On the existence for viscoelastodynamic problems with unilateral boundary conditions

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This note deals with a damped wave equation and the evolution of a Kelvin–Voigt viscoelastic material, both problems being subject to unilateral boundary conditions. The functional properties of all the traces are precisely identified through Fourier analysis, which implies the existence of a solution satisfying almost everywhere the unilateral boundary conditions.

1 The damped wave equation with unilateral boundary conditions

We consider a damped wave equation taking place in a half-space, with an obstacle at the boundary. Let $u(x,t)$ denote the displacement at time t of the material point of spatial coordinate $x=(x_1,x')\in[0,\infty)\times\mathbb{R}^{d-1}$ at rest with $d\geq 2$. We will agree that if we write a function of space and time as a function of three variables, then the first variable is the normal space variable x_1, the second variable is the tangential space variable x', and the last variable is time. Let $f(x,t)$ be a density of forces, depending on space and time. The mathematical problem is formulated as follows:

$$u_{tt}(x,t)-\Delta u(x,t)-\Delta u_t(x,t)=f(x,t),\quad x\in(-\infty,0]\times\mathbb{R}^{d-1},\quad t>0, \tag{1}$$

with Signorini boundary conditions

$$u(0,\cdot,\cdot)\geq 0,\quad u_{x_1}(0,\cdot,\cdot){+}u_{x_1t}(0,\cdot,\cdot)\geq 0,\quad u(0,\cdot,\cdot)(u_{x_1}(0,\cdot,\cdot){+}u_{x_1t}(0,\cdot,\cdot))=0, \tag{2}$$

and Cauchy initial data

$$u(\cdot,0)=u_0\quad\text{and}\quad u_t(\cdot,0)=u_1, \tag{3}$$

where $(\cdot)_t\overset{\text{def}}{=}\frac{\partial}{\partial t}(\cdot)$ and $(\cdot)_{x_1}\overset{\text{def}}{=}\frac{\partial}{\partial x_1}(\cdot)$. The existence result for (1)–(3) is proved by the penalty method. More precisely, let u^ϵ be a solution of (1) and (3) with the rigid constraint (2) which is now replaced by a very stiff response, i.e.

$$u^\epsilon_{x_1}(0,\cdot,\cdot)+u^\epsilon_{x_1t}(0,\cdot,\cdot)=\tfrac{1}{\epsilon}(u^\epsilon(0,\cdot,\cdot))^-, \tag{4}$$

where $(u^\epsilon(0,\cdot,\cdot))^-\overset{\text{def}}{=}\min(u^\epsilon(0,\cdot,\cdot),0)$. A priori estimates on the penalized problem allow to pass to the limit with respect to the penalty parameter ϵ and to deduce the existence of a weak solution to (1)–(3). The reader can find the detailed proof in [2, 3].

The trace space is characterized by using Fourier analysis. To do so, we introduce $v^\epsilon\overset{\text{def}}{=}\mathrm{e}^{-\nu t}(u^\epsilon{-}\bar u)$, $\nu>0$, which is a solution of

$$(\nu{+}\partial_t)^2v^\epsilon(x,t)-(1{+}\nu{+}\partial_t)\Delta v^\epsilon(x,t)=0,\quad x\in(-\infty,0]\times\mathbb{R}^{d-1},\quad t>0, \tag{5a}$$

$$(1{+}\nu{+}\partial_t)v^\epsilon_{x_1}(0,\cdot,\cdot)=-\mathrm{e}^{-\nu t}(\bar u_{x_1}(0,\cdot,\cdot){+}\bar u_{x_1t}(0,\cdot,\cdot))-\tfrac{1}{\epsilon}(v^\epsilon(0,\cdot,\cdot){+}\mathrm{e}^{-\nu t}\bar u(0,\cdot,\cdot))^-, \tag{5b}$$

$$v^\epsilon(\cdot,0)=0\quad\text{and}\quad v^\epsilon_t(\cdot,0)=0. \tag{5c}$$

Here $\bar u$ is the solution of (1) with Dirichlet data and initial data (3). We define λ_1 as the inverse Fourier transform of the causal determination of $(1{+}\nu{+}i\omega)\sqrt{|\xi|^2{+}\frac{(\nu+i\omega)^2}{1+\nu+i\omega}}$ where ξ and ω denote, respectively, the dual variables to x' and t. Then (5) can be written as follows

