APPROXIMATING DYNAMIC PHASE-FIELD FRACTURE IN VISCOELASTIC MATERIALS WITH A FIRST-ORDER FORMULATION FOR VELOCITY AND STRESS

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Abstract. We investigate a model for dynamic fracture in viscoelastic materials at small strains. The sharp crack interface is regularized with a phase-field approximation, and for the phase-field variable a viscous evolution with a quadratic dissipation potential is employed. A non-smooth penalization prevents material healing. The viscoelastic momentum balance is formulated as a first order system and coupled in a nonlinear way to the non-smooth evolution equation of the phase field. We give a full discretization in time and space, using a discontinuous Galerkin method for the first-order system. Based on this, existence of discrete solutions is shown and, as the step size in space and time tends to zero, their convergence to a suitable notion of weak solution of the system is discussed.

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1. INTRODUCTION

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2. An energetic small strain elastic and visco-elastic phase-field fracture model

Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or $d = 3$ be a bounded Lipschitz domain with boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$.

We want to determine the displacement vector $\mathbf{u}$, the velocity $\mathbf{v} = \partial_t \mathbf{u}$, the (linearized) strain $\boldsymbol{\varepsilon} = \mathrm{sym}(\mathrm{D}\mathbf{u})$ and strain rate $\dot{\boldsymbol{\varepsilon}} = \partial_t \boldsymbol{\varepsilon} = \mathrm{sym}(\mathrm{D}\mathbf{v})$, and the phase field z such that in Ω for all $t \in (0, T)$

$$\mathbf{0} = \varrho_0 \partial_t \mathbf{v} - \operatorname{div} \boldsymbol{\sigma} - \mathbf{f}\,, \tag{1a}$$

$$0 \in \tau_\mathrm{r} \partial_t z + \partial\chi_{(-\infty,0]}(\partial_t z) + \frac{1}{2} g'(z) \mathbf{C}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - G_\mathrm{c}\big(1 - z + l_\mathrm{c}^2 \Delta z\big) \tag{1b}$$

is satisfied in weak form with degradated visco-elastic stress response given by

$$\boldsymbol{\sigma} = g(z)\mathbf{C}\boldsymbol{\varepsilon} + \mathbf{D}\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}_\mathrm{E} + \boldsymbol{\sigma}_\mathrm{D}\,, \qquad \boldsymbol{\sigma}_\mathrm{E} = g(z)\mathbf{C}\boldsymbol{\varepsilon}\,, \qquad \boldsymbol{\sigma}_\mathrm{D} = \mathbf{D}\dot{\boldsymbol{\varepsilon}}\,. \tag{1c}$$

The elastodynamics is determined by the mass density $\varrho_0 > 0$ and the applied volume force density $\mathbf{f}$, $\mathbf{C}$ is the Hookean elasticity tensor, damping is described by the tensor $\mathbf{D}$ with $\mathbf{C}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} > 0$ for $\boldsymbol{\varepsilon} \neq 0$ and $\mathbf{D}\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \le 0$ for $\dot{\boldsymbol{\varepsilon}} \neq 0$. Here we use $\mathbf{C}\boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I}$ with Lamé constants $\mu, \lambda > 0$ and $\mathbf{D}\dot{\boldsymbol{\varepsilon}} = \eta\dot{\boldsymbol{\varepsilon}}$ with $\eta \ge 0$. For simplicity, we assume that the material is homogeneous, i.e., all material parameters are constant in Ω. The analysis includes the case $\mathbf{D} = \mathbf{0}$ without viscosity, but then the regulatity of the solution is reduced.

The crack evolution is driven by the elastic driving force $Y = -g'(z)\mathbf{C}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$ and the geometric regularization decribed by $G_\mathrm{c}\big(1 - z + l_\mathrm{c}^2\Delta z\big)$. It depends on a retardation time $\tau_\mathrm{r} > 0$, and length scale $l_\mathrm{c} > 0$, and a scaling factor $G_\mathrm{c} > 0$ which is a material parameter which encodes the energy release rate by crack opening[1]. The material degradation is encoded in the degradation function $g \in \mathrm{C}^1(\mathbb{R})$ with $g' \ge 0$, $g(0) = g_* > 0$, $g(1) = 1$, $g'(1) > 0$, and $g'(z) = 0$ for $z \le 0$ and $z \ge 2$. Then, $0 < g_* < g^*$ and $g^{**} > 0$ exists such that $0 < g_* \le g(z) \le g^*$ and $0 \le g'(z) \le g^{**}$, so that $g(z), g(z)^{-1} \in \mathrm{L}_\infty(\Omega)$ for $z \in \mathrm{H}^1(\Omega)$.

