Thermally driven phase transformation in shape-memory alloys

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Outline

- Introduction
Outline

- Introduction
- Mathematical formulation
Outline

- Introduction
- Mathematical formulation
- A priori estimates
Outline

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- Mathematical formulation
- A priori estimates
- Existence result
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- A priori estimates
- Existence result
- Conclusion
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Previous works:

- A. Souza, E. Mamiya, N. Zouain’98,
- A. Mielke, F. Theil’04,
- A. Mainik, A. Mielke’05,
- G. Francfort, A. Mielke’06,
- A. Mielke’05,
Mathematical formulation

We consider a body with reference configuration $\Omega \subset \mathbb{R}^d$.

- $u : \Omega \to \mathbb{R}^d$: the phase transformation and deformations,
- $z : \Omega \to \mathbb{R}^{d \times d}_{\text{dev}} := \{ z \in \mathbb{R}^{d \times d}_{\text{sym}} : \text{tr}(z) = 0 \}$: the internal variable,

Notations

- $\mathbb{R}^{d \times d}_{\text{sym}} := \{ z \in \mathbb{R}^{d \times d} : z = z^T \}$
  - $a : b := \text{tr}(ab) = a_{ij}b_{ij}$: the scalar product,
  - $|a|^2 := a : a = a_{ij}a_{ij}$: the norm,
- $(\cdot)^T$: the transpose of the matrix $(\cdot)$,
- $\text{tr}(\cdot)$: the trace of the matrix $(\cdot)$. 
The potential energy has the following form:

$$
\mathcal{E}(t, u, z) := \int_{\Omega} W(e(u), z, \theta) + \frac{\sigma}{2} |\nabla z|^2 \, dx - \langle l(t), u \rangle,
$$

**Notations**

- **The stored energy density** is defined by
  
  $$
  W(e(u), z, \theta) := \frac{1}{2} (e(u) - z) : \mathbb{C}(\theta) : (e(u) - z) + h(z, \theta),
  $$

- **$e(u) := \nabla u + \nabla u^T$** : the linearized deformation satisfies the Korn's inequality, i.e.
  
  $$
  \int_{\Omega} |e(u)|^2 \, dx \geq c_{\text{Korn}} \|u\|_{W^{1,2}}^2, \ c_{\text{Korn}} > 0,
  $$

- **$\mathbb{C}(\theta) : \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}^{d \times d}_{\text{sym}}$** : the elasticity tensor (symmetric positive linear map) that depends on the temperature $\theta$ and is defined as follows:
  
  $$
  \mathbb{C}(\theta) : a := \lambda(\theta) \text{tr}(a) + 2\mu(\theta) a,
  $$

  * $\lambda(\theta), \mu(\theta)$: the Lamé coefficients depending on the temperature $\theta$.

- **$h(z, \theta) := c_1(\theta)|z|^2 + c_2(\theta)\sqrt{\delta^2 + |z|^2} + (|z|^2 - c_3(\theta))^3$**,

- **$\sigma > 0$** : measures some nonlocal interaction effect for $z$, 
- $l(t)$: the applied mechanical loading is defined as follows:

$$\langle l(t), u \rangle = \int_{\Omega} f_{\text{appl}}(t, x) \cdot u(x) \, dx + \int_{\partial\Omega} g_{\text{appl}}(t, x) \cdot u(x) \, d\gamma.$$ 

The dissipation potential is defined by

$$\mathcal{R}(\dot{z}) := \int_{\Omega} \rho |\dot{z}| \, dx = \rho \|\dot{z}\|_{L^1(\Omega)}, \ \rho > 0.$$ 

**Remark 1.**

- We do not solve an associated heat equation,
- This approximation used in engineering models:

**Assumptions:**

- the changes of the loading are slow,
- the body is small in at least one direction,

$\Rightarrow$ excess heat can be transported very fast to the surface.
We specify now the set of admissible deformations $\mathcal{F}$ by choosing a suitable Sobolev space $W^{1,2}(\Omega; \mathbb{R}^d)$ and by describing Dirichlet data at the part $\Gamma_{\text{Dir}}$ of the boundary $\partial \Omega$

$$\mathcal{F} := \{ u \in W^{1,2}(\Omega; \mathbb{R}^d) : u|_{\Gamma_{\text{Dir}}} = 0 \},$$

and the internal variable $z$ live in $\mathcal{Z} := L^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}).$
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Assumptions: Initial data $(u(0), z(0)) = (u_0, z_0) \in \mathcal{F} \times \mathcal{Z}$ are given.
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**Energetic formulation:**

