Existence and approximation for a 3D model of thermally induced phase transformations in shape-memory alloys

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The phase transformations of the crystallographic lattice are characterized by an internal variable $z : \Omega \to Z = \mathrm{conv}\{e_1, \ldots, e_N\}$.

Interpretation: if $z(x) = \sum_{k=1}^{N} \lambda_k(x) e_k$, $\lambda_k(x)$= volume fraction of phase k.

The model is described by

- a stored energy density $W(e(u), z, \theta)$ where u denotes the displacements, $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, and θ is the temperature,
- a dissipation $\mathcal{D}(z_1, z_2)$ due to phase changes.

We assume that $\mathcal{D}$ is given by

$$\mathcal{D}(z_1, z_2) = \int_\Omega \psi(x, z_2 - z_1)\, dx$$

where $\psi(x, \cdot)$ is convex, l.s.c. and positively homogeneous of degree 1 for a.e. $x \in \Omega$.

We study the quasi-static evolution in small strain regime, within the framework of the variational theory of rate-independent processes.

The potential energy is given by

$$\mathcal{E}(t,u,z,\theta) = \int_\Omega \Big(W(e(u),z,\theta) + \frac{\sigma}{2}|\nabla z|^2 \Big)\,dx - \langle \ell(t),u \rangle$$

where ℓ is a time-dependent applied loading and $\frac{\sigma}{2}|\nabla z|^2$, with $\sigma > 0$, takes into account some non-local effects for the internal variable z.

We assume that $\theta = \theta_{\mathrm{appl}}(t,x)$ is a given data (valid if Ω is small in at least one direction) and that $u = u_{\mathrm{Dir}}(t)$ on $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$.

We define the set $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ by

$$\begin{aligned} \mathcal{F} &= \{\widetilde{u} \in H^1(\Omega;\mathbb{R}^d);\, \widetilde{u}_{|\Gamma_{\mathrm{Dir}}} = 0\}, \\ \mathcal{Z} &= \{z \in H^1(\Omega;\mathbb{R}^N);\, z(x) \in Z \text{ a.e. } x \in \Omega\}. \end{aligned}$$

We look for an energetic solution for the rate independent problem associated to $\mathcal{E}$ and $\mathcal{D}$, i.e $q = (\widetilde{u} = u - u_{\rm Dir}, z) : [0,T] \to \mathcal{Q}$ satisfying the global stability condition (S) and the global energy balance (E)

$$\text{(S)}\ \forall \bar{q} = (\bar{u}, \bar{z}) \in \mathcal{Q}:\ \widetilde{\mathcal{E}}(t, q(t)) \leq \widetilde{\mathcal{E}}(t, \bar{q}) + \mathcal{D}(z(t), \bar{z}),$$

$$\text{(E)}\ \widetilde{\mathcal{E}}(t, q(t)) + \mathrm{Var}_{\mathcal{D}}(z; 0, t) = \widetilde{\mathcal{E}}(0, q(0)) + \int_0^t \partial_s \widetilde{\mathcal{E}}(s, q(s))\, \mathrm{d}s.$$

with

$$\widetilde{\mathcal{E}}(t, q(t)) = \mathcal{E}(t, \widetilde{u}(t) + u_{\rm Dir}(t), z(t), \theta_{\rm appl}(t))$$

and

$$\mathrm{Var}_{\mathcal{D}}(z; r, s) = \sup\Big\{\sum_{j=1}^{p} \mathcal{D}\big(z(t_{j-1}), z(t_j)\big)\ \Big|\ p \in \mathbb{N},\ r \leq t_0 < t_1 < \ldots < t_p \leq s\Big\}$$

for all $(r, s) \in [0,T]^2$ such that $r < s$.

Let us observe that (S) is equivalent to

$$q(t) \in \mathcal{S}(t) = \big\{q \in \mathcal{Q}; \widetilde{\mathcal{E}}(t, q) \leq \widetilde{\mathcal{E}}(t, \bar{q}) + \mathcal{D}(z, \bar{z}) \quad \bar{q} \in \mathcal{Q}\big\} \quad \text{for all } t \in [0,T].$$

In order to construct approximate solutions we consider closed subspaces $\mathcal{F}_h$ and V_h of $\mathcal{F}$ and $V = H^1(\Omega; \mathrm{I\!R}^N)$ respectively (e.g. finite dimensional subspaces) and we define

$$\mathcal{Q}_h = \mathcal{F}_h \times \mathcal{Z}_h, \quad \mathcal{Z}_h = \{z_h \in V_h; z_h(x) \in Z \text{ a.e. in } \Omega\} = \mathcal{Z} \cap V_h.$$

We assume that for all $q = (\widetilde{u}, z) \in \mathcal{Q}$ there exists a sequence $(q_h)_{h>0}$ such that

$$q_h = (\widetilde{u}_h, v_h) \in \mathcal{Q}_h \quad \forall h > 0, \quad q_h \to q \quad \text{strongly in } \mathcal{Q}.$$

We consider also a partition $\Pi = (t_k^\tau)_{0 \le k \le k^\tau}$ of $[0, T]$, i.e.

$$0 = t_0^\tau < t_1^\tau < \ldots < t_{k^\tau}^\tau = T$$

such that $\Delta(\Pi) = \sup\{t_k^\tau - t_{k-1}^\tau,\ 1 \le k \le k^\tau\} \le \tau$, with $\tau \in (0, T)$.