For the elasticity system the corresponding first-order system for $(\mathbf{v}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ is given by

$$\varrho_0 \partial_t \mathbf{v} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{f}\,, \tag{2a}$$

$$\partial_t \boldsymbol{\varepsilon} - \mathrm{sym}(\mathrm{D}\mathbf{v}) = \mathbf{0}\,, \tag{2b}$$

$$\boldsymbol{\sigma} - g(z)\mathbf{C}\boldsymbol{\varepsilon} - \mathbf{D}\partial_t\boldsymbol{\varepsilon} = \mathbf{0}\,. \tag{2c}$$

The wave propagation is complemented by initial and boundary conditions on $\partial\Omega = \partial_\mathrm{N}\Omega \cup \partial_\mathrm{D}\Omega$ together with free Neumann boundary conditions for the phase field, i.e.,

$$\mathbf{v}(0) = \mathbf{v}_0 \ \text{ in } \Omega\,, \qquad \boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0 \ \text{ in } \Omega\,, \qquad z(0) = 1 \ \text{ in } \Omega\,, \tag{3a}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g}_\mathrm{N} \text{ on } (0,T) \times \partial_\mathrm{N}\Omega\,, \qquad \mathbf{v} = \mathbf{v}_\mathrm{D} \text{ on } (0,T) \times \partial_\mathrm{D}\Omega\,, \qquad \nabla z \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega\,. \tag{3b}$$

The configuration depends on initial data $\mathbf{v}_0$ and $\boldsymbol{\varepsilon}_0$, volume forces $\mathbf{f}$, and boundary data $\mathbf{g}_\mathrm{N}$ and $\mathbf{v}_\mathrm{D}$.

Depending on $\mathbf{u}_0$ with $\boldsymbol{\varepsilon}_0 = \mathrm{sym}(\mathrm{D}\mathbf{u}_0)$, we obtain $\mathbf{u}(t) = \mathbf{u}_0 + \displaystyle\int_0^t \mathbf{v}(s)\,\mathrm{d}s$.

The energetic framework relies on the energetic potentials

$$\mathcal{E}^\mathrm{el}(z, \boldsymbol{\varepsilon}) = \frac{1}{2}\int_\Omega g(z)\mathbf{C}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}\,\mathrm{d}\mathbf{x}\,, \qquad \mathcal{E}^\mathrm{pf}(z) = \frac{G_\mathrm{c}}{2}\int_\Omega \big((1-z)^2 + l_\mathrm{c}^2|\nabla z|^2\big)\,\mathrm{d}\mathbf{x}\,, \tag{4}$$

the kinetic energy and the external energy

$$\mathcal{E}^\mathrm{kin}(\mathbf{v}) = \frac{1}{2}\int_\Omega \varrho_0|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,, \qquad \mathcal{E}^\mathrm{ext}(t, \mathbf{u}) = \int_\Omega \mathbf{f}(t)\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} + \int_{\partial_\mathrm{N}\Omega} \mathbf{g}_\mathrm{N}(t)\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \tag{5a}$$

and the elastic dissipation potential and the viscous dissipation potential for the phase field

$$\mathcal{R}^\mathrm{el}(z, \dot{\boldsymbol{\varepsilon}}) = \frac{1}{2}\int_\Omega \mathbf{D}\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}\mathbf{x}\,, \qquad \mathcal{R}^\mathrm{pf}(\dot z) = \frac{1}{2}\int_\Omega \big(\tau_\mathrm{r}|\dot z|^2 + \partial\chi_{(-\infty,0]}(\dot z)\big)\,\mathrm{d}\mathbf{x}\,. \tag{5b}$$

[1] Here, G_c depends on the Griffits constant but by the phase-field approach an additional scaling with respect to the length scale l_c is required, see [Marigo et al., 2016, Sect. 3.3.2].

In our model, the elastic driving force is conjugated to the stress response, i.e., $\boldsymbol{\sigma}_{\mathrm{E}} = \partial_{\boldsymbol{\varepsilon}} \mathcal{E}^{\mathrm{el}}(z, \boldsymbol{\varepsilon})$ and $Y = -\partial_z \mathcal{E}^{\mathrm{el}}(z, \boldsymbol{\varepsilon})$. This is essential for the following analysis.

3. A weak formulation in space and time

In the time-space cylinder $Q = (0,T)\times\Omega$ we define smooth test spaces

$$\mathcal{V} = \mathrm{C}^1(\overline{Q};\mathbb{R}^d)\,, \qquad \mathcal{V}_{T,\mathrm{D}} = \big\{\mathbf{w}\in\mathcal{V}\colon \mathbf{w}(T)=\mathbf{0} \text{ in } \Omega\,,\ \mathbf{w}=\mathbf{0} \text{ on } (0,T)\times\Gamma_\mathrm{D}\big\}\,, \tag{6a}$$

$$\mathcal{W} = \mathrm{C}^1(\overline{Q};\mathbb{R}^{d\times d}_{\mathrm{sym}})\,,\quad \mathcal{W}_T = \big\{\mathbf{\Phi}\in\mathcal{W}\colon \mathbf{\Phi}(T)=\mathbf{0} \text{ in } \Omega\big\}\,,\quad \mathcal{W}_{T,\mathrm{N}} = \big\{\mathbf{\Psi}\in\mathcal{W}_T\colon \mathbf{\Psi}\mathbf{n}=\mathbf{0} \text{ on } (0,T)\times\Gamma_\mathrm{N}\big\}\,, \tag{6b}$$

$$\mathcal{Z} = \big\{\varphi\in\mathrm{C}^1(\overline{Q})\colon \varphi\le 0 \text{ a.e. in } Q\big\}\,. \tag{6c}$$

