and $\delta_k^{\tau,h} := \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h})$.

**Step 1: A priori estimates**

**Lemma 1.** $\|\overline{q}^{\tau,h}(t)\|_{\mathcal{Q}}$, $|\overline{\eta}^{\tau,h}(t)|$, $|\partial_t \widetilde{\mathcal{E}}(t, \overline{q}^{\tau,h}(t))|$, $\mathrm{Var}(\overline{\eta}^{\tau,h}; [0, t])$ and $|\overline{\delta}^{\tau,h}(t)|$ are bounded independently of $\tau$, $h$ and $t$.

## Step 2. Selection of subsequences

**Proposition 3. (Helly's selection principle)** Assume that

$$\left.\begin{array}{l}\forall (z,\widetilde{z}) \in \mathcal{Z}^2,\ \forall (z^h,\widetilde{z}^h) \in \mathcal{Z}_h^2 : \\ z = \lim_{h\to 0} z^h \text{ and } \widetilde{z} = \lim_{h\to 0} \widetilde{z}^h\end{array}\right\} \Rightarrow \mathcal{D}(z,\widetilde{z}) \le \liminf_{h\to 0} \mathcal{D}(z^h,\widetilde{z}^h),$$

$$\left.\begin{array}{l}\forall z \in \mathcal{Z},\ \forall K \subset \mathcal{Z} \text{ sequentially compact },\forall (z_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}} : \\ \min\big(\mathcal{D}(z_n,z),\mathcal{D}(z,z_n)\big) \to 0 \text{ for } n \to \infty\end{array}\right\} \Rightarrow z = \lim_{n\to\infty} z_n.$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that

$$\exists C > 0\ \forall n \in \mathbb{N} : \mathrm{Var}_{\mathcal{D}}(z_n;[0,T]) \le C,$$

$$\exists K \subset \mathcal{Z} \text{ sequentially compact } \forall n \in \mathbb{N}\ \forall t \in [0,T] : \ z_n(t) \in K.$$

Then, there exists $(z_{n_j})_{j\in\mathbb{N}}$, a nondecreasing function $\delta : [0,T] \to \mathbb{R}$, and $z : [0,T] \to \mathcal{Z}$ such that for all $(s,t) \in [0,T]^2$ with $s \le t$

$$z(t) = \lim_{j\to\infty} z_{n_j}(t),\ \delta(t) = \lim_{j\to\infty} \mathrm{Var}_{\mathcal{D}}(z_{n_j};[0,t]),\ \mathrm{Var}_{\mathcal{D}}(z;[s,t]) \le \delta(t) - \delta(s).$$

Assumptions on $\psi \Rightarrow \mathcal{D}$ satisfies the properties given in Proposition 1 with Lemma 1 $\Rightarrow$ there exists a subsequence $(\tau_n, h_n)_{n\in\mathbb{N}}$ such that for all $(s,t) \in [0,T]^2$, $s \leq t$:

$$\overline{\eta}^{\tau_n,h_n}(t) \to \eta(t),\ \overline{\delta}^{\tau_n,h_n}(t) \to \delta(t),\ \overline{z}^{\tau_n,h_n}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z},$$
$$\mathrm{Var}_{\mathcal{D}}(z;[s,t]) \leq \delta(t) - \delta(s),\ \partial_t\mathcal{E}(\cdot,\overline{q}^{\tau_n,h_n}) \rightharpoonup \xi_* \text{ weakly} * \text{ in } \mathrm{L}^\infty([0,T]).$$

with $\eta \in \mathrm{BV}([0,T];\mathbb{R})$, $\delta : [0,T] \to \mathbb{R}$ a non decreasing function, $z : [0,T] \to \mathcal{Z}$.

**Step 3: Stability of the limit process.**

Since $\|\overline{q}^{\tau,h}(t)\|_{\mathcal{Q}}$ is bounded independently of $\tau$ and $h$, there exists a subsequence $(n_j^t)_{j\in\mathbb{N}}$ (depending on $t$) such that

$$\overline{q}^{\tau_{n_j^t},h_{n_j^t}}(t) \rightharpoonup q(t) \text{ weakly in } \mathcal{Q},$$

and thus $q(t) = (\widetilde{u}(t), z(t))$.