We approximate the initial condition q_0 by $[q_0]^h \in \mathcal{Q}_h$ and we solve the following incremental problems:

$$(\mathrm{IP})^h_\Pi \begin{cases} \text{for } k = 1, \ldots, k^\tau \text{ find} \\ q_k^{\tau,h} = (\widetilde{u}_k^{\tau,h}, z_k^{\tau,h}) \in \mathrm{Argmin}\Big\{\widetilde{\mathcal{E}}(t_k^\tau, \bar{q}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, \bar{z}^{\tau,h}) \,\big|\, \bar{q}^{\tau,h} = (\bar{u}^{\tau,h}, \bar{z}^{\tau,h}) \in \mathcal{Q}_h\Big\}. \end{cases}$$

Let us assume that :

$$(\mathbf{H1}) \qquad \begin{cases} \theta_{\mathrm{appl}} \in C^1([0,T]; L^\infty(\Omega; [\theta_{\min}, \theta_{\max}])) \\ \ell \in C^1([0,T]; (H^1(\Omega; \mathbb{R}^d))') \\ u_{\mathrm{Dir}} \in C^1([0,T]; H^1(\Omega; \mathbb{R}^d)) \end{cases}$$

$$(\mathbf{H2}) \qquad \exists c_\psi, C_\psi > 0 \quad \text{s.t.} \quad c_\psi |v|_1 \le \psi(x,v) \le C_\psi |v|_1 \quad \forall v \in \mathbb{R}_0^N, \quad |v|_1 = \sum_{j=1}^N |v_j|$$

$$(\mathbf{H3}) \qquad \begin{cases} W \in C^0(\mathbb{R}^{d\times d}_{\mathsf{sym}} \times Z \times [\theta_{\mathsf{min}}, \theta_{\mathsf{max}}]; \mathbb{R}) \\ W(\cdot, z, \theta) \text{ is stricly convex for all } (z,\theta) \in Z \times [\theta_{\min}, \theta_{\max}] \\ \exists c, C > 0 \quad \text{s.t.} \quad c\big(|e|^2 + |z|^2\big) - C \le W(e,z,\theta) \le c\big(|e|^2 + |z|^2\big) + C \\ \forall (e,z,\theta) \in \mathbb{R}^{d\times d}_{\mathsf{sym}} \times Z \times [\theta_{\mathsf{min}}, \theta_{\mathsf{max}}] \end{cases}$$

Proposition 1. The following properties hold:

- $\mathcal{D}$ is continuous for the weak topology of $H^1(\Omega)$,
- for all $t \in [0,T]$, $\widetilde{\mathcal{E}}(t,\cdot)$ has weakly compact sublevels,
- there exist positive real numbers C_0, c_0, C_1, c_1 such that

$$C_0\|q\|_{\mathcal{Q}}^2 - c_0 \leq \widetilde{\mathcal{E}}(t,q) \leq C_1\|q\|_{\mathcal{Q}}^2 - c_1$$

for all $(t,q) \in [0,T] \times Q$.

We infer that the incremental problems $(\mathrm{IP})_{\Pi}^{h}$ admit a solution and, for all $k = 1,\ldots,k^\tau$, we have

$$\widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) \leq \widetilde{\mathcal{E}}(t_k^\tau, \bar{q}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, \bar{z}^{\tau,h}) - \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}) \leq \widetilde{\mathcal{E}}(t_k^\tau, \bar{q}^{\tau,h}) + \mathcal{D}(z_k^{\tau,h}, \bar{z}^{\tau,h})$$

for all $\bar{q}^{\tau,h} \in \mathcal{Q}_h$ i.e.

$$q_k^{\tau,h} \in \mathcal{S}_h(t_k^\tau) = \left\{q^h \in \mathcal{Q}_h ; \widetilde{\mathcal{E}}(t,q^h) \leq \widetilde{\mathcal{E}}(t,\bar{q}^h) + \mathcal{D}(z^h,\bar{z}^h) \quad \bar{q}^h \in \mathcal{Q}_h\right\}.$$

Let us assume moreover that $\partial_\theta W \in C^0(\mathbb{R}^{d\times d}_{\text{sym}} \times Z \times [\theta_{\text{min}}, \theta_{\text{max}}]; \mathbb{R})$ and $\partial_e W \in C^0(\mathbb{R}^{d\times d}_{\text{sym}} \times Z \times [\theta_{\text{min}}, \theta_{\text{max}}]; \mathbb{R}^{d\times d}_{\text{sym}})$. Then

$$\widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) - \widetilde{\mathcal{E}}(t_{k-1}^\tau, q_{k-1}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}) \le \int_{t_{k-1}^\tau}^{t_k^\tau} \partial_t \widetilde{\mathcal{E}}(t, q_{k-1}^{\tau,h})\,\mathrm{d}t,$$

for all $1 \le k \le k^\tau$ (discrete upper energy inequality) and

$$\widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) - \widetilde{\mathcal{E}}(t_{k-1}^\tau, q_{k-1}^{\tau,h}) + \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}) \ge \int_{t_{k-1}^\tau}^{t_k^\tau} \partial_t \widetilde{\mathcal{E}}(t, q_k^{\tau,h})\,\mathrm{d}t,$$

for all $2 \le k \le k^\tau$ (discrete lower energy inequality).