If $(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ is sufficiently smooth solving (2a), (2b), and (2c), testing with $(\mathbf{w},\mathbf{\Phi},\mathbf{\Psi})\in\mathcal{V}_{T,\mathrm{D}}\times\mathcal{W}_{T,\mathrm{N}}\times\mathcal{W}_T$ yields the variational characterization

$$\begin{aligned}
0 &= \big(\varrho_0\partial_t\mathbf{v}-\operatorname{div}\boldsymbol{\sigma}-\mathbf{f},\mathbf{w}\big)_Q + \big(\partial_t\boldsymbol{\varepsilon}-\operatorname{sym}(\mathrm{D}\mathbf{v}),\mathbf{\Phi}\big)_Q + \big(\boldsymbol{\sigma}-g(z)\mathbf{C}\boldsymbol{\varepsilon}-\mathbf{D}\partial_t\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q\\
&= \big(\varrho_0\partial_t\mathbf{v},\mathbf{w}\big)_Q + \big(\partial_t\boldsymbol{\varepsilon},\mathbf{\Phi}-\mathbf{D}\mathbf{\Psi}\big)_Q - \big(\operatorname{div}\boldsymbol{\sigma},\mathbf{w}\big)_Q - \big(\operatorname{sym}(\mathrm{D}\mathbf{v}),\mathbf{\Phi}\big)_Q + \big(\boldsymbol{\sigma}-g(z)\mathbf{C}\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q - \big(\mathbf{f},\mathbf{w}\big)_Q\\
&= -\big(\varrho_0\mathbf{v},\partial_t\mathbf{w}\big)_Q + \big(\varrho_0\mathbf{v}(T),\mathbf{w}(T)\big)_\Omega - \big(\varrho_0\mathbf{v}(0),\mathbf{w}(0)\big)_\Omega\\
&\qquad - \big(\boldsymbol{\varepsilon},\partial_t\mathbf{\Phi}-\mathbf{D}\partial_t\mathbf{\Psi}\big)_Q + \big(\boldsymbol{\varepsilon}(T),\mathbf{\Phi}(T)-\mathbf{D}\mathbf{\Psi}(T)\big)_\Omega - \big(\boldsymbol{\varepsilon}(0),\mathbf{\Phi}(0)-\mathbf{D}\mathbf{\Psi}(0)\big)_\Omega\\
&\qquad + \big(\boldsymbol{\sigma},\operatorname{sym}(\mathrm{D}\mathbf{w})\big)_Q - \big(\boldsymbol{\sigma}\mathbf{n},\mathbf{w}\big)_{0,T\times\partial\Omega} + \big(\mathbf{v},\operatorname{div}\mathbf{\Phi}\big)_Q - \big(\mathbf{v},\mathbf{\Phi}\mathbf{n}\big)_{0,T\times\partial\Omega}\\
&\qquad + \big(\boldsymbol{\sigma}-g(z)\mathbf{C}\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q - \big(\mathbf{f},\mathbf{w}\big)_Q\\
&= -\big(\varrho_0\mathbf{v},\partial_t\mathbf{w}\big)_Q - \big(\varrho_0\mathbf{v}_0,\mathbf{w}(0)\big)_\Omega - \big(\boldsymbol{\varepsilon},\partial_t\mathbf{\Phi}-\mathbf{D}\partial_t\mathbf{\Psi}\big)_Q - \big(\boldsymbol{\varepsilon}_0,\mathbf{\Phi}(0)-\mathbf{D}\mathbf{\Psi}(0)\big)_\Omega\\
&\qquad + \big(\boldsymbol{\sigma},\operatorname{sym}(\mathrm{D}\mathbf{w})\big)_Q - \big(\mathbf{g}_\mathrm{N},\mathbf{w}\big)_{0,T\times\Gamma_\mathrm{N}} + \big(\mathbf{v},\operatorname{div}\mathbf{\Phi}\big)_Q - \big(\mathbf{v}_\mathrm{D},\mathbf{\Phi}\mathbf{n}\big)_{0,T\times\Gamma_\mathrm{D}} + \big(\boldsymbol{\sigma}-g(z)\mathbf{C}\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q - \big(\mathbf{f},\mathbf{w}\big)_Q
\end{aligned}$$

using the initial and boundary conditions of ansatz and test functions. Here, the L_2 inner product is denoted by $(\cdot,\cdot)_Q$. This is now used to derive a weak formulation in space and time. Therefore, we introduce the bilinear forms

$$\begin{aligned}
m_Q\big((\mathbf{v},\boldsymbol{\varepsilon}),(\mathbf{w},\boldsymbol{\eta})\big) &= \big(\varrho_0\mathbf{v},\mathbf{w}\big)_Q + \big(\boldsymbol{\varepsilon},\boldsymbol{\eta}\big)_Q\,,\\
a_Q\big((\mathbf{v},\boldsymbol{\sigma}),(\mathbf{w},\mathbf{\Phi})\big) &= \big(\boldsymbol{\sigma},\operatorname{sym}(\mathrm{D}\mathbf{w})\big)_Q + \big(\mathbf{v},\operatorname{div}\mathbf{\Phi}\big)_Q\,,\\
r_Q\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\mathbf{\Psi}\big) &= \big(\boldsymbol{\sigma}-g(z)\mathbf{C}\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q\,,\\
b_Q(z,\varphi) &= -G_\mathrm{c}\,\big(1-z,\varphi\big)_Q + G_\mathrm{c}l_\mathrm{c}^2\,\big(\nabla z,\nabla\varphi\big)_Q
\end{aligned}$$

and, depending on the data $\mathbf{f}$, $\mathbf{v}_0$, $\boldsymbol{\varepsilon}_0$, $\mathbf{v}_\mathrm{D}$, and $\mathbf{g}_\mathrm{N}$, the linear form

$$\ell_Q(\mathbf{w},\mathbf{\Phi},\mathbf{\Psi}) = \big(\mathbf{f},\mathbf{w}\big)_Q + \big(\varrho_0\mathbf{v}_0,\mathbf{w}(0)\big)_\Omega + \big(\boldsymbol{\varepsilon}_0,\mathbf{\Phi}(0)-\mathbf{D}\mathbf{\Psi}(0)\big)_\Omega + \big(\mathbf{v}_\mathrm{D},\mathbf{\Phi}\mathbf{n}\big)_{(0,T)\times\Gamma_\mathrm{D}} + \big(\mathbf{g}_\mathrm{N},\mathbf{w}\big)_{(0,T)\times\Gamma_\mathrm{N}}\,.$$

