# On the convergence for kinetic variational inequality to quasi-static variational inequality with application to elastic-plastic systems with hardening

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joint work with Alexander Mielke

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*Mathematics for key technologies*

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- Exterior loading $l(t) := \begin{pmatrix} f_u(t) \\ f_z(t) \end{pmatrix}$
- Dissipation functional $\mathcal{R} : H \to [0, \infty]$
  - convex, lower-semicontinuous, homogeneous of degree 1
  - the subdifferential is defined by
    $\partial\mathcal{R}(v) = \{\sigma \in H^* \,|\, \forall w \in H : \mathcal{R}(w) \geq \mathcal{R}(v) + \langle \sigma, w - v \rangle\}$
  - $\mathcal{R}(\dot{q}) := \widetilde{\mathcal{R}}(\dot{z})$

The **variational inequality**

$$\forall v \in H: \ \langle M\ddot{q} + Aq - l(t), v - \dot{q}\rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \geq 0. \tag{2}$$

The **quasi-static** (Q.S.) system ($M = 0$ in (1))

$$A\bar{q} + \partial\mathcal{R}(\dot{\bar{q}}) \ni l(t). \tag{3}$$

The **variational inequality**

$$\forall \bar{v} \in H: \ \langle A\bar{q} - l, \bar{v} - \dot{\bar{q}}\rangle + \mathcal{R}(\bar{v}) - \mathcal{R}(\dot{\bar{q}}) \geq 0. \tag{4}$$

**Energetic solution:** $q = (u, z) : [0, T] \to V$

$$(S) \quad \forall \widehat{q} \in H: \ \mathcal{E}(t, \bar{q}(t), \dot{\bar{q}}(t)) \leq \mathcal{E}(t, \widehat{q}, \dot{\widehat{q}}) + \mathcal{R}(\widehat{q} - \bar{q}(t)),$$

$$(E) \quad \mathcal{E}(t, \bar{q}, \dot{\bar{q}}) + \int_0^t \mathcal{R}(\dot{\bar{q}}(s))\,\mathrm{d}s = \mathcal{E}(0, \bar{q}_0, \dot{\bar{q}}_0) - \langle \dot{l}(s), \bar{q}\rangle.$$

The **governing dynamic** system

$$\begin{cases} m^{1/2}\dot{u} - v = 0, \\ m^{1/2}\dot{v} + a_{11}u + a_{12}z = f_u(t), \\ \dot{z} - \partial\mathcal{R}^*(-a_{21}u - a_{22}z) \in f_z(t), \end{cases} \tag{5}$$

with initial conditions

$$(\dot{u}(0), q(0)) = (\dot{u}(0), u(0), z(0)) = (\dot{u}_0, u_0, z_0) = (\dot{u}_0, q_0) \in H_1 \times V. \tag{6}$$

The **quasi-static** system

$$\begin{cases} a_{11}\bar{u} + a_{12}\bar{z} = f_u(t), \\ \dot{\bar{z}} - \partial\mathcal{R}^*(-a_{21}\bar{u} - a_{22}\bar{z}) \in f_z(t), \end{cases} \tag{7}$$

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$$\bar{q}(0) = (\bar{u}(0), \bar{z}(0)) = (\bar{u}_0, \bar{z}_0) = \bar{q}_0 \in V. \tag{8}$$

**Notations**:

▷ $\mathcal{R}^*(\cdot)$ is the **Legendre transform** of $\widetilde{\mathcal{R}}(\cdot)$

▷ $V = V_1 \times V_2$

**Existence** and **uniqueness** of solution to

$$\forall t \in [0, T]: \; w(t) \in \mathcal{D}(\mathbb{A}) := \{w \in Y \mid \mathbb{A}w \neq \emptyset\}, \tag{9a}$$

$$\dot{w}(t) + \mathbb{A}w(t) \ni g(t), \; t > 0, \tag{9b}$$

$$w(0) = w_0. \tag{9c}$$

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**Proposition 1.** *Assume that $\mathbb{A}$ is a maximal monotone operator in the Hilbert space $Y$, $g \in W^{1,1}([0, T]; Y)$ and $w_0 \in \mathcal{D}(\mathbb{A})$. Then there exists a unique solution $w \in W^{1,\infty}([0, T]; Y)$ of* (9).