We define the approximate solution $q_R^{\tau,h}$ by

$$q_{\mathrm{R}}^{\tau,h}(t) = q_{k-1}^{\tau,h} \quad \forall t \in [t_{k-1}^\tau, t_k^\tau),\ \forall k = 1, \ldots, k^\tau, \quad q_{\mathrm{R}}^{\tau,h}(T) = q_{k^\tau}^{\tau,h}.$$

Next we investigate the asymptotics as h and τ tend to zero.

Proposition 2. Let us assume that there exist positive constants C_0^W, C_1^W, C^θ, C_0^θ, C_1^θ, C^e, C_0^e, C_1^e and a nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ with $\lim_{\tau\to 0^+}\omega(\tau) = 0$ such that for all $e, e_1, e_2 \in \mathbb{R}^{d\times d}_{\text{sym}}$, $z, z_1, z_2 \in Z$ and $\theta, \theta_1, \theta_2 \in [\theta_{\text{min}}, \theta_{\text{max}}]$, we have

$$
\begin{aligned}
&|\partial_e W(e,z,\theta)|^2 + |\partial_\theta W(e,z,\theta)| \le C_1^W(W(e,z,\theta)+C_0^W),\\
&|\partial_\theta W(e,z,\theta_1)-\partial_\theta W(e,z,\theta_2)| \le C_1^\theta(W(e,z,\theta_1)+C_0^\theta)\,\omega(|\theta_1-\theta_2|),\\
&|\partial_e W(e,z,\theta_1)-\partial_e W(e,z,\theta_2)|^2 \le C_1^e(W(e,z,\theta_1)+C_0^e)\,\omega(|\theta_1-\theta_2|),\\
&|\partial_\theta W(e_1,z_1,\theta)-\partial_\theta W(e_2,z_2,\theta)| \le C^\theta(|e_1-e_2|+|z_1-z_2|)(1+|e_1+e_2|+|z_1+z_2|),\\
&|\partial_e W(e_1,z_1,\theta)-\partial_e W(e_2,z_2,\theta)| \le C^e(|e_1-e_2|+|z_1-z_2|).
\end{aligned}
$$

Then
(i) there exist $C_0^{\mathcal{E}}, C_1^{\mathcal{E}} > 0$ such that

$$|\partial_t\widetilde{\mathcal{E}}(t,q)| \le C_1^{\mathcal{E}}(\widetilde{\mathcal{E}}(t,q)+C_0^{\mathcal{E}}) \quad \forall (t,q)\in[0,T]\times\mathcal{Q}$$

(ii) for all $E\in\mathbb{R}$ there exists a nondecreasing function $\omega_E : [0,\infty)\to[0,\infty)$ with $\lim_{\rho\to 0^+}\omega_E(\rho) = 0$ such that

$$|\partial_t\widetilde{\mathcal{E}}(s,q)-\partial_t\widetilde{\mathcal{E}}(t,q)| \le \omega_E(|t-s|) \quad \forall (s,t,q)\in[0,T]^2\times\mathcal{Q} \text{ s.t. } \widetilde{\mathcal{E}}(0,q)\le E.$$

Let us introduce the following notations:

$$\delta_k^{\tau,h} = \mathcal{D}(z_{k-1}^{\tau,h}, z_k^{\tau,h}), \quad \eta_k^{\tau,h} = \widetilde{\mathcal{E}}(t_k^\tau, q_k^{\tau,h}) \quad \forall k \in \{0, \ldots, k^\tau\}$$

and

$$\delta_{\mathrm{R}}^{\tau,h}(t) = \mathsf{Var}_{\mathcal{D}}(z_{\mathrm{R}}^{\tau,h}; 0, t), \quad \eta_{\mathrm{R}}^{\tau,h}(t) = \widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau,h}(t)) \quad \forall t \in [0, T].$$

Step 1: A priori estimates

Lemma 1. $\|q_R^{\tau,h}(t)\|_{\mathcal{Q}}$, $|\eta_{\mathrm{R}}^{\tau,h}(t)|$, $|\partial_t \widetilde{\mathcal{E}}(t, q_R^{\tau,h}(t))|$, $\mathsf{Var}(\eta_{\mathrm{R}}^{\tau,h}; 0, t)$ and $|\delta_{\mathrm{R}}^{\tau,h}(t)|$ are bounded independently of τ, h and t.