Then, a weak solution of the model described by (2) is defined as follows: Find

$$(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma})\in\mathrm{L}_2(Q;\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{sym}})\,,\qquad z\in\mathrm{H}^1(0,T;\mathrm{L}_2(\Omega))\cap\mathrm{L}_2(0,T;\mathrm{H}^1(\Omega))\quad\text{with}\quad z(0)=1 \text{ in } \Omega \tag{7}$$

satisfying for all smooth test functions $(\mathbf{w},\mathbf{\Phi},\mathbf{\Psi})\in\mathcal{V}_{T,\mathrm{D}}\times\mathcal{W}_{T,\mathrm{N}}\times\mathcal{W}_T$ and $\varphi\in\mathcal{Z}$

$$-m_Q\big((\mathbf{v},\boldsymbol{\varepsilon}),(\partial_t\mathbf{w},\partial_t\mathbf{\Phi}-\mathbf{D}\partial_t\mathbf{\Psi})\big) + a_Q\big((\mathbf{v},\boldsymbol{\sigma}),(\mathbf{w},\mathbf{\Phi})\big) + r_Q\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\mathbf{\Psi}\big) - \ell_Q(\mathbf{w},\mathbf{\Phi},\mathbf{\Psi}) = 0\,, \tag{8a}$$

$$\tau_\mathrm{r}\,\big(\partial_t z,\varphi\big)_Q + \frac{1}{2}\big(g'(z)\mathbf{C}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon},\varphi\big)_Q + b_Q(z,\varphi)\ge 0\,. \tag{8b}$$

Note that (7) implies $z(t)\in\mathrm{L}_2(\Omega)$ for all $t\in[0,T]$ and thus $g\big(z(t)\big), g'\big(z(t)\big)\in\mathrm{L}_\infty(\Omega)$ with $g(z(t,\mathbf{x}))\in[g_*,g^*]$ and $g'(z(t,\mathbf{x}))\ge 0$ for a.a. $\mathbf{x}\in\Omega$, so that $g'\big(z(t)\big)\mathbf{C}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}\in\mathrm{L}_2(\Omega)$ is well-defined and $g'\big(z(t)\big)\mathbf{C}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}\ge 0$ a.e. in Ω.

For the elasticity system $(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ the initial and boundary values are included in the right-hand side ℓ_Q, and the for the weak solution the initial and boundary conditions are satisfied only weakly. If the weak solution is also a strong solution, additional regularity $\big(\mathbf{v}(0),\boldsymbol{\varepsilon}(0),\boldsymbol{\sigma}(0)\big)\in\mathrm{L}_2(\Omega;\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\mathbf{v}|_{(0,T)\times\Gamma_\mathrm{D}}\in\mathrm{L}_2((0,T)\times\Gamma_\mathrm{D};\mathbb{R}^d)$, and $\boldsymbol{\sigma}\mathbf{n}|_{(0,T)\times\Gamma_\mathrm{D}}\in\mathrm{L}_2((0,T)\times\Gamma_\mathrm{N};\mathbb{R}^d)$ is required to obtain the initial and boundary conditions (3a) also strongly.

Our aim is to show that a fully discrete approximation in space and time of this problem is uniformly bounded and that a weak limit is a weak solution satisfying (8). This proves in case of homogeneous boundary data the following result.

Theorem 1. *A weak solution* $(\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma},z)$ *of the dynamic phase field fracture model satifying* (7) *and* (8) *exists.*

Therefore, we reformulate the result [Thomas and Tornquist, 2021] for the second-order formulation of the wave equation to the weak first-order setting.

4. Approximation in space

The visco-elastic wave equation is approximated with a discontinuous Galerkin (DG) method, the phase field with lowest order conforming finite elements.

On a mesh $\Omega_h = \bigcup_{K\in\mathcal{K}_h} K$ with elements K, let $V_h^{\rm dg} = \prod_{K\in\mathcal{K}_h}\mathbb{P}_k(K;\mathbb{R}^d)$ and $W_h^{\rm dg} = \prod_{K\in\mathcal{K}_h}\mathbb{P}_k(K;\mathbb{R}^{d\times d}_{\rm sym})$ be the discontinuous finite element space of polynomial degree $k\geq 1$, and let $V_h^{\rm cf}\subset \mathbb{P}(\Omega_h)\cap {\rm C}^0(\overline{\Omega})$ be the lowest order conforming finite elements, so that $\varphi_h\in V_h^{\rm cf}$ is uniquely defined by the values $\big(\varphi_h(\mathbf{x})\big)_{\mathbf{x}\in\mathcal{N}_h}$ at the element vertices $\mathcal{N}_h = \bigcup_{K\in\mathcal{K}_h}\mathcal{N}_K\subset\overline{\Omega}$. Then, we have

$$\min_{\mathbf{x}\in\mathcal{N}_K}\varphi_h(\mathbf{x}) = \min_{\mathbf{x}\in\overline{K}}\varphi_h(\mathbf{x}) \quad\text{and}\quad \max_{\mathbf{x}\in\mathcal{N}_K}\varphi_h(\mathbf{x}) = \max_{\mathbf{x}\in\overline{K}}\varphi_h(\mathbf{x})\,,\qquad K\in\mathcal{K}_h\,.$$

We assume that the mesh is shape regular and that $\operatorname{diam}(K)\leq h$ for $K\in\mathcal{K}_h$.