A function $(u, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ is an energetic solution of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}$ if for all $t \in [0, T]$, the global stability condition ($S$) and the global energy conservation ($E$) are satisfied, i.e.

$$(S) \quad \forall (\bar{u}, \bar{z}) \in \mathcal{F} \times \mathcal{Z} : \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, \bar{u}, \bar{z}) + \mathcal{R}(\bar{z} - z(t)),$$

$$(E) \quad \mathcal{E}(t, u(t), z(t)) + \int_{0}^{t} \mathcal{R}(\dot{z}(s)) \, ds$$

$$= \mathcal{E}(0, u_0, z_0) + \int_{0}^{t} \partial_s \mathcal{E}(s, u(s), z(s)) \, ds.$$
A priori estimates

We clarify now the assumptions and we establish some preliminary results that we will use in the next section.

**Lemma 1.** Assume $c_j(\theta), j = 1, \ldots, 3,$ belong to $C^1(0, T)$. Then there exist $c_j^w, j = 1, 2,$ such that for all $j = 1, 2$,

$$|\partial^j_\theta W(e(u), z, \theta)| \leq c_1^w (W(e(u), z, \theta) + c_0^w).$$
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**Idea of the proof.** These estimates result from the application of Young’s inequality.
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$$|\partial^j \mathcal{W}(e(u), z, \theta)| \leq c_1^W \left( \mathcal{W}(e(u), z, \theta) + c_0^W \right).$$

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**Lemma 2.** Under the assumptions of Lemma 1, for all $\theta_1 \in [\theta_{\text{min}}, \theta_{\text{max}}]$, we have

$$\mathcal{W}(e(u), z, \theta_1) + c_0^W \leq \exp(c_1^W |\theta_1 - \theta|) \left( \mathcal{W}(e(u), z, \theta) + c_0^W \right).$$
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**Lemma 2.** Under the assumptions of Lemma 1, for all $\theta_1 \in [\theta_{\min}, \theta_{\max}]$, we have

$$W(e(u), z, \theta_1) + c_0^W \leq \exp(c_1^W |\theta_1 - \theta|)(W(e(u), z, \theta) + c_0^W).$$

*Idea of the proof.* Estimates obtained in the Lemma 1 for $j = 1$ and then the application of classical **Gronwall’s lemma** yield the desired result.
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**Idea of the proof.** Estimates obtained in the Lemma 1 for $j = 1$ and then the application of classical Gronwall’s lemma yield the desired result.

**Remark 1.** There exist $c > 0$ and $C > 0$ such that

$$W(e(u), z, \theta) \geq c|e(u)|^2 - C.$$
Existence result

**Assumptions:** The temperature $\theta_{\text{appl}}$ and the external loading $l$ with
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- $\theta_{\text{appl}} \in C^1([0, T]; L^\infty(\Omega; [\theta_{\text{min}}, \theta_{\text{max}}]))$, 

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Proposition 1. Under the above assumptions the following holds:
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1. $\theta_{\text{appl}} \in C^1([0, T]; L^\infty(\Omega; [\theta_{\text{min}}, \theta_{\text{max}}]))$,
2. $l \in C^1([0, T]; W^{1,2}(\Omega; \mathbb{R}^d)^*)$.