Step 2: Passage to the limit

We will use Helly's selection principle:

Lemma (Helly's selection principle). Assume that the following properties hold:

$$\left.\begin{array}{l} \forall (z,\widetilde{z}) \in \mathcal{Z}^2,\ \forall (z^h,\widetilde{z}^h) \in \mathcal{Z}_h^2 : \\ z = \lim_{h\to 0} z^h \ \text{ and } \ \widetilde{z} = \lim_{h\to 0} \widetilde{z}^h \end{array}\right\} \Rightarrow \mathcal{D}(z,\widetilde{z}) \le \liminf_{h\to 0} \mathcal{D}(z^h,\widetilde{z}^h),$$

and

$$\left.\begin{array}{l} \forall z \in \mathcal{Z},\ \forall K \subset \mathcal{Z} \text{ sequentially compact },\ \forall (z_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}} : \\ \min\big(\mathcal{D}(z_n,z),\mathcal{D}(z,z_n)\big) \to 0 \ \text{ for } \ n \to \infty \end{array}\right\} \Rightarrow z = \lim_{n\to\infty} z_n.$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that $z_n : [0,T] \to \mathcal{Z}$ for all $n \in \mathbb{N}$ and satisfying

$$\begin{array}{l} \exists C > 0,\ \forall n \in \mathbb{N} : \mathrm{Var}_{\mathcal{D}}(z_n;0,T) \le C, \\ \exists K \subset \mathcal{Z} \text{ sequentially compact },\ \forall n \in \mathbb{N},\ \forall t \in [0,T] :\ z_n(t) \in K. \end{array}$$

Then, there exists a subsequence $(z_{n_j})_{j\in\mathbb{N}}$, a nondecreasing function $\delta : [0,T] \to \mathbb{R}$, and a limit process $z : [0,T] \to \mathcal{Z}$ such that for all $(s,t) \in [0,T]^2$ with $s \le t$, we have

$$z(t) = \lim_{j\to\infty} z_{n_j}(t),\ \delta(t) = \lim_{j\to\infty} \mathrm{Var}_{\mathcal{D}}(z_{n_j};0,t),\ \mathrm{Var}_{\mathcal{D}}(z;s,t) \le \delta(t) - \delta(s).$$

Assumption (H2) implies that $\mathcal{D}$ satisfies the previous properties and with lemma 1 we infer that there exists a subsequence $(\tau_n, h_n)_{n\in\mathbb{N}}$ such that

$$\eta_{\mathrm{R}}^{\tau_n,h_n}(t) \to \eta(t),\ \ \delta_{\mathrm{R}}^{\tau_n,h_n}(t) \to \delta(t),\ \ z_{\mathrm{R}}^{\tau_n,h_n}(t) \rightharpoonup z(t)\ \text{ in }\ \mathcal{Z} \quad \forall t \in [0,T]$$

with $\eta \in BV([0,T];\mathbb{R})$, $\delta : [0,T] \to \mathbb{R}$ a non decreasing function and $z : [0,T] \to \mathcal{Z}$ such that

$$\mathrm{Var}_{\mathcal{D}}(z; s,t) \leq \delta(t) - \delta(s) \quad \text{for all } (s,t) \in [0,T]^2 \text{ with } s \leq t.$$

Moreover, possibly extracting another subsequence, we have

$$\partial_t \widetilde{\mathcal{E}}(\cdot, q_{\mathrm{R}}^{\tau_n,h_n}) \rightharpoonup \xi_* \text{ weakly} * \ \text{ in } L^\infty([0,T]).$$

Let $t \in [0,T]$. Since $\|q_R^{\tau,h}(t)\|_{\mathcal{Q}}$ is bounded independently of τ and h, there exists a subsequence $(n_j^t)_{j\in\mathbb{N}}$ (depending on t) such that

$$q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t) \rightharpoonup q(t) \text{ weakly in } \mathcal{Q}$$

and thus $q(t) = (\widetilde{u}(t), z(t))$.

Lemma 2. We have

$$q(t) \in \mathcal{S}(t) = \left\{q \in \mathcal{Q}; \widetilde{\mathcal{E}}(t,q) \le \widetilde{\mathcal{E}}(t,\bar{q}) + \mathcal{D}(z,\bar{z}) \quad \forall \bar{q} \in \mathcal{Q}\right\}.$$

Sketch of the proof. Let $\bar{q} \in \mathcal{Q}$ and define

$$t_j = \max\{t_k^{\tau_{n_j^t},h_{n_j^t}} \le t,\ k = 0,\dots,k_{\tau_{n_j^t}}\}.$$

We have $\lim_{j\to\infty} t_j = t$ and $q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t) \in \mathcal{S}_{h_{n_j^t}}(t_j)$.

So using (i) in proposition 2, we get

$$\widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t)) \leq \exp(C_1^{\mathcal{E}}|t-t_j|)\widetilde{\mathcal{E}}(t_j, q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t)) + C_0^{\mathcal{E}}\big(\exp(C_1^{\mathcal{E}}|t-t_j|)-1\big)$$
$$\leq \exp(C_1^{\mathcal{E}}|t-t_j|)\big(\widetilde{\mathcal{E}}(t_j, q^{h_{n_j^t}})+\mathcal{D}(z_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t), z^{h_{n_j^t}})\big) + C_0^{\mathcal{E}}\big(\exp(C_1^{\mathcal{E}}|t-t_j|)-1\big)$$

for all $q^{h_{n_j^t}} \in \mathcal{Q}_{h_{n_j^t}}$. Then, we choose $(q^{h_{n_j^t}})_{j\in\mathbb{N}}$ such that $q^{h_{n_j^t}} \longrightarrow \bar{q}$ in $\mathcal{Q}$ and we pass to the limit.