$$\lambda_1*v^\epsilon(0,\cdot,\cdot)=-\mathrm{e}^{-\nu t}(\bar u_{x_1}(0,\cdot,\cdot){+}\bar u_{x_1t}(0,\cdot,\cdot))+\tfrac{1}{\epsilon}(v^\epsilon(0,\cdot,\cdot){+}\mathrm{e}^{-\nu t}\bar u(0,\cdot,\cdot))^-, \tag{6}$$

where $v^\epsilon(0,\cdot,\cdot)$ vanishes for all $t\leq 0$. Recall that $u(0,x',t)$ belongs to the Sobolev space $\mathrm{H}^{a,b}_{\mathrm{loc}}(\mathbb{R}^{d-1}{\times}[0,\infty))$, $(a,b)\in\mathbb{R}^2$, iff $|\xi|^a\widehat u(0,\xi,\omega)$ and $|\omega|^b\widehat u(0,\xi,\omega)$ belong to $\mathrm{L}^2(\mathbb{R}^d)$. Here $\widehat u(0,\xi,\omega)$ denotes the Fourier transform of $u(0,x',t)$. If u_0

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belongs to $\mathrm{H}^{5/2}((-\infty,0]\times\mathbb{R}^{d-1})$, u_1 belongs to $\mathrm{H}^1((-\infty,0]\times\mathbb{R}^{d-1})$, and f belongs to $\mathrm{L}^2_{\text{loc}}([0,\infty);\mathrm{L}^2((-\infty,0]\times\mathbb{R}^{d-1}))$, then the trace $\bar{u}_{x_1}(0,\cdot,\cdot)+\bar{u}_{x_1t}(0,\cdot,\cdot)$ increases at most polynomially with respect to time in $\mathrm{L}^2([0,\infty);\mathrm{L}^2(\mathbb{R}^{d-1}))$. Hence multiplying (6) by $(1+\nu)v^\epsilon(0,\cdot,\cdot)+v^\epsilon_t(0,\cdot,\cdot)$ and estimating the pseudodifferential term in the Fourier variable, we obtain the following lemma:

Lemma 1.1 *Let u^ϵ be the solution of* (1), (3) *and* (4)*. Then we may extract a subsequence, still denoted by u^ϵ such that $u^\epsilon(0,\cdot,\cdot)$ converges weakly to $u(0,\cdot,\cdot)$ in $\mathrm{H}^{1/2,5/4}_{\text{loc}}((-\infty,0]\times\mathbb{R}^{d-1})$. Moreover u is a strong solution of* (1)–(3)*.*

2 The evolution of a Kelvin–Voigt material with unilateral boundary conditions

We treat now the evolution of a Kelvin–Voigt material occupying a three dimensional half-space, satisfying unilateral boundary conditions and Cauchy data at $t=0$. We make the assumptions of small deformations and we consider here a homogeneous and isotropic material. Let $\varepsilon_{ij}(u)\stackrel{\text{def}}{=}\frac{1}{2}(u_{j,x_i}+u_{i,x_j})$ be the strain tensor, and let there be given two Hooke tensors defined by using Lamé constants λ^n and μ^n, $a^n_{ijkl}\stackrel{\text{def}}{=}\lambda^n\delta_{ij}\delta_{kl}+2\mu^n\delta_{ik}\delta_{jl}$, $n=0,1$, where δ is the Kronecker symbol. Using the summation convention on repeated indices, we define the two stress tensors σ^n_{ij} corresponding, respectively, to the elastic and the viscous parts of the stress $\sigma^n_{ij}(u)\stackrel{\text{def}}{=}a^n_{ijkl}\varepsilon_{kl}(u)$. The displacement field u satisfies the system

$$\rho u_{i,tt}(x,t)-\sigma^0_{ij,x_j}(u(x,t))-\sigma^1_{ij,x_j}(u_t(x,t))=f_i(x,t),\quad i=1,2,3,\quad x\in[0,\infty)\times\mathbb{R}^2,\quad t>0. \tag{7}$$