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**Notations:**

▷ $w := (\widetilde{a}^{1/2}u, v, a_{22}^{-1/2}e)$ with $\widetilde{a} := (a_{11} - a_{12}a_{22}^{-1}a_{21})$, $e := f_z - (a_{21}u + a_{22}z)$

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- ▷ $\mathbb{A} := \begin{pmatrix} 0 & -\widetilde{a}^{1/2}m^{-1/2} & 0 \\ m^{-1/2}\widetilde{a}^{1/2} & 0 & -m^{-1/2}a_{12}a_{22}^{-1/2} \\ 0 & a_{22}^{-1/2}a_{21}m^{-1/2} & a_{22}^{1/2}\partial\mathcal{R}^*(a_{22}^{1/2}(\cdot)) \end{pmatrix}$

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- $g := \begin{pmatrix} 0 \\ m^{-1/2}(f_u(t) + a_{12}a_{22}^{-1}f_z(t)) \\ a_{22}^{-1}\dot{f}_z(t) \end{pmatrix}$

**Corollary 1.** *Assume that* $l \in W^{1,1}([0,T];H)$ *and* $(\dot{u}_0, q_0) \in H_1 \times V$ *satisfy* $f_z(0) \in \partial\mathcal{R}(0) + a_{21}u_0 + a_{22}z_0$. *Then there exists a unique solution* $q \in W^{1,\infty}([0,T];V)$ *that solves* (5) *and* (6). *This solution additionally satisfies* $Mq \in W^{2,\infty}([0,T];H)$.

The **energy** associated to (3) (Q.S.) is given by

$$\mathcal{E}(t,\bar{q}) = \frac{1}{2}\langle A\bar{q}, \bar{q}\rangle - \langle l(t), \bar{q}\rangle. \tag{10}$$

**Lemma 1.** *Assume that* $l \in C^1([0,T];V^*)$ *and* $\bar{q}_0 \in V$ *satisfying* $\partial\mathcal{R}(0) \ni -A\bar{q}_0 + l(0)$. *Then, the variational inequality* (4) *and hence also* (7)-(8) *have a unique solution* $\bar{q} \in C^{\mathrm{Lip}}([0,T];V)$.

*Idea of the proof.*

- $\mathcal{E}(t,\cdot) \in C^3(V)$ is uniformly convex
- $\mathcal{R}$ is convex, lower-semicontinuous, homogeneous of degree 1
- Use time dicretization and solve a vartiational inequality in each time step

**Notations:**

- $\|q\| := \sqrt{\langle Aq, q\rangle}$, $\|l\|_* := \sqrt{\langle l, A^{-1}l\rangle}$, $|q|_M := \|M^{1/2}q\|_H$
- The bilinear form $B : V \to H \to V^* \to \mathbb{R}$:

$$B[q, \dot q, l] := |\dot q|_M^2 + \|q - A^{-1}l\|^2 + \|l\|_*^2$$

  which verifies that

  - $B[q(t), \dot q(t), l(t)] = 2\mathcal{E}(t, q(t), \dot q(t)) + 2\|l(t)\|_*^2$
  - $\frac{1}{g^2}\left(\|q\|^2 + |\dot q|_M^2 + \|l\|_*^2\right) \leq B[q, \dot q, l] \leq g^2\left(\|q\|^2 + |\dot q|_M^2 + \|l\|_*^2\right)$
    with $g := \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden number.

