On the convergence for kinetic variational inequality to quasi-static variational inequality with application to elastic-plastic systems with hardening

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In this note, a priori estimates for the kinetic problem are obtained that imply, that the kinetic solutions converge to the quasi-static ones, when the size of initial perturbations and the rate of application of the forces tend to 0. An application to three-dimensional elastic-plastic systems with hardening is given.

1 Mathematical formulation

We start with a Hilbert space $H \stackrel{\text{def}}{=} H_1 \times H_2$ with dual H^*, the dual pairing and the norm are respectively denoted by $\langle\cdot,\cdot\rangle : H \times H^* \to \mathbb{R}$ and $\|\cdot\|_H$. Let $V \stackrel{\text{def}}{=} V_1 \times V_2$ be such that $V \subset H \subset V^*$ with the dual V^*. We use below the following norms: $\|\cdot\|^2 \stackrel{\text{def}}{=} \langle A\,\cdot,\cdot\rangle$, $\|\cdot\|_*^2 \stackrel{\text{def}}{=} \langle\,\cdot\,, A^{-1}\cdot\rangle$ and $|\cdot|_M \stackrel{\text{def}}{=} \|M^{1/2}\cdot\|_H$. We denote by $A : H \to H^*$ a strictly positive operator. We consider the variational inclusion

$$\varepsilon^2 M\ddot{q}(t) + Aq(t) + \partial\mathcal{R}(\dot{q}(t)) \ni l(t) \text{ where } M \stackrel{\text{def}}{=} \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix},\ A \stackrel{\text{def}}{=} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{1.1}$$

$q(t) \stackrel{\text{def}}{=} (u(t), z(t))^{\mathsf{T}}$ with $u \in H_1$ and $z \in H_2$, $l(t) \stackrel{\text{def}}{=} (f_u(t), f_z(t))^{\mathsf{T}}$, ε is a parameter which eventually tends to 0 and $\partial\mathcal{R}$ is the subdifferential of the dissipation functional $\mathcal{R}$ which is assumed to be convex, lower-semicontinuous, homogeneous of degree 1. Here we assume to be given initial data $(\dot{u}(0), q(0)) \in H_1 \times V$. The corresponding quasi-static solution $\bar{q} \stackrel{\text{def}}{=} (\bar{u}, \bar{z})$ solves (1.1) with $\varepsilon = 0$ together with $\bar{q}(0) \in V$. In what concerns the kinetic problem, the theory of maximal monotone operators is used to prove existence of a unique solution whereas for the quasi-static problem, we refer to [3] or the recently developed theory in [6].

2 A priori estimates

The aim of this section is to estimate the distance between q and $\bar{q}$ by introducing a special intermediate solution $\widehat{q} = (\widehat{u}, \widehat{z})$ satisfies (1.1) together with initial conditions $(\dot{\widehat{u}}(0), \widehat{q}(0)) = (0, \bar{q}(0))$. Notice that the existence of a unique $\widehat{q}$ is obtained using the theory of maximal monotone operators. First, we provide a priori estimates for the problem which allow us to control the term $M\ddot{\widehat{q}}$ in H instead of the usual estimates in V^*. The problem occurs through the fact that $\partial\mathcal{R}$ is nonsmooth and classical techniques for smooth problems do not suffice. One way to handle this is to use Yosida regularization (see [4]). Here we choose a different technique that is based on difference quotients. Our a priori estimates can be derived most easily by using the bilinear form $B : V \times H \times V^* \to \mathbb{R}$ defined via $B[q,\dot{q},l] \stackrel{\text{def}}{=} |\varepsilon\dot{q}|_M^2 + \|q - A^{-1}l\|^2 + \|l\|_*^2$. The construction is such that for solutions we have $B[q,\dot{q},l] = 2\mathcal{E}(t,q,\dot{q}) + 2\|l\|_*^2$ with $\mathcal{E}(t,q,\dot{q}) = \frac{1}{2}\langle\varepsilon^2 M\dot{q},\dot{q}\rangle + \frac{1}{2}\langle Aq,q\rangle - \langle l(t),q\rangle$. Moreover, B defines an equivalent norm on $V \times H \times V^*$, since $\frac{1}{g^2}\big(\|q\|^2 + |\varepsilon\dot{q}|_M^2 + \|l\|_*^2\big) \le B[q,\dot{q},l] \le g^2\big(\|q\|^2 + |\varepsilon\dot{q}|_M^2 + \|l\|_*^2\big)$ with $g \stackrel{\text{def}}{=} \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. Then, one can deduce the following proposition