The boundary conditions on $\{0\}\times\mathbb{R}^2\times[0,\infty)$ are

$$u_1\le 0,\quad \sigma^0_{11}(u)+\sigma^1_{11}(u_t)\le 0,\quad u_1(\sigma^0_{11}(u)+\sigma^1_{11}(u_t))=0, \tag{8a}$$

$$\sigma^0_{12}(u)+\sigma^1_{12}(u_t)=0\quad\text{and}\quad \sigma^0_{13}(u)+\sigma^1_{13}(u_t)=0, \tag{8b}$$

and the initial data are given by

$$u(\cdot,0)=v_0\quad\text{and}\quad u_t(\cdot,0)=v_1. \tag{9}$$

As for the damped wave equation with unilateral boundary conditions, the existence result is obtained by the penalty method. We introduce the penalized problem, namely, let u^ϵ be a solution of (7), (8b), (9) and

$$\sigma^0_{11}(u)+\sigma^1_{11}(u_t)=-\tfrac{1}{\epsilon}(u^\epsilon_1)^+\quad\text{on}\quad\{0\}\times\mathbb{R}^2\times[0,\infty), \tag{10}$$

where $(u^\epsilon_1)^+\stackrel{\text{def}}{=}\max(u^\epsilon_1,0)$. A priori estimates on the penalized problem and care relative due to the unboundedness of $[0,\infty)\times\mathbb{R}^2$ allow to pass to the limit in the penalized variational formulation and to infer the existence of a solution to (7)–(9), for the detailed proof the reader is referred to [3]. Note that the Korn's inequality plays a crucial role to obtain these a priori estimates. A much more general and complicated case is treated in [1], since it allows for contact, a given friction at the boundary, a nonlinear constitutive law for viscoelasticity, and a general geometry.

We characterize now the trace spaces. Let $\bar{u}$ be the solution of (7), (8b), (9) with Dirichlet boundary data at $x_1=0$. If v_0 belongs to $(\mathrm{H}^{5/2}([0,\infty)\times\mathbb{R}^2))^3$, v_1 belongs to $(\mathrm{H}^1([0,\infty)\times\mathbb{R}^2))^3$, and f belongs to $(\mathrm{L}^2_{\text{loc}}([0,\infty);\mathrm{L}^2([0,\infty)\times\mathbb{R}^2)))^3$, then the trace $-(a^0_{11kl}\varepsilon_{kl}(\bar{u})+a^1_{11kl}\varepsilon_{kl}(\bar{u}_t))$ on $\{0\}\times\mathbb{R}^2\times[0,\infty)$ increases exponentially with respect to time in $\mathrm{L}^2_{\text{loc}}(\{0\}\times\mathbb{R}^2\times[0,\infty))$ and not polynomially as in the case of the damped wave equation with Dirichlet boundary. The trace spaces are determined by using analogous techniques already developed for the damped wave equation, but here a Fourier transform in the tangential variables (x_2,x_3,t) and a Laplace transform in x_1 lead to the following lemma:

Lemma 2.1 *Let $u^\epsilon=(u^\epsilon_1,u^\epsilon_2,u^\epsilon_3)^\top$ be the solution of* (7), (8a), (9) *and* (10)*. Then we may extract a subsequence, still denoted by u^ϵ_1, such that $u^\epsilon_1(0,\cdot,\cdot)$ converges weakly to $u_1(0,\cdot,\cdot)$ in $\mathrm{H}^{1/2,5/4}_{\text{loc}}(\mathbb{R}^2\times[0,\infty))$. Moreover u is a strong solution of* (7)–(9)*.*

Remark 2.2 Nothing is known about the uniqueness for the damped wave equation as well as for the evolution of Kelvin–Voigt material in the case where both problems are subjected to unilateral boundary conditions.

Acknowledgements A.P. was supported by the Deutsche Forschungsgemeinschaft through the project C18 “Analysis and numerics of multidimensional models for elastic phase transformation in a shape-memory alloys” of the Research Center MATHEON.

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