Using $\langle \sigma, \dot q\rangle \geq 0$ for all $\sigma \in \partial\mathcal{R}(\dot q)$

$$\frac{d}{dt}\mathcal{E}(t, q(t), \dot q(t)) \leq -\langle \dot l(t), q(t)\rangle.$$

Then, we deduce that

$$\frac{d}{dt}B[q(t), \dot q(t), l(t)] \leq \|\dot l(t)\|_* 2g\sqrt{2}\sqrt{B[q(t), \dot q(t), l(t)]}.$$

Dividing by $\sqrt{B[q(t),\dot q(t),l(t)]}$ and integrating both sides, for $0 \le s \le t \le T$:

$$\sqrt{B[q(t),\dot q(t),l(t)]} \le \sqrt{B[q(s),\dot q(s),l(s)]} + g\sqrt{2}\int_s^t \|\dot l(\tau)\|_*\,\mathrm{d}\tau,$$

which implies that

$$\left(\|q(t)\|^2+|\dot q(t)|_M^2\right)^{1/2} \le g^2\left(\|q(0)\|^2+|\dot q(0)|_M^2+\|l(0)\|_*^2\right)^{1/2} + g^2\sqrt{2}\int_0^t \|\dot l(\tau)\|_*\mathrm{d}\tau.$$

Similarly, using (13) we obtain *a priori* bounds on the energy, namely

$$\mathcal{E}(t,q(t),\dot q(t)) \le \left(\sqrt{\mathcal{E}(0,q(0),\dot q(0)) + \|l(0)\|_*^2} + g\int_0^t \|\dot l(\tau)\|_*\mathrm{d}\tau\right)^2 - \|l(t)\|_*^2.$$

**Proposition 2.** *Let $l_1, l_2 \in W^{1,1}([0,T];V^*)$ and $q_1$ and $q_2$ be solutions of (1) with right-hand sides $l_1$ and $l_2$ respectively, then $w = q_1 - q_2$ satisfies for all $t \in [0,T]$ the following estimate*

$$\begin{aligned} B[w(t),\dot w(t),l_1(t)-l_2(t)]^{1/2} \le{} & B[w(0),\dot w(0),l_1(0)-l_2(0)]^{1/2} \\ & + g\sqrt{2}\int_0^t \|\dot l_1(\tau)-\dot l_2(\tau)\|_*\mathrm{d}\tau. \end{aligned} \tag{11}$$

For arbitrary functions $y \in L^\infty([0, T]; Y)$, $h > 0$, $t \in [0, T-h]$

$$\delta_h y(t) := \frac{1}{h}\big(y(t+h) - y(t)\big).$$

For all $p \in [1, \infty]$ and $y \in W^{1,p}([0, T]; Y)$ we have

$$\|\delta_h y\|_{L^p([0,T-h];Y)} \leq \|\dot{y}\|_{L^p([0,T];Y)}.$$

Taking $q_1(t) = q(t+h)$ and $q_2(t) = q(t)$ in Proposition 1, for all $0 \leq s \leq t$:

$$\begin{aligned} B[\delta_h q(t), \delta_h \dot{q}(t), \delta_h l(t)]^{1/2} \leq & B[\delta_h q(s), \delta_h \dot{q}(s), \delta_h l(s)] \\ & + g\sqrt{2}\int_s^t \|\delta_h \dot{l}(\tau)\|_* \mathrm{d}\tau. \end{aligned} \tag{12}$$

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- **In general** $q(0) = q_0 \in V$ and $\dot{u}(0) = \dot{u}_0 \in H_1$ do not guarantee that

$$\limsup_{h \searrow 0}\big(\|\delta_h q(0)\| + |\delta_h \dot{q}(0)|_M\big) < \infty.$$

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- ▷ **The additional assumptions** $\dot{u}_0 \in V_1$, $l(0) \in \partial R(0) + Aq(0)$ do not help.

**Assumption:**

$$\exists\, \rho > 0\ \exists\, l \in W^{2,1}([-\rho,0];V^*)\ \exists\, q=(u,z) \in W^{1,\infty}([-\rho,0];V): \\ u \in W^{2,\infty}([-\rho,0];H_1),\ \text{(1) is satisfied},\ q(0)=q_0,\, \dot{u}(0)=\dot{u}_0. \tag{13}$$

$\Rightarrow$ The stability condition $l(0) \in \partial R(0) + Aq_0$ holds

$\Rightarrow$ The following limits for $h \searrow 0$ exist:
$\delta_h q(-h) \to \dot{Q}$ in $V$, $\quad \delta_h \dot{u}(-h) \to \ddot{U}$ in $H_1$, $\quad \delta_h l(-h) \to \dot{L}$ in $V^*$

**Remark 1.** There are two cases where this condition can be easily satisfied. The first one will be essential in the next section.