**Proposition 1.** Under the above assumptions the following holds:

1. If for some $(t, u, z) \in [0, T] \times \mathcal{F} \times \mathcal{Z}$ we have $\mathcal{E}(t, u, z) < +\infty$, then $\mathcal{E}(\cdot, u, z)$ are bounded in $C^1([0, T])$ and

   $$\partial_t \mathcal{E}(t, u, z) = \int_\Omega \partial_\theta \mathcal{W}(e(u), z, \theta_{\text{appl}}(t)) \dot{\theta}_{\text{appl}}(t) \, dx - \langle \dot{l}(t), u \rangle. \quad (1)$$
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2. There exist two constants $c_0^E > 0$ and $c_1^E > 0$ such that $\mathcal{E}(t, u, z) < +\infty$ implies

$$|\partial_t \mathcal{E}(t, u, z)| \leq c_1^E (\mathcal{E}(t, u, z) + c_0^E). \tag{2}$$
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   $$\partial_t E(t, u, z) = \int_\Omega \partial_\theta W(e(u), z, \theta_{\text{appl}}(t)) \dot{\theta}_{\text{appl}}(t) \, dx - \langle l(t), u \rangle. \quad (1)$$

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   $$|\partial_t E(t, u, z)| \leq c_1^E (E(t, u, z) + c_0^E). \quad (2)$$

3. For each strictly positive $\varepsilon$ and $E \in \mathbb{R}$ there exists $\delta$ such that $E(t_1, u, z) \leq E$ and $|t_1 - t_2| < \delta$ imply

   $$|\partial_t E(t_1, u, z) - \partial_t E(t_2, u, z)| \leq \varepsilon. \quad (3)$$
Proof:

1. The Korn’s inequality and Remark 1 ⇒ ∃c > 0, C > 0 such that

$$\mathcal{E}(t, u, z) \geq c_0 \|u\|_{W^{1,2}}^2 - C_0.$$  

For all $h \neq 0$ and $t + h \in [0, T]$ the mean-value theorem provides that

$$\frac{1}{h}(\mathcal{E}(t + h, u, z) - \mathcal{E}(t, u, z)) = \partial_t \mathcal{E}(t + sh, u, z)$$

$$= \int_\Omega \partial_\theta W(e(u), z, \theta_{appl}) \dot{\theta}_{appl}(t + sh) \, dx - \langle l(t + h) - l(t), u \rangle, s \in (0, 1).$$

The Lebesgue’s theorem ⇒ the differentiability of $\mathcal{E}(t, u, z)$ ($\partial_t \mathcal{E}(t, u, z)$).

$(\mathcal{E}(t, u, z) < +\infty ⇒ 0 < W(e(u), z, \theta_{appl}(t)) \in L^1(\Omega))$
Proof:

1. The Korn’s inequality and Remark 1 ⇒ \( \exists c > 0, C > 0 \) such that
   \[
   \mathcal{E}(t, u, z) \geq c_0 \|u\|^2_{W^{1,2}} - C_0.
   \]
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   \]
   \[
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2. Lemma 1 for \( j = 1 \) and Cauchy-Schwarz’s inequality ⇒ (2).
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2. Lemma 1 for \( j = 1 \) and Cauchy-Schwarz’s inequality ⇒ (2).

3. Observe now that
   \[
   \left| \partial_t \mathcal{E}(t_1, u, z) - \partial_t \mathcal{E}(t_2, u, z) \right|
   \leq \int_{\Omega} \left| \partial_\theta W(e(u), z, \theta_{appl}(t_1)) - \partial_\theta W(e(u), z, \theta_{appl}(t_2)) \right| \, dx \| \dot{\theta}_{appl} \|_{L^\infty}
   \]
   \[
   + \int_{\Omega} \left| \partial_\theta W(e(u), z, \theta_{appl}) \right| \, dx \| \dot{\theta}_{appl}(t_1) - \dot{\theta}_{appl}(t_2) \|_{L^\infty}
   \]
   \[
   + \| \dot{l}(t_1) - \dot{l}(t_2) \|_{(W^{1,2})'} \| u \|_{W^{1,2}}.
   \]

   The mean-value theorem, Lemma 1 and Lemma 2 ⇒ (3).
**Goal:** the *energetic formulation* $(S)$ and $(E)$ has at least one solution.
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**Remark 2.** Existence theory for $(S)$ and $(E)$, based on the *incremental minimization problem*, was developed in [Mainik/Mielke’05, Mielke’05, Francfort/Mielke’06]
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**Notations**