Proposition 2.1 *Let $l_1, l_2 \in W^{1,1}([0,T];V^*)$ and q_1 and q_2 be solutions of (1.1) with right-hand sides l_1 and l_2 respectively, then $w = q_1 - q_2$ satisfies the estimate*

$$\forall t \in [0,T]:\ B[w(t),\dot{w}(t),l_1(t)-l_2(t)]^{1/2} \le\ B[w(0),\dot{w}(0),l_1(0)-l_2(0)]^{1/2} + g\sqrt{2}\int_0^t \|\dot{l}_1(\tau)-\dot{l}_2(\tau)\|_*\,\mathrm{d}\tau. \tag{2.1}$$

Since $\dot{\widehat{u}}(0) = 0$, we may choose $\widehat{q}(t) = \widehat{q}(0)$ for all $t \in [-\rho, 0]$, $\rho > 0$, and let $l(t) = A\widehat{q}(0)$. Hence, we deduce that the stability condition $l(0) \in \partial R(0) + A\widehat{q}(0)$ holds and that the following limits for $h \searrow 0$ exist:

$$\frac{1}{h}\big(\widehat{q}(0) - \widehat{q}(-h)\big) \to 0 \text{ in } V,\ \frac{1}{h}\big(\dot{\widehat{u}}(0) - \dot{\widehat{u}}(-h)\big) \to 0 \text{ in } H_1,\ \frac{1}{h}\big(l(0) - l(-h)\big) \to 0 \text{ in } V^*. \tag{2.2}$$

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Using Proposition 2.1 and (2.2), we find

$$|\varepsilon\ddot{\widehat{q}}(t)|_M \le gB[\dot{\widehat{q}}(t),\ddot{\widehat{q}}(t),\dot{l}(t)]^{1/2} \le gB[0,0,0]^{1/2} + g^2\sqrt{2}\Big(\|\dot{l}(0)\|_* + \int_0^t \|\ddot{l}(s)\|_* \,\mathrm{d}s\Big). \tag{2.3}$$

On the other hand, we use the standard trick of adding variational inequalities leads to

$$|\varepsilon\dot{\widehat{q}}(t)|^2_M + \|\widehat{q}(t)-\bar{q}(t)\|^2 \le\ 2\bar{C}\operatorname*{ess\,sup}_{s\in[0,T]}|\varepsilon\ddot{\widehat{q}}(s)|_M \int_0^t \varepsilon\|l(s)\|_*\,\mathrm{d}s. \tag{2.4}$$

The difference between the solution q and the special solution $\widehat{q}$ comes from Proposition 2.1. Hence, (2.3) and (2.4) imply the following theorem:

Theorem 2.2 *Let the above assumptions on M, A and $\mathcal{R}$ hold and assume $l=(f_u,f_z)\in W^{2,1}([0,T],V^*)$. For $\bar{q}(0)\in V$ with $l(0)\in\partial\mathcal{R}(0)+A\bar{q}(0)\subset\{0\}\times H_2$ let $\bar{q}$ be the unique solution of* (1.1) *with $\varepsilon=0$. For arbitrary $q(0)=(u(0),z(0))\in V$ and $\dot{u}(0)\in H_1$ satisfying $f_z(0)\in a_{21}u(0)+a_{22}z(0)+\partial\mathcal{R}(0)$ let q be the unique solution of* (1.1)*. Then there exists $C>0$ such that*

$$\big(|\varepsilon\dot{q}(t)|^2_M + \|q(t)-\bar{q}(t)\|^2\big)^{1/2} \le \big(|\varepsilon\dot{u}(0)|^2_M+\|q(0)-\bar{q}(0)\|^2\big)^{1/2} + C\sqrt{\varepsilon}. \tag{2.5}$$