For the discontinuous functions, we define jump terms on the faces $\mathcal{F}_h = \bigcup_K\mathcal{F}_K$, where $\mathcal{F}_K$ are the faces on every element K. For inner faces $f\in\mathcal{F}_h\cap\Omega$, let K_f be the neighboring cell such that $\overline{f} = \partial K\cap\partial K_f$. On boundary faces $f\in\mathcal{F}_h\cap\partial\Omega$ we set $K_f = K$. Let $\mathbf{n}_K$ be the outer unit normal vector on ∂K. We define the jump $[\mathbf{v}_h]_{K,f} = \mathbf{v}_{h,K_f} - \mathbf{v}_{h,K}$ on inner faces, where $\mathbf{v}_{h,K}$ denotes the continuous extension of $\mathbf{v}_h|_K$ to $\overline{K}$. In the same way, the jump for the stress tensor is defined. On Dirichlet boundary faces, we set $[\mathbf{v}_h]_{K,f} = -2\mathbf{v}_h$ and $[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n} = \mathbf{0}$. On Neumann boundaries, set $[\mathbf{v}_h]_{K,f} = \mathbf{0}$ and $[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n} = -2\,\boldsymbol{\sigma}_h\mathbf{n}$.

The defines the DG approximation [Hochbruck et al., 2015, Dörfler and Wieners, 2019, Weinberg and Wieners, 2021] for the discontinuous functions $(\mathbf{v}_h,\boldsymbol{\sigma}_h), (\mathbf{w}_h,\boldsymbol{\Phi}_h)\in V_h^{\rm dg}\times W_h^{\rm dg}$ depending on the phase field $z_h\in V_h^{\rm cf}$ by

$$\begin{aligned} a_h^{\rm dg}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) &= \big(\boldsymbol{\sigma}_h,\operatorname{sym}(\mathrm{D}\mathbf{w}_h)\big)_{\Omega_h} + \big(\mathbf{v}_h,\operatorname{div}\boldsymbol{\Phi}_h\big)_{\Omega_h}\\ &\quad-\frac12\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\rm P}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\cdot\big(Z_{\rm P}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K - [\mathbf{w}_h]_{K,f}\big)\Big)_f\\ &\quad-\frac12\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\rm P}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\times\big(Z_{\rm P}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K - [\mathbf{w}_h]_{K,f}\big)\Big)_f \end{aligned}$$

and the right-hand side

$$\begin{aligned} \ell_h^{\rm dg}\big(t,z_h;(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) &= \big(\mathbf{f}(t),\mathbf{w}_h\big)_\Omega + \big(\mathbf{v}_{\rm D}(t),\boldsymbol{\Phi}_h\mathbf{n}\big)_{\Gamma_{\rm D}} + \big(\mathbf{g}_{\rm N}(t),\mathbf{w}_h\big)_{\Gamma_{\rm N}}\\ &\quad-\big(\mathbf{v}_{\rm D}(t),Z_{\rm P}(z_h)(\mathbf{n}\cdot\mathbf{w}_h)\mathbf{n} + Z_{\rm S}(z_h)(\mathbf{n}\times\mathbf{w}_h)\big)_{\Gamma_{\rm D}}\\ &\quad-\big(\mathbf{g}_{\rm N}(t),Z_{\rm P}(z_h)(\mathbf{n}\cdot\boldsymbol{\Phi}_h\mathbf{n})\mathbf{n} + Z_{\rm S}(z_h)\mathbf{n}\times(\boldsymbol{\Phi}_h\mathbf{n})\big)_{\Gamma_{\rm N}} \end{aligned}$$

depending on the impedances $Z_{\rm P}(z_h) = \sqrt{g(z_h)\varrho_0(2\mu+\lambda)}$ and $Z_{\rm S}(z_h) = \sqrt{g(z_h)\varrho_0\mu}$ of compressional waves and shear waves, respectively. Note that this depends on the degraded material parameters.