As a consequence

$$\widetilde{u}(t) \in \operatorname{Argmin}\big\{\widetilde{\mathcal{E}}\big(t, \widetilde{u}, z(t)\big),\ \widetilde{u} \in \mathcal{F}\big\}.$$

Since $\widetilde{\mathcal{E}}\big(t, \cdot, z(t)\big)$ is strictly convex, the whole sequence $\big(\widetilde{u}_R^{\tau_n,h_n}(t)\big)_{n\in\mathbb{N}}$ converges weakly in $\mathcal{Q}$.

So we have defined a limit process $q \in L^\infty\big([0,T];\mathcal{Q}\big)$ which satisfies the global stability property.

Step 3: Energy estimate

With the discrete upper energy estimate and step 1, there exists $C > 0$ such that

$$\eta_{\mathrm{R}}^{\tau,h}(t) + \delta_{\mathrm{R}}^{\tau,h}(t) \le \eta_0^h + \int_0^t \partial_t \widetilde{\mathcal{E}}(s, q_{\mathrm{R}}^{\tau,h}(s))\,\mathrm{d}s + C(\exp(C_1^{\mathcal{E}}\tau) - 1)$$

for all $t \in [0,T]$, for all τ and h, which yields at the limit

$$\eta(t) + \mathsf{Var}_{\mathcal{D}}(q;0,t) \le \eta(t) + \delta(t) \le \eta(0) + \int_0^t \xi_*(s)\,\mathsf{d}s \quad \forall t \in [0,T].$$

Moreover

Lemma 3. For all $t \in [0,T]$, the following convergences hold:

$$\lim_{n\to+\infty} \widetilde{\mathcal{E}}\big(t, q_R^{\tau_n,h_n}(t)\big) = \eta(t) = \widetilde{\mathcal{E}}\big(t, q(t)\big),$$
$$\lim_{n\to+\infty} \partial_t\widetilde{\mathcal{E}}\big(t, q_R^{\tau_n,h_n}(t)\big) = \partial_t\widetilde{\mathcal{E}}\big(t, q(t)\big).$$

Sketch of the proof. Let $t \in [0,T]$. Since $\widetilde{\mathcal{E}}(t,\cdot)$ is lower-semicontinuous we have

$$\widetilde{\mathcal{E}}(t,q(t)) \leq \liminf_{n\to+\infty} \widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_n,h_n}(t)) = \liminf_{n\to+\infty} \eta_{\mathrm{R}}^{\tau_n,h_n}(t) = \eta(t).$$

But $q_{\mathrm{R}}^{\tau_n,h_n}(t) \in \mathcal{S}_{h_n}(t_j)$ and with the same computations as in step 2 with $q^{h_n} \longrightarrow q(t)$ in $\mathcal{Q}$, we get

$$\eta(t) = \limsup_{n\to+\infty} \widetilde{\mathcal{E}}(t, q_R^{\tau_n,h_n}(t)) \leq \widetilde{\mathcal{E}}(t,q(t)) + \mathcal{D}(z(t),z(t)) = \widetilde{\mathcal{E}}(t,q(t))$$

and thus $\lim_{n\to+\infty} \widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_n,h_n}(t)) = \eta(t) = \widetilde{\mathcal{E}}(t,q(t))$.
Then, using property (ii) of proposition 2, we infer that

$$\lim_{n\to+\infty} \partial_t \widetilde{\mathcal{E}}(t, q_R^{\tau_n,h_n}(t)) = \partial_t \widetilde{\mathcal{E}}(t,q(t)).$$

Next, we fix $t \in [0,T]$ and let $\Pi^p = \{0 = t_0^p < t_1^p < \ldots < t_{N_p}^p = t\}$ be a sequence of partitions of $[0,t]$, such that $\lim_{p\to+\infty} \Delta(\Pi^p) = 0$ and

$$\int_0^t \partial_t \widetilde{\mathcal{E}}(\sigma, q(\sigma))\, \mathrm{d}\sigma = \lim_{p\to+\infty} \sum_{j=1}^{N_p} \partial_t \widetilde{\mathcal{E}}(t_j^p, q(t_j^p))(t_j^p - t_{j-1}^p).$$

We have $q(t_{j-1}^p) \in \mathcal{S}(t_{j-1}^p)$ so, by choosing $\bar{q} = (\widetilde{u}(t_j^p), z(t_j^p))$, we obtain

$$\widetilde{\mathcal{E}}(t_{j-1}^p, q(t_j^p)) - \widetilde{\mathcal{E}}(t_{j-1}^p, q(t_{j-1}^p)) + \mathcal{D}(q(t_{j-1}^p), q(t_j^p)) \geq \int_{t_{j-1}^p}^{t_j^p} \partial_t \widetilde{\mathcal{E}}(s, q(t_j^p))\, \mathrm{d}s$$

and after summation over j

$$\widetilde{\mathcal{E}}(t, q(t)) - \widetilde{\mathcal{E}}(0, q(0)) + \mathrm{Var}_{\mathcal{D}}(q; 0, t) \geq \sum_{j=1}^{N_p} \partial_t \widetilde{\mathcal{E}}(t_j^p, q(t_j^p))(t_j^p - t_{j-1}^p) + \sum_{j=1}^{N_p} \mu_j^p,$$