- $\text{Argmin}\{\varphi(u) : u \in \mathcal{H}\}$: the set of all minimizers of a functional $\varphi : \mathcal{H} \to \mathbb{R}_\infty$.
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We define the incremental problem as follows:

\[
(IP)_\Pi \begin{cases} 
\text{for } k = 1, \ldots, d \text{ find} \\
(u_k, z_k) \in \text{Argmin}\{\mathcal{E}(t_k, \tilde{u}, \tilde{z}) + \mathcal{R}(\tilde{z} - z_k) : (\tilde{u}, \tilde{z}) \in \mathcal{F} \times \mathcal{Z}\}.
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- \((IP)_\Pi\) has always solutions,
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\end{array} \right.$$ 

- $(IP)_\Pi$ has always solutions,
- we are able to define the piecewise constant interpolant $(u^\Pi, z^\Pi): [0, T] \to \mathcal{F} \times \mathcal{Z}$ with $(u^\Pi(t), z^\Pi(t)) = (u_j, z_j)$ for $t \in [t_{j-1}, t_j)$ for $j = 0, \ldots, N$. 

\[ \Rightarrow \phi : H \rightarrow \mathbb{R}_\infty \]
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- $(IP)_\Pi$ has always solutions,
- we are able to define the piecewise constant interpolant $(u^n, z^n) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ with $(u^n(t), z^n(t)) = (u_j, z_j)$ for $t \in [t_{j-1}, t_j)$ for $j = 0, \ldots, N$.

**Assumption:** $(u_0, z_0) \in \mathcal{F} \times \mathcal{Z}$ are given stable initial datum, i.e. $(u_0, z_0)$ satisfies the global stability condition $(S)$ at $t = 0$. 

$\varphi$ 
$H$ 
$R$ 
$\Pi$ 
$F \times Z$
Theorem 1. Assume that $\mathcal{E}$ and $\mathcal{R}$ satisfy the assumptions from above. Then, for each stable $(u(0), z(0)) = (u_0, z_0)$, there exists an energetic solution $(u, z) : [0, T] \rightarrow \mathcal{F} \times \mathcal{Z}$ such that

$$
\begin{align*}
    u & \in L^{\infty}([0, T]; \mathcal{W}^{1, 2}(\Omega; \mathbb{R}^d)), \\
    z & \in BV([0, T]; L^1(\Omega; \mathbb{R}^{d \times d}_{\text{dev}})).
\end{align*}
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    z \in BV([0, T]; L^1(\Omega; \mathbb{R}^{d \times d})).
\]

Moreover, let $\Pi_k = \{0 = t_0^k < t_1^k < \ldots < t_N^k = T\}$, $k \in \mathbb{N}$, be a sequence of partitions with fineness $\Delta(\Pi_k) := \max\{t_j^k - t_{j-1}^k : j = 1, \ldots, N_k\}$ tends to zero and $(u^{\Pi_k}, z^{\Pi_k}) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ be piecewise constant interpolants of the solution of the incremental problem $(IP)_{\Pi_k}$, then there exists a subsequence $(\bar{u}_n, \bar{z}_n) := (u_n^{\Pi_k}, z_n^{\Pi_k})$ such that for all $t \in [0, T]$ the following holds

\[
    \bar{z}_n(t) \to z(t) \text{ in } \mathcal{Z}, \\
    \mathcal{E}(t, \bar{u}_n(t), \bar{z}_n(t)) \to \mathcal{E}(t, u(t), z(t)), \\
    \int_0^t \mathcal{R}(\dot{\bar{z}}_n(s)) \, ds \to \int_0^t \mathcal{R}(\dot{z}(s)) \, ds,
\]

there exists a subsequence $(N_i^t)_{i \in \mathbb{N}}$ such that

\[
    \bar{u}_{N_i^t}(t) \to u(t) \text{ in } \mathcal{F} \text{ for } l \to 0.
\]
Conclusion

- Uniqueness result,
- Existence result for the same problem with an associated heat equation.
Assumptions on $\theta_{\text{appl}}$ and $I$ imply that

$$
\|\dot{l}(t_1) - \dot{l}(t_2)\|_{W^{1,2}} + \|\dot{\theta}_{\text{appl}}(t_1) - \dot{\theta}_{\text{appl}}(t_2)\|_{L^\infty} \leq \omega(|t_1 - t_2|),
$$

where $\omega : [0, +\infty) \to [0, +\infty)$ is a modulus of continuity with $\omega(0) = 0$. 