The DG approximation is monotone satisfying

$$\begin{aligned} a_h^{\rm dg}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{v}_h,\boldsymbol{\sigma}_h)\big) = \frac14\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\bigg(&\big\|Z_{\rm P}(z_h)^{-1/2}\mathbf{n}_K\cdot[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K\|_f^2 + \big\|Z_{\rm P}(z_h)^{1/2}\mathbf{n}_K\dot{[}\mathbf{v}_h]_{K,f}\|_f^2\\ &+\big\|Z_{\rm S}(z_h)^{-1/2}\mathbf{n}_K\times[\boldsymbol{\sigma}_h]_{K,f}\mathbf{n}_K\|_f^2 + \big\|Z_{\rm S}(z_h)^{1/2}\mathbf{n}_K\times[\mathbf{v}_h]_{K,f}\|_f^2\bigg)\geq 0\,, \end{aligned}\tag{9}$$

and is consistent satisfying for smooth test functions $(\mathbf{w},\boldsymbol{\Phi})\in\mathcal{V}_{T,{\rm D}}\times\mathcal{W}_{\rm N}$ and $t\in(0,T)$

$$a_h^{\rm dg}\big(z_h;(\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w},\boldsymbol{\Phi})(t)\big) = \big(\boldsymbol{\sigma}_h,\operatorname{sym}(\mathrm{D}\mathbf{w})(t)\big)_\Omega + \big(\mathbf{v}_h,\operatorname{div}\boldsymbol{\Phi}(t)\big)_\Omega\,, \tag{10a}$$

$$\ell_h^{\rm dg}\big(t,z_h;(\mathbf{w},\boldsymbol{\Phi})(t)\big) = \big(\mathbf{f}(t),\mathbf{w}\big)_\Omega + \big(\mathbf{v}_{\rm D}(t),\boldsymbol{\Phi}(t)\mathbf{n}\big)_{\Gamma_{\rm D}} + \big(\mathbf{g}_{\rm N}(t),\mathbf{w}(t)\big)_{\Gamma_{\rm N}}\,. \tag{10b}$$

Remark 2. *The method can be simplified by using fixed impedances $Z_P = \sqrt{\varrho_0(2\mu+\lambda)}$ and $Z_S = \sqrt{\varrho_0\mu}$ independently of the degradation; the following arguments only rely on the properties (9) and (10).*

5. Approximation in time

In the discrete formulation, the condition $\partial_t z \le 0$ is approximated using a Yosida regularization $\theta_h M_+^2(\dot z)$ defined by $M_+^2(\dot z) = \frac{1}{2}\max\{\dot z, 0\}^2$ and a penalty parameter $\theta_h > 0$. Note that $\partial M_+^2(\dot z) = \dot z$ for $\dot z > 0$ and $\partial M_+^2(\dot z) = 0$ for $\dot z \le 0$.
For the limit analysis, we use the penalty parameter $\theta_h = \frac{\theta_0}{h}$ and the regularization of the viscous dissipation potential for the phase field

$$\mathcal{R}_h^{\mathrm{pf}}(\dot z) = \int_\Omega \Big(\frac{\tau_{\mathrm r}}{2}|\dot z|^2 + \theta_h M_+^2(\dot z)\Big)\,\mathrm{d}\mathbf{x}\,. \tag{11}$$

In Ω, we define

$$\begin{aligned}
m_\Omega\big((\mathbf{v},\boldsymbol{\varepsilon}),(\mathbf{w},\boldsymbol{\eta})\big) &= \big(\varrho_0\mathbf{v},\mathbf{w}\big)_\Omega + \big(\boldsymbol{\varepsilon},\boldsymbol{\eta}\big)_\Omega\,, && \mathbf{v},\mathbf{w}\in \mathrm{L}_2(\Omega;\mathbb{R}^d)\,,\quad \boldsymbol{\varepsilon},\boldsymbol{\eta}\in \mathrm{L}_2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,,\\
r_\Omega\big(z;(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\boldsymbol{\Psi}\big) &= \big(\boldsymbol{\sigma} - g(z)\mathbf{C}\boldsymbol{\varepsilon},\boldsymbol{\Psi}\big)_\Omega\,, && z\in \mathrm{L}_\infty(\Omega)\,,\quad \boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\Psi}\in \mathrm{L}_2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,,\\
b_\Omega(z,\varphi) &= -G_{\mathrm c}\big((1-z),\varphi\big)_\Omega + G_{\mathrm c} l_{\mathrm c}^2\big(\nabla z,\nabla\varphi\big)_\Omega\,, && z,\varphi\in \mathrm{H}^1(\Omega)\,,
\end{aligned}$$

and, depending on $z_h^{n-1}\in V_h^{\mathrm{cf}}$ and $\boldsymbol{\varepsilon}_h^{n-1}$, the coercive functional

$$\begin{aligned}
\mathcal{G}_h^n(z_h) &= \frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(z_h - z_h^{n-1}) + \mathcal{E}^{\mathrm{el}}(z_h,\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\mathrm{pf}}(z_h)\\
&= \int_\Omega \Big(\frac{\tau_{\mathrm r}}{2\triangle t_h^n}\big(z_h - z_h^{n-1}\big)^2 + \frac{\theta_h}{\triangle t_h^n}M_+^2\big(z_h - z_h^{n-1}\big) + \frac12 g(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1} + \frac{G_{\mathrm c}}{2}\big((1-z_h)^2 + l_{\mathrm c}^2|\nabla z_h|^2\big)\Big)\,\mathrm{d}\mathbf{x}\\
&\ge \frac{1}{\triangle t_h^n}\mathcal{R}_h^{\mathrm{pf}}(z_h - z_h^{n-1}) \ge \frac{\tau_{\mathrm r}}{2\triangle t_h^n}\big\|z_h - z_h^{n-1}\big\|_\Omega^2\,,\qquad z_h\in V_h^{\mathrm{cf}}\,.
\end{aligned} \tag{12}$$