**Lemma 2.** We have

$$q(t) \in \mathcal{S}(t) = \left\{ q \in \mathcal{Q} \;\middle|\; \forall \overline{q} \in \mathcal{Q} : \widetilde{\mathcal{E}}(t,q) \leq \widetilde{\mathcal{E}}(t,\overline{q}) + \mathcal{D}(z,\overline{z}) \right\}.$$

**Sketch of the proof.** Let $\overline{q} \in \mathcal{Q}$ and define

$$t_j = \max\Big\{ t_k^{\tau_{n_j^t}, h_{n_j^t}} \le t,\ k = 0, \dots, k_{\tau_{n_j^t}} \Big\}.$$

We have $\lim_{j\to\infty} t_j = t$ and $\overline{q}^{\tau_{n_j^t}, h_{n_j^t}}(t) \in \mathcal{S}_{h_{n_j^t}}(t_j)$.

Using (P1), we have

$$\begin{aligned} \widetilde{\mathcal{E}}(t, \overline{q}^{\tau_{n_j^t}, h_{n_j^t}}(t)) &\le \exp(C_1^{\mathcal{E}}|t-t_j|)\widetilde{\mathcal{E}}(t_j, \overline{q}^{\tau_{n_j^t}, h_{n_j^t}}(t)) + C_0^{\mathcal{E}}\big(\exp(C_1^{\mathcal{E}}|t-t_j|)-1\big) \\ &\le \exp(C_1^{\mathcal{E}}|t-t_j|)\big(\widetilde{\mathcal{E}}(t_j, q^{h_{n_j^t}}) + \mathcal{D}(\overline{z}^{\tau_{n_j^t}, h_{n_j^t}}(t), z^{h_{n_j^t}})\big) + C_0^{\mathcal{E}}\big(\exp(C_1^{\mathcal{E}}|t-t_j|)-1\big) \end{aligned}$$

for all $q^{h_{n_j^t}} \in \mathcal{Q}_{h_{n_j^t}}$. Then, we choose $(q^{h_{n_j^t}})_{j\in\mathbb{N}}$ such that $q^{h_{n_j^t}} \to \overline{q}$ in $\mathcal{Q}$ and we pass to the limit. As a consequence $\widetilde{u}(t) \in \mathrm{Argmin}\big\{\widetilde{\mathcal{E}}(t, \widetilde{u}, z(t)) \,\big|\, \widetilde{u} \in \mathcal{F}\big\}$.
Since $\widetilde{\mathcal{E}}(t, \cdot, z(t))$ is strictly convex, the whole sequence $\big(\overline{u}^{\tau_n, h_n}(t)\big)_{n\in\mathbb{N}}$ converges weakly in $\mathcal{Q}$.
So we have defined a limit process $q \in \mathrm{L}^\infty([0,T]; \mathcal{Q})$ which satisfies the global stability property.

**Step 4: Upper energy estimate**

With the discrete upper energy estimate and step 1, there exists $C > 0$ such that

$$\overline{\eta}^{\tau,h}(t) + \overline{\delta}^{\tau,h}(t) \le \eta_0^h + \int_0^t \partial_t \widetilde{\mathcal{E}}(s, \overline{q}^{\tau,h}(s))\,\mathrm{d}s + C(\exp(C_1^{\mathcal{E}}\tau) - 1)$$

for all $t \in [0, T]$, for all $\tau$ and $h$, which yields at the limit

$$\forall t \in [0, T]: \ \eta(t) + \mathrm{Var}_{\mathcal{D}}(q; [0, t]) \le \eta(t) + \delta(t) \le \eta(0) + \int_0^t \xi_*(s)\,\mathrm{d}s.$$

**Lemma 3.** For all $t \in [0, T]$:

$$\lim_{n\to\infty} \widetilde{\mathcal{E}}\big(t, \overline{q}^{\tau_n,h_n}(t)\big) = \eta(t) = \widetilde{\mathcal{E}}\big(t, q(t)\big),$$

$$\lim_{n\to\infty} \partial_t\widetilde{\mathcal{E}}\big(t, \overline{q}^{\tau_n,h_n}(t)\big) = \partial_t\widetilde{\mathcal{E}}\big(t, q(t)\big).$$

**Sketch of the proof.** Since $\widetilde{\mathcal{E}}(t,\cdot)$ is l.s.c., we have

$$\widetilde{\mathcal{E}}(t,q(t)) \le \liminf_{n\to\infty} \widetilde{\mathcal{E}}(t,\overline{q}^{\tau_n,h_n}(t)) = \liminf_{n\to\infty} \overline{\eta}^{\tau_n,h_n}(t) = \eta(t).$$