where $|\mu_j^p| = \left|\int_{t_{j-1}^p}^{t_j^p} \left(\partial_t \widetilde{\mathcal{E}}(s, q(t_j^p)) - \partial_t \widetilde{\mathcal{E}}(t_j^p, q(t_j^p))\right) \mathrm{d}s\right| \leq (t_j^p - t_{j-1}^p)\omega_E(\Delta(\Pi^p))$ with $\lim_{\rho\to 0} \omega_E(\rho) = 0$. Then passing to the limit as $\Delta(\Pi^p)$ tends to zero, we get

$$\widetilde{\mathcal{E}}(t, q(t)) - \widetilde{\mathcal{E}}(0, q(0)) + \mathrm{Var}_{\mathcal{D}}(q; 0, t) \geq \int_0^t \partial_t \widetilde{\mathcal{E}}(s, q(s))\, \mathrm{d}s.$$

Finally we have

$$\widetilde{\mathcal{E}}\big(0,q(0)\big) + \int_0^t \partial_t\widetilde{\mathcal{E}}\big(s,q(s)\big)\,\mathsf{d}s \le \widetilde{\mathcal{E}}(t,q(t)) + \mathsf{Var}_{\mathcal{D}}(q;0,t)$$
$$\le \eta(t) + \delta(t) \le \widetilde{\mathcal{E}}\big(0,q(0)\big) + \int_0^t \xi_*(s)\,\mathsf{d}s$$

and

$$\xi_* = \partial_t\widetilde{\mathcal{E}}(\cdot,q) \quad \text{a.e. in } [0,T].$$

Hence all the inequalities are in fact equalities, thus

$$\delta(t) = \mathsf{Var}_{\mathcal{D}}(q;0,t) \quad \forall t \in [0,T]$$

and

$$\partial_t\widetilde{\mathcal{E}}(\cdot,q_{\mathrm{R}}^{\tau_n,h_n}) \to \partial_t\widetilde{\mathcal{E}}(\cdot,q) \quad \text{strongly in} \quad L^1([0,T]).$$

Theorem 1 (Convergence of the approximate solutions).

There exist a subsequence $\{(\tau_n, h_n)\}_{n\in\mathbb{N}}$ tending to $(0,0)$ and an energetic solution $q = (\widetilde{u}, z) : [0,T] \to \mathcal{Q}$ of (S) and (E) such that $q(0) = q_0$,

$$\widetilde{u} \in L^\infty\big([0,T]; H^1(\Omega;\mathbb{R}^d)\big),$$
$$z \in L^\infty\big([0,T]; H^1(\Omega;Z)\big) \cap BV\big([0,T]; L^1(\Omega;Z)\big),$$

and, for all $t \in [0,T]$, the following convergences hold:

$$z_{\mathrm{R}}^{\tau_n,h_n}(t) \to z(t) \text{ strongly in } \mathcal{Z},$$
$$\widetilde{u}_{\mathrm{R}}^{\tau_n,h_n}(t) \rightharpoonup \widetilde{u}(t) \text{ weakly in } \mathcal{F},$$
$$\widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_n,h_n}(t)) \to \widetilde{\mathcal{E}}(t, q(t)),$$
$$\mathrm{Var}_{\mathcal{D}}(z_{\mathrm{R}}^{\tau_n,h_n}; 0, t) \to \mathrm{Var}_{\mathcal{D}}(z; 0, t),$$
$$\partial_t\widetilde{\mathcal{E}}(\cdot, q_{\mathrm{R}}^{\tau_n,h_n}) \to \partial_t\widetilde{\mathcal{E}}(\cdot, q) \text{ strongly in } L^1([0,T]).$$

Proposition 3. Let us assume moreover that W is α_W-uniformly convex in its first two arguments. Then q is Lipschitz continuous.

Sketch of the proof. The uniform convexity of W implies that there exists $\kappa > 0$ s.t.

$$\forall t \in [0,T],\ \forall (q_1,q_2) \in \mathcal{Q}^2 :\ \frac{\kappa}{2}\|q_2 - q_1\|_{\mathcal{Q}}^2 \le \widetilde{\mathcal{E}}(t,q_2) - \widetilde{\mathcal{E}}(t,q_1) - \langle D_q\widetilde{\mathcal{E}}(t,q_1), q_2 - q_1\rangle.$$

On the other hand, (S) implies

$$(\mathrm{Sloc})\ \forall s \in [0,T], \forall v = (\widetilde{u}, z) \in \mathcal{Q} :\ \langle D_q\widetilde{\mathcal{E}}(s), v - q(s)\rangle + \mathcal{D}(z(s), z) \ge 0.$$

So, with $q_1 = q(s)$, $q_2 = q(t)$, $0 \le s \le t \le T$

$$\begin{aligned} &\frac{\kappa}{2}\|q(t) - q(s)\|_{\mathcal{Q}}^2 \le \widetilde{\mathcal{E}}(s,q(t)) - \widetilde{\mathcal{E}}(s,q(s)) + \mathrm{Var}_{\mathcal{D}}(z;s,t) \\ &= -\int_s^t \partial_r\widetilde{\mathcal{E}}(r,q(t))\,\mathrm{d}r + \int_s^t \partial_r\widetilde{\mathcal{E}}(r,q(r))\,\mathrm{d}r \le C\int_s^t \|q(r) - q(t)\|_{\mathcal{Q}}\,\mathrm{d}r. \end{aligned}$$

Then we infer that $\|q(t) - q(s)\|_{\mathcal{Q}} \le \dfrac{2C}{\kappa}(t-s)$.