We assume that the loading is much slower than the wave speed. Thus we start with large time steps $\triangle t_{\mathrm{qs}} > 0$ for quasi-static increments. If waves are initiated by crack opening, the time step is decreased to $\triangle t_{\mathrm{pf}}\in(0,\triangle t_{\mathrm{qs}})$ such that $c_{\mathrm P}\triangle t_{\mathrm{pf}} \approx h$ with wave speed $c_{\mathrm P} = \sqrt{(2\mu+\lambda)/\rho)}$.
We start with initial values $(\mathbf{v}_h^0,\boldsymbol{\varepsilon}_h^0,\boldsymbol{\sigma}_h^0)\in V_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}$ and $z_h^0\in V_h^{\mathrm{cf}}$ in the material without fracture, i.e., $z_h^0 = 1$.
We set $t_h^0 = 0$, $t_h^1 = \triangle t_{\mathrm{qs}}$, and $\triangle t_h^1 = t_h^1 - t_h^0$.
In every time step $n = 1,2,3,\dots$ we proceed as follows:

(S1) Depending on $(\boldsymbol{\varepsilon}_h^{n-1}, z_h^{n-1})$, we approximate the phase field $z_h^n\in V_h^{\mathrm{cf}}$ by the implicit Euler method, i.e., by computing a critical point of $\mathcal{G}_h^n(\cdot)$ by solving the nonlinear equation

$$\frac{\tau_{\mathrm r}}{\triangle t_h^n}\big(z_h^n - z_h^{n-1},\varphi_h\big)_\Omega + \frac{\theta_h}{\triangle t_h^n}\big(\partial M_+^2(z_h^n - z_h^{n-1}),\varphi_h\big)_\Omega + \frac12\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1},\varphi_h\big)_\Omega + b_\Omega(z_h^n,\varphi_h) = 0\,,\quad \varphi_h\in V_h^{\mathrm{cf}}$$

such that $\mathcal{G}_h^n(z_h^n)\le \mathcal{G}_h^n(z_h^{n-1})$; this can be achieved by starting the iterative solution method with z_h^{n-1}.

(S2) Depending on $(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1},\boldsymbol{\sigma}_h^{n-1})\in V_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}$ and $z_h^n\in V_h^{\mathrm{cf}}$ we compute the solution for the next time step $(\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n)\in V_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}$ by the implicit Euler method, i.e., by solving the linear equation

$$\begin{aligned}
&m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{w}_h,\boldsymbol{\Phi}_h - \mathbf{D}\boldsymbol{\Psi}_h)\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) + \triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\boldsymbol{\Psi}_h\big)\\
&\qquad = m_\Omega\big(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big) + \triangle t_h^n \ell_h^{\mathrm{dg}}\big(t_h^n, z_h^n;(\mathbf{w}_h,\boldsymbol{\Phi}_h)\big)\,,\qquad (\mathbf{w}_h,\boldsymbol{\Phi}_h,\boldsymbol{\Psi}_h)\in V_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}\times W_h^{\mathrm{dg}}\,.
\end{aligned}$$

(S3) If the relaxed energy is small and $z_h^n\approx z_h^{n-1}$, we expect that the next time step will also be quasi-static, and we set $\triangle t_h^{n+1} = \triangle t_{\mathrm{qs}}$; otherwise, we set $\triangle t_h^{n+1} = \triangle t_{\mathrm{pf}}$.
Then, we set $t_h^{n+1} = t_h^n + \triangle t_h^{n+1}$, and we continue with the next time step $n := n+1$ proceeding with (S1).

For simplicity of the presentation, we consider in the following only the case of homogeneous boundary data $\mathbf{v}_{\mathrm D} = \mathbf{0}$ and $\mathbf{g}_{\mathrm N} = \mathbf{0}$, and the volume forces are approximated by the L_2 projection $\mathbf{f}_h^n\in V_h^{\mathrm{dg}}$ in $(t_h^{n-1},t_h^n)\times\Omega$, i.e.,

$$\big(\mathbf{f}_h^n,\mathbf{w}_h\big)_{(t_h^{n-1},t_h^n)\times\Omega} = \big(\mathbf{f},\mathbf{w}_h\big)_{(t_h^{n-1},t_h^n)\times\Omega}\,,\qquad \mathbf{w}_h\in V_h^{\mathrm{dg}}\,, \tag{13}$$

and we use for the following analysis the discrete right-hand side

$$\ell_h^{\mathrm{dg}}\big(t_h^n, z_h^n; (\mathbf{w}_h, \mathbf{\Phi}_h)\big) = \big(\mathbf{f}_h^n, \mathbf{w}_h^n\big)_\Omega \,. \tag{14}$$

We also assume that $(\mathbf{v}_h^0, \boldsymbol{\varepsilon}_h^0)$ are the L_2 projections of the initial values $(\mathbf{v}_0, \boldsymbol{\varepsilon}_0)$.