3 Application: elastic-plastic systems with hardening

We consider a body with reference configuration $\Omega\subset\mathbb{R}^d$. This body may undergo displacements $u:\Omega\to\mathbb{R}^d$. The plastic strain will be characterized by $e^{\mathrm{pl}}:\Omega\to\mathbb{S}^d_0$ where $\mathbb{S}^d_0$ is the space of symmetric $d\times d$ tensors such that the trace of e^{pl} vanishes. The set of admissible displacements $\mathcal{F}$ is chosen as a suitable subspace of $W^{1,2}(\Omega;\mathbb{R}^d)$ by describing Dirichlet data at the part Γ_{Dir} of $\partial\Omega$ and the plastic variable e^{pl} lives in $\mathcal{Z}\overset{\mathrm{def}}{=}L^2(\Omega;\mathbb{S}^d_0)$. We denote by $e(u)\overset{\mathrm{def}}{=}\frac{1}{2}(\nabla u+\nabla u^{\mathsf{T}})$ and $\mathbb{E}$ respectively the linearized strain tensor and elasticity tensor. We define the dissipation potential by $\mathcal{R}(\dot{e}^{\mathrm{pl}})\overset{\mathrm{def}}{=}\int_\Omega R(x,\dot{e}^{\mathrm{pl}}(x))\,\mathrm{d}x$ where $R(x,\cdot)$ is 1-homogeneous, convex and satisfies $0<r_1|v|\le R(x,v)\le r_2|v|$. We consider the governing system

$$\begin{cases}\varepsilon^2\rho\ddot{u} - \operatorname{div}\big(\mathbb{E}{:}(e(u)-e^{\mathrm{pl}})\big) = l_{\mathrm{ext}}(t),\\ -\dot{e}^{\mathrm{pl}} + \partial R^*\big(\mathbb{E}{:}(e(u)-e^{\mathrm{pl}}) - \mathbb{H}e^{\mathrm{pl}}\big)\ni 0.\end{cases} \tag{3.1}$$

We assume to be given initial data $(u(0),\dot{u}(0),e^{\mathrm{pl}}(0))\in\mathcal{F}\times\mathcal{F}\times\mathcal{Z}$. Here $\rho>0$ is the density, $l_{\mathrm{ext}}(t)$ is the applied mechanical loading and R^* is the Legendre transform of R. Let $(\bar{u},\bar{e}^{\mathrm{pl}})$ solves (3.1) with $\varepsilon=0$ and $(\bar{u}(0),\bar{e}^{\mathrm{pl}}(0))\in\mathcal{F}\times\mathcal{Z}$. Notice that the existence result for the kinetic problem was established in [7, 8] whereas existence and uniqueness of a strong solution to the corresponding quasi-static problem was obtained in [1, 2]. On the other hand, applying Theorem 2.2 and using Korn's inequality, we deduce the following Corollary:

Corollary 3.1 *Assume that $(u(0),e^{\mathrm{pl}}(0))$ and $(\bar{u}(0),\bar{e}^{\mathrm{pl}}(0))$ satisfy $0\in\mathbb{E}{:}(e^{\mathrm{pl}}(0)-e(u(0)))+\mathbb{H}e^{\mathrm{pl}}(0)+\partial\mathcal{R}(0)$ and $(l_{\mathrm{ext}}(0),0)^{\mathsf{T}}\in A(\bar{u}(0),\bar{e}^{\mathrm{pl}}_0)^{\mathsf{T}}+\{0\}\times\partial R(0)$, respectively. Then there exist $c,C>0$ such that*

$$\begin{aligned}\forall\varepsilon>0:\ &\big(\|\varepsilon\rho^{1/2}\dot{u}(t)\|^2_{L^2}+\|u(t)-\bar{u}(t)\|^2_{W^{1,2}}+\|e^{\mathrm{pl}}(t)-\bar{e}^{\mathrm{pl}}(t)\|^2_{L^2}\big)^{1/2}\\ &\le c\big(\|\varepsilon\rho^{1/2}\dot{u}(0)\|^2_{L^2}+\|u(0)-\bar{u}(0)\|^2_{W^{1,2}}+\|e^{\mathrm{pl}}(0)-\bar{e}^{\mathrm{pl}}(0)\|^2_{L^2}\big)^{1/2}+C\sqrt{\varepsilon}.\end{aligned}$$

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