The **strict** convexity assumption is not necessary to prove a convergence result.

Indeed, let us assume that $W(\cdot, z, \theta)$ is convex for a.e. $(z, \theta) \in Z \times [\theta_{\min}, \theta_{\max}]$, then

- proposition 1 and proposition 2 are still true,
- the incremental problems $(\mathrm{IP})^h_{\Pi}$ still admit a solution $q_k^{\tau,h}$ for all $k = 1, \dots, k^\tau$ such that $q_k^{\tau,h} \in \mathcal{S}_h(t_k^\tau)$ and which satisfies the discrete upper/lower energy inequalities,
- the approximate solution $q_R^{\tau,h}$ still satisfies the a priori estimates,
- we can apply Helly's selection principle and extract a subsequence $(\tau_n, h_n)_{n\in\mathrm{I\!N}}$ such that

$$\eta_{\mathrm{R}}^{\tau_n,h_n}(t) \to \eta(t),\ \delta_{\mathrm{R}}^{\tau_n,h_n}(t) \to \delta(t),\ z_{\mathrm{R}}^{\tau_n,h_n}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z} \quad \forall t \in [0,T]$$

and

$$\partial_t \widetilde{\mathcal{E}}(\cdot, q_{\mathrm{R}}^{\tau_n,h_n}) \rightharpoonup \xi_* \text{ weakly} * \text{ in } L^\infty([0,T])$$

with $\eta \in BV([0,T];\mathrm{I\!R})$and $\delta : [0,T] \to \mathrm{I\!R}$ a non decreasing function such that

$$\mathrm{Var}_{\mathcal{D}}(z; s, t) \le \delta(t) - \delta(s) \quad \text{for all } (s,t) \in [0,T]^2 \text{ with } s \le t.$$

• For any fixed $t \in [0,T]$, we can define a limit displacement $\widetilde{u}(t)$ by considering a subsequence $q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t)$ weakly converging in $\mathcal{Q}$. Once again

$$q(t) = \big(\widetilde{u}(t), z(t)\big) \in \mathcal{S}(t), \quad \widetilde{u}(t) \in \mathrm{Argmin}\big\{\widetilde{\mathcal{E}}\big(t, \widetilde{u}, z(t)\big),\ \widetilde{u} \in \mathcal{F}\big\}.$$

But $\widetilde{\mathcal{E}}\big(t, \cdot, z(t)\big)$ is not strictly convex any more and we can not infer that the whole sequence $\big(\widetilde{u}_R^{\tau_n,h_n}(t)\big)_{n\in\mathbb{N}}$ converges weakly in $\mathcal{Q}$.

• We still have a upper energy estimate (which is a consequence of the discrete upper energy estimate and the a priori estimates), i.e.

$$\eta(t) + \mathrm{Var}_{\mathcal{D}}(q;0,t) \le \eta(t) + \delta(t) \le \eta(0) + \int_0^t \xi_*(s)\,\mathrm{d}s \quad \forall t \in [0,T].$$

• We have also a lower energy estimate (which is a consequence of the stability of the limit process), i.e.

$$\widetilde{\mathcal{E}}(t,q(t)) - \widetilde{\mathcal{E}}(0,q(0)) + \mathsf{Var}_{\mathcal{D}}(q;0,t) \geq \int_0^t \partial_t \widetilde{\mathcal{E}}(s,q(s))\,\mathsf{d}s.$$

Thus we obtain

$$\widetilde{\mathcal{E}}(0,q(0)) + \int_0^t \partial_t \widetilde{\mathcal{E}}\big(s,q(s)\big)\,\mathsf{d}s \leq \widetilde{\mathcal{E}}(t,q(t)) + \mathsf{Var}_{\mathcal{D}}(q;0,t)$$
$$\leq \eta(t) + \delta(t) \leq \widetilde{\mathcal{E}}(0,q(0)) + \int_0^t \xi_*(s)\,\mathsf{d}s$$

and it remains to compare ξ_* and $\partial_t \widetilde{\mathcal{E}}(\cdot,q)$. So we go back to the proof of lemma 3.