6. Well-posedness and stability in space and time of the discrete solution

We show that the discrete problems in the staggered scheme has a solution and we provide bounds for $(\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n, z_h^n)$. Therefore, we set $\triangle\boldsymbol{\varepsilon}_h^n = \boldsymbol{\varepsilon}_h^n - \boldsymbol{\varepsilon}_h^{n-1}$ and $\triangle z_h^n = z_h^n - z_h^{n-1}$ for $n = 1, \dots, N$, and the projection $\Pi_h^{\mathrm{dg}} \colon \mathrm{L}_1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \longrightarrow W_h^{\mathrm{dg}}$ is defined by

$$\big(\Pi_h^{\mathrm{dg}}\boldsymbol{\Phi}, \boldsymbol{\Psi}_h\big)_\Omega = \big(\boldsymbol{\Phi}, \boldsymbol{\Psi}_h\big)_\Omega\,, \qquad \boldsymbol{\Phi} \in \mathrm{L}_1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})\,,\ \boldsymbol{\Psi}_h \in W_h^{\mathrm{dg}}\,.$$

Lemma 3. *For $n = 1, \dots, N$ a solution $z_h^n \in V_h^{\mathrm{cf}}$ in* (S1) *and a unique solution $(\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$ in* (S2) *exists satisfying $\boldsymbol{\sigma}_h^n = \Pi_h^{\mathrm{dg}}\big(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n\big) + \frac{1}{\triangle t_h^n}\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n$. In case of homogeneous boundary data $\mathbf{v}_{\mathrm{D}} = \mathbf{0}$, $\mathbf{g}_{\mathrm{N}} = \mathbf{0}$, the discrete solution is bounded by the discrete energy-dissipation inequality*

$$\begin{aligned}\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + \mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\mathrm{pf}}(z_h^n) + \sum_{k=1}^n \Big(\frac{2}{\triangle t_k}\mathcal{R}^{\mathrm{el}}(\triangle\boldsymbol{\varepsilon}_h^k) + \frac{1}{\triangle t_k}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^k)\Big)&\\ \le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0) + \sum_{k=1}^n \triangle t_k \big(\mathbf{f}_h^k, \mathbf{v}_h^k\big)_\Omega\,.&\end{aligned} \tag{15}$$

Proof. Since the coercive functional $\mathcal{G}_h^n$ is bounded from below by a quadratic functional and V_h^{cf} is discrete, in (S1) a minimizer z_h^n exists, and the minimizer is a critical point of $\mathcal{G}_h^n$ solving the nonlinear equation in (S1).

Next we show that the discrete linear system in (S2) has a unique solution. Therefore, we show that the homogeneous problem only admits the trivial solution: assume that $(\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$ solves

$$m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), (\mathbf{w}_h, \boldsymbol{\Phi}_h - \mathbf{D}\boldsymbol{\Psi}_h)\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0), (\mathbf{w}_h, \boldsymbol{\Phi}_h)\big) + \triangle t_h^n r_\Omega\big(z_h^n; (\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0), \boldsymbol{\Psi}_h\big) = 0$$

for all $(\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h) \in V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}$. Testing with $(\mathbf{0}, \mathbf{0}, \boldsymbol{\Psi}_h)$ yields

$$\begin{aligned}0 &= -\big(\boldsymbol{\Phi}_h^0, \mathbf{D}\boldsymbol{\Psi}_h\big)_\Omega + \triangle t_h^n\big(\boldsymbol{\Psi}_h - g(z_h^n)\mathbf{C}\boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h\big)_\Omega\\ &= -\big(\mathbf{D}\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h\big)_\Omega + \triangle t_h^n\big(\boldsymbol{\Psi}_h - \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\Phi}_h), \boldsymbol{\Psi}_h\big)_\Omega\,, \qquad \boldsymbol{\Psi} \in W_h^{\mathrm{dg}}\,,\end{aligned} \tag{16}$$

i.e., $\boldsymbol{\Psi}_h^0 = \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\Phi}_h^0) + (\triangle t_h^n)^{-1}\mathbf{D}\boldsymbol{\Phi}_h^0$. Now testing with $(\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0, \mathbf{0})$ yields, using (9),

$$\begin{aligned}0 &= m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0)\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0), (\mathbf{w}_h, \boldsymbol{\Psi}_h)\big)\\ &\ge m_\Omega\big((\mathbf{w}_h^0, \boldsymbol{\Phi}_h^0), (\mathbf{w}_h^0, \boldsymbol{\Psi}_h^0)\big) = \varrho_0\,\big(\mathbf{w}_h^0, \mathbf{w}_h^0\big)_\Omega + \big(\boldsymbol{\Phi}_h^0, \boldsymbol{\Psi}_h^0\big)_\Omega\\ &= \varrho_0\,\big(\mathbf{w}_h^0, \mathbf{w}_h^0\big)_\Omega + \big(\boldsymbol{\Phi}_h^0, g(z_h^n)\mathbf{C}\boldsymbol{\Phi}_h^0\big)_\Omega + (\triangle t_h^n)^{-1}\big(\boldsymbol{\Phi}_h^0, \mathbf{D}\boldsymbol{\Phi}_h^0\big)_\Omega \ge \varrho_0\,\|\mathbf{w}_h\|_\Omega^2 + \big\|g(z_h^n)^{1/2}\mathbf{C}^{1/2}\boldsymbol{\Phi}_h\big\|_\Omega^2\end{aligned}$$

which implies $\mathbf{w}_h^0 = \mathbf{0}$ and, using $g(z_h^n) \ge g_* > 0$, also $\boldsymbol{\Phi}_h^0 = \mathbf{0}$. Inserting in (16) yields $\boldsymbol{\Psi}_h^0 = \mathbf{0}$, so that indeed the solution of the homogeneous problem is $(\mathbf{0}, \mathbf{0}, \mathbf{0})$.

Testing the unique solution $(\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n)$ in (S2) with $(\mathbf{0}, \mathbf{0}, \boldsymbol{\Psi