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▷ if $\dot{u}_0 = 0$, then we may choose $q(t) = q_0$ for all $t \in [-\rho, 0]$ and let $l(t) = Aq_0$. Then $\dot{Q} = 0$, $\ddot{U} = 0$, $\dot{L} = 0$.

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**Remark 1.** There are two cases where this condition can be easily satisfied. The first one will be essential in the next section.

- if $\dot u_0 = 0$, then we may choose $q(t) = q_0$ for all $t \in [-\rho,0]$ and let $l(t) = Aq_0$. Then $\dot Q = 0$, $\ddot U = 0$, $\dot L = 0$.
- If $\dot u_0 \in V_1$ and if the block structure is present, we may choose $q(t) = q_0 + t(\dot u_0, 0)^T$ and let $l(t) = l(0) + tA(\dot u_0,0)^T$. Then $\dot Q = (\dot u_0, 0)^T$, $\ddot U = 0$, and $\dot L = A(\dot u_0,0)^T$.

**Theorem 1.** *Let* $l \in W^{2,1}([0,T];V^*)$ *and* $(q_0, \dot{u}_0) \in V \times V_1$ *be given such that condition* (13) *holds. Then, the unique solution* $q$ *of* (5) *and* (6) *satisfies the a priori estimate*

$$\begin{aligned} & B[\dot{q}(t), (\ddot{u}(t), 0)^T, \dot{l}(t)]^{1/2} \\ & \leq B[\dot{Q}, (\ddot{U}, 0)^T, \dot{L}]^{1/2} + g\sqrt{2}\Big( \|\dot{L} - \dot{l}(0)\|_* + \int_0^t \|\ddot{l}(\tau)\|_* \mathrm{d}\tau \Big). \end{aligned} \tag{14}$$

*Idea of the proof.* Concatenate the artifical solution $q \in W^{1,\infty}([-\rho, 0]; V)$ and the given solution $q \in W^{1,\infty}([0,T]; V)$ as well as the loadings.

**Goal:** Compare the solution $q$ of the kinetic equation

$$M\ddot{q} + \partial\mathcal{R}(\dot{q}) + Aq \ni l(t), \quad q(0) = q_0, \quad \dot{u}(0) = \dot{u}_0,$$

with the corresponding solution $\bar{q}$ of the quasistatic equation

$$\partial\mathcal{R}(\dot{\bar{q}}) + A\bar{q} \ni l(t), \quad \bar{q}(0) = q_0. \tag{15}$$

**Proposition 3.** Assume that $l \in W^{1,1}([0,T]; V^*)$, $(M\dot{q}_0, q_0) \in H \times V$, $\bar{q}_0 \in V$. Then we have

$$\begin{aligned} |\dot{q}(t)|_M^2 + \|q(t) - \bar{q}(t)\|^2 \le |\dot{q}_0|_M^2 + \|q_0 - \bar{q}_0\|^2 \\ +2 \operatorname*{ess\,sup}_{s\in[0,T]} |\ddot{q}(s)|_M \int_0^t |\dot{\bar{q}}(s)|_M \mathrm{d}s. \end{aligned} \tag{16}$$

*Idea of the proof.*

- Add variational inequalities ($v = \dot{\bar{q}}$ and $\bar{v} = \dot{q}$ resp. in (2) and (4))
- Use Cauchy-Schwarz's inequality.

**Very slow dynamics:** Rescale the time as $\widetilde{t} = \varepsilon t$ (loading given $l(t) = \widetilde{l}(\varepsilon t)$)
We consider now the problem (replace $M$ by $\varepsilon^2 \widetilde{M}$)

$$\varepsilon^2 \widetilde{M} \ddot{q}_\varepsilon + A q_\varepsilon + \partial \mathcal{R}(\dot{q}_\varepsilon) \ni l(t), \quad q(0) = q_0, \quad M \dot{q}_\varepsilon(0) = M \dot{q}_0. \tag{17}$$

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▷ Existence and uniqueness follows from Corrollary 1.