But $\overline{q}^{\tau_n,h_n}(t) \in \mathcal{S}_{h_n}(t_j)$ and with the same computations as in step 2 with $\overline{q}^{\tau_n,h_n} \to q(t)$ in $\mathcal{Q}$, we get

$$\eta(t) = \limsup_{n\to\infty} \widetilde{\mathcal{E}}(t,\overline{q}^{\tau_n,h_n}(t)) \le \widetilde{\mathcal{E}}(t,q(t)) + \mathcal{D}(z(t),z(t)) = \widetilde{\mathcal{E}}(t,q(t)),$$

and thus $\lim_{n\to\infty} \widetilde{\mathcal{E}}(t,\overline{q}^{\tau_n,h_n}(t)) = \eta(t) = \widetilde{\mathcal{E}}(t,q(t))$. Then, using (P2), we find

$$\lim_{n\to\infty} \partial_t \widetilde{\mathcal{E}}(t,\overline{q}^{\tau_n,h_n}(t)) = \partial_t \widetilde{\mathcal{E}}(t,q(t)).$$

**Step 5: Lower energy estimate.**
**Notations:**

- $\Pi^n := \{0 = t_0^n < t_1^n < \ldots < t_{N_n}^n = t\}$: a sequence of partitions of $[0, t]$, such that $\lim_{n\to\infty} \Delta(\Pi^n) = 0$ and

$$\int_0^t \partial_t \widetilde{\mathcal{E}}(\sigma, q(\sigma))\,\mathrm{d}\sigma = \lim_{n\to\infty} \sum_{j=1}^{N_n} \partial_t \widetilde{\mathcal{E}}(t_j^n, q(t_j^n))(t_j^n - t_{j-1}^n).$$

- $\mu_j^n := \int_{t_{j-1}^n}^{t_j^n} \left(\partial_t \widetilde{\mathcal{E}}(s, q(t_j^n)) - \partial_t \widetilde{\mathcal{E}}(t_j^n, q(t_j^n))\right) \mathrm{d}s$

We have $q(t_{j-1}^n) \in \mathcal{S}(t_{j-1}^n)$. Choosing $\overline{q} = (\widetilde{u}(t_j^n), z(t_j^n))$, we obtain

$$\widetilde{\mathcal{E}}(t_{j-1}^n, q(t_j^n)) - \widetilde{\mathcal{E}}(t_{j-1}^n, q(t_{j-1}^n)) + \mathcal{D}(q(t_{j-1}^n), q(t_j^n)) \geq \int_{t_{j-1}^n}^{t_j^n} \partial_t \widetilde{\mathcal{E}}(s, q(t_j^n))\,\mathrm{d}s$$

and after summation over $j$

$$\widetilde{\mathcal{E}}(t, q(t)) - \widetilde{\mathcal{E}}(0, q(0)) + \mathrm{Var}_{\mathcal{D}}(q; [0, t]) \geq \sum_{j=1}^{N_n} \partial_t \widetilde{\mathcal{E}}(t_j^n, q(t_j^n))(t_j^n - t_{j-1}^n) + \sum_{j=1}^{N_n} \mu_j^n,$$

Notice that

$$|\mu_j^n| \le (t_j^n - t_{j-1}^n)\omega_E(\Delta(\Pi^n))$$

with $\lim_{\rho\to 0}\omega_E(\rho) = 0$.
Then passing to the limit as $\Delta(\Pi^n)$ tends to zero, we get

$$\widetilde{\mathcal{E}}(t, q(t)) - \widetilde{\mathcal{E}}(0, q(0)) + \mathrm{Var}_{\mathcal{D}}(q; [0, t]) \ge \int_0^t \partial_t \widetilde{\mathcal{E}}(s, q(s))\,\mathrm{d}s.$$