Let $t \in [0,T]$. Since $\widetilde{\mathcal{E}}(t,\cdot)$ is lower-semicontinuous we have

$$\widetilde{\mathcal{E}}(t,q(t)) \leq \liminf_{j\to+\infty} \widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t)) = \liminf_{j\to+\infty} \eta_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t) = \eta(t).$$

Moreover $q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t) \in \mathcal{S}_{h_{n_j^t}}(t_j)$ and with the same computations as in step 2 with $q^{h_{n_j^t}} \longrightarrow q(t)$ in $\mathcal{Q}$, we get

$$\eta(t) = \limsup_{j\to+\infty} \widetilde{\mathcal{E}}(t, q_R^{\tau_{n_j^t},h_{n_j^t}}(t)) \leq \widetilde{\mathcal{E}}(t,q(t)) + \mathcal{D}(z(t),z(t)) = \widetilde{\mathcal{E}}(t,q(t))$$

and thus $\lim_{j\to+\infty} \widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_{n_j^t},h_{n_j^t}}(t)) = \eta(t) = \widetilde{\mathcal{E}}(t,q(t))$.
Then, using property (ii) of proposition 2, we infer that

$$\lim_{j\to+\infty} \partial_t\widetilde{\mathcal{E}}(t, q_R^{\tau_{n_j^t},h_{n_j^t}}(t)) = \partial_t\widetilde{\mathcal{E}}(t,q(t))$$

but we can not conclude directly that $\xi_* = \mathrm{w}* - \lim_{L^\infty([0,T])} \partial_t\widetilde{\mathcal{E}}(\cdot, q_R^{\tau_n,h_n})$ is equal to $\partial_t\widetilde{\mathcal{E}}(\cdot,q)$ a.e. in $[0,T]$.

So we choose the subsequence $\big(q_R^{\tau_{n_j^t},h_{n_j^t}}(t)\big)_{j\in\mathbb{N}}$ in such a way that

$$\lim_{j\to+\infty} \partial_t\widetilde{\mathcal{E}}(t, q_R^{\tau_{n_j^t},h_{n_j^t}}(t)) = \xi_{\sup}(t)$$

with $\xi_{\sup} \in L^1\big([0,T]\big)$ given by

$$\forall s \in [0,T]:\ \ \xi_{\sup}(s) = \limsup_{n\to+\infty} \partial_t\widetilde{\mathcal{E}}(s, q_R^{\tau_n,h_n}(s)).$$

Then, $\xi_{\sup}(t) = \partial_t\widetilde{\mathcal{E}}(t, q(t))$ for all $t \in [0,T]$ and $\xi_*(t) \le \xi_{\sup}(t)$ for a.e. t in $[0,T]$. Thus

$$\widetilde{\mathcal{E}}(0,q(0)) + \int_0^t \xi_{\sup}(s)\,\mathsf{d}s = \widetilde{\mathcal{E}}(0,q(0)) + \int_0^t \partial_t\widetilde{\mathcal{E}}\big(s,q(s)\big)\,\mathsf{d}s \le \widetilde{\mathcal{E}}(t,q(t)) + \mathsf{Var}_{\mathcal{D}}(q;0,t)$$
$$\le \eta(t) + \delta(t) \le \widetilde{\mathcal{E}}(0,q(0)) + \int_0^t \xi_*(s)\,\mathsf{d}s \le \widetilde{\mathcal{E}}(0,q(0)) + \int_0^t \xi_{\sup}(s)\,\mathsf{d}s.$$

Hence all the inequalities are in fact equalities, i.e.

$$\delta(t) = \mathsf{Var}_{\mathcal{D}}(q; 0, t) \quad \forall t \in [0, T], \quad \xi_* = \xi_{\mathsf{sup}} = \partial_t \widetilde{\mathcal{E}}(\cdot, q) \quad \text{a.e. in } [0, T]$$

and

$$\partial_t \widetilde{\mathcal{E}}(\cdot, q_{\mathrm{R}}^{\tau_n, h_n}) \to \partial_t \widetilde{\mathcal{E}}(\cdot, q) \quad \text{strongly in} \quad L^1([0, T]).$$

Remark.

This construction of $\widetilde{u}(t)$ does not ensure the measurability of the mapping $t \mapsto \widetilde{u}(t)$ but we still have an uniform estimate of $\|\widetilde{u}(t)\|_{\mathcal{Q}}$ on $[0, T]$.

Theorem 2 (Convergence of the approximate solutions).
There exist a subsequence $\{(\tau_n, h_n)\}_{n\in\mathbb{N}}$ tending to $(0,0)$ and an energetic solution $q = (\widetilde{u}, z) : [0,T] \to \mathcal{Q}$ of (S) and (E) such that $q(0) = q_0$,

$$\|\widetilde{u}(t)\|_{\mathcal{F}} \text{ is uniformly bounded on } [0,T],$$
$$z \in L^\infty\big([0,T]; H^1(\Omega; Z)\big) \cap BV\big([0,T]; L^1(\Omega; Z)\big),$$

and, for all $t \in [0,T]$, the following convergences hold:

$$z_{\mathrm{R}}^{\tau_n,h_n}(t) \rightharpoonup z(t) \text{ weakly in } \mathcal{Z},$$
$$\widetilde{\mathcal{E}}(t, q_{\mathrm{R}}^{\tau_n,h_n}(t)) \to \widetilde{\mathcal{E}}(t, q(t)),$$
$$\mathrm{Var}_{\mathcal{D}}(q_{\mathrm{R}}^{\tau_n,h_n}; 0, t) \to \mathrm{Var}_{\mathcal{D}}(q; 0, t),$$
$$\partial_t \widetilde{\mathcal{E}}(\cdot, q_{\mathrm{R}}^{\tau_n,h_n}) \to \partial_t \widetilde{\mathcal{E}}(\cdot, q) \text{ strongly in } L^1([0,T]).$$

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