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- Existence and uniqueness follows from Corrollary 1.
- *A priori* estimates are also valid with

$$|q|_{\widetilde{M}} := \|\widetilde{M}^{1/2} q\|_H, |q|_M = |q|_{\varepsilon^2 \widetilde{M}} = \varepsilon |q|_{\widetilde{M}}.$$

**Theorem 2.** Let the above assumptions on $\widetilde{M}$, $A$, $\mathcal{R}$ hold. Assume $l \in W^{2,1}([0,T],V^*)$ and $\bar{q}_0 \in V$ with $l(0) \in \partial\mathcal{R}(0) + A\bar{q}_0$. Let $\bar{q}$ be the unique solution (15), and $q$ the unique solution of (17) for arbitrary $q_0 \in V$ and $\dot{u}_0 \in H_1$ satisfying $l(0) \in \partial\mathcal{R}(0) + Aq_0$. Then

$$\left(\varepsilon^2|\dot{q}_\varepsilon(t)|^2_{\widetilde{M}} + \|q_\varepsilon(t)-\bar{q}(t)\|^2\right)^{1/2} \le \left(\varepsilon^2|\dot{q}_0|^2_{\widetilde{M}} + \|q_0-\bar{q}_0\|^2\right)^{1/2} + \sqrt{\varepsilon C[l](t)},$$

with $C[l](t):=g^2\sqrt{2}\mu\int_0^t \|\dot{l}(s)\|_*\mathrm{d}s\left(\|\dot{l}(0)\|_* + \int_0^t \|\ddot{l}(s)\|_*\mathrm{d}s\right)$, $\mu := \sup_{\|v\|=1} |v|_{\widetilde{M}}$.

*Idea of the proof.*

- ▷ Introduce special dynamic solution $\widehat{q}_\varepsilon$ defined by $\widehat{q}_\varepsilon(0) = \bar{q}_0$, $\dot{\widehat{u}}_\varepsilon(0) = 0$.
- ▷ Use Proposition 3. to obtain the difference between $q_\varepsilon$ and $\widehat{q}_\varepsilon$.
- ▷ Use Proposition 2. to obtain the difference between $\bar{q}$ and $\widehat{q}_\varepsilon$.

**Notations and assumptions:**

- $\mathbb{C} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $B \in \mathbb{R}^{m}_{\mathrm{sym}}$ are a symmetric positive definite maps
- the dissipation $\mathcal{R}(z) := \int_\Omega \varphi(x, z(x))\,\mathrm{d}x$ with $\varphi \in C^0(\bar{\Omega} \times \mathbb{R}^m)$ such that $R_1|v| \le \varphi(x,v) \le R_2|v|$ for all $(x,v) \in \bar{\Omega} \times \mathbb{R}^m$ with $\varphi(x,\cdot) : \mathbb{R}^m \to [0,\infty)$ is 1-homogeneous and convex.

**Very slow time** ($\widetilde{t} = \varepsilon t$): the elastic-plastic system with hardening can be written as

$$\begin{cases} \varepsilon^2 m\ddot{u}_\varepsilon - \mathrm{div}(\mathbb{C}{:}(\mathcal{E}(u_\varepsilon) - z_\varepsilon)) = l_{\mathrm{ext}}(t), \\ -\mathbb{C}{:}(\mathcal{E}(u_\varepsilon) - z_\varepsilon) + Bz_\varepsilon + \partial\mathcal{R}(\dot{z}_\varepsilon) \ni 0, \end{cases} \tag{18}$$

with the Dirichlet boundary conditions and initial conditions $(u_0, \dot{u}_0, z_0)$.

- For quasi-static problem take $\varepsilon = 0$.
- Existence and uniqueness (see Showalter and Shi '99).
- The Theorem 2. leads to ($q{:=}(u,z)$)

$$\left(\varepsilon^2|\dot{u}_\varepsilon(t)|^2_{\widetilde{M}} + \|q_\varepsilon(t) - \bar{q}(t)\|^2\right)^{1/2} \le \left(\varepsilon^2|\dot{u}_0|^2_{\widetilde{M}} + \|q_0 - \bar{q}_0\|^2\right)^{1/2} + \sqrt{\varepsilon C[l](t)}.$$