Finally we have

$$\begin{aligned} &\widetilde{\mathcal{E}}(0, q(0)) + \int_0^t \partial_t \widetilde{\mathcal{E}}(s, q(s))\,\mathrm{d}s \\ &\le \widetilde{\mathcal{E}}(t, q(t)) + \mathrm{Var}_{\mathcal{D}}(q; [0, t]) \le \eta(t) + \delta(t) \le \widetilde{\mathcal{E}}(0, q(0)) + \int_0^t \xi_*(s)\,\mathrm{d}s, \end{aligned}$$

and $\xi_* = \partial_t \widetilde{\mathcal{E}}(\cdot, q)$ a.e. in $[0, T]$. Hence all the inequalities are in fact equalities, thus

$$\forall t \in [0, T]: \; \delta(t) = \mathrm{Var}_{\mathcal{D}}(q; [0, t]).$$

and

$$\partial_t \widetilde{\mathcal{E}}(\cdot, \overline{q}^{\tau_n, h_n}) \to \partial_t \widetilde{\mathcal{E}}(\cdot, q) \text{ strongly in } \mathrm{L}^1([0, T]).$$

**Assumption:** Let $[q_0]^h \in \mathcal{Q}_h$ be such that $[q_0]^h \to q_0$ in $\mathcal{Q}$.

**Theorem 1 (Convergence of the approximate solutions).** There exist a subsequence $(\tau_n, h_n)_{n\in\mathbb{N}}$ tending to $(0,0)$ and an energetic solution $q = (\widetilde{u}, z) : [0, T] \to \mathcal{Q}$ of (S) and (E) such that $q(0) = q_0$,

$$\widetilde{u} \in \mathrm{L}^\infty([0, T]; \mathrm{H}^1(\Omega; \mathbb{R}^d)),$$
$$z \in \mathrm{L}^\infty([0, T]; \ \mathrm{H}^1(\Omega; Z)) \cap \mathrm{BV}([0, T]; \mathrm{L}^1(\Omega; Z)),$$

and, for all $t \in [0, T]$, the following convergences hold:

$$\overline{q}^{\tau_n, h_n}(t) \to q(t) \text{ strongly in } \mathcal{Q},$$
$$\widetilde{\mathcal{E}}(t, \overline{q}^{\tau_n, h_n}(t)) \to \widetilde{\mathcal{E}}(t, q(t)),$$
$$\mathrm{Var}_{\mathcal{D}}(\overline{z}^{\tau_n, h_n}; [0, t]) \to \mathrm{Var}_{\mathcal{D}}(z; [0, t]),$$
$$\partial_t \widetilde{\mathcal{E}}(\cdot, \overline{q}^{\tau_n, h_n}) \to \partial_t \widetilde{\mathcal{E}}(\cdot, q) \text{ strongly in } \mathrm{L}^1([0, T]).$$

**Assumption:**

- $W$ is $\alpha_W$-uniformly convex in its first two arguments

**Proposition 3.** $q$ is Lipschitz continuous.
**Sketch of the proof.** The uniform convexity of $W$ implies that there exists $\kappa > 0$ such that $(t, q_1, q_2) \in [0, T] \times (\mathcal{F} \times V)^2$:

$$\frac{\kappa}{2}\|q_2 - q_1\|_{\mathcal{Q}}^2 \leq \widetilde{\mathcal{E}}(t, q_2) - \widetilde{\mathcal{E}}(t, q_1) - \langle D_q\widetilde{\mathcal{E}}(t, q_1), q_2 - q_1\rangle.$$

On the other hand, (S) implies

$$(\mathrm{S})_{\mathrm{loc}} \quad \forall s \in [0, T], \forall v = (\widetilde{u}, z) \in \mathcal{Q}: \ \langle D_q\widetilde{\mathcal{E}}(s), v - q(s)\rangle + \mathcal{D}(z(s), z) \geq 0.$$

So, with $q_1 = q(s)$, $q_2 = q(t)$, $0 \leq s \leq t \leq T$

$$\frac{\kappa}{2}\|q(t) - q(s)\|_{\mathcal{Q}}^2 \leq \widetilde{\mathcal{E}}(s, q(t)) - \widetilde{\mathcal{E}}(s, q(s)) + \mathrm{Var}_{\mathcal{D}}(z; s, t)$$
$$= -\int_s^t \partial_r\widetilde{\mathcal{E}}(r, q(t))\,\mathrm{d}r + \int_s^t \partial_r\widetilde{\mathcal{E}}(r, q(r))\,\mathrm{d}r \leq C\int_s^t \|q(r) - q(t)\|_{\mathcal{Q}}\,\mathrm{d}r.$$

Then we infer that $\|q(t) - q(s)\|_{\mathcal{Q}} \leq \frac{2C}{\kappa}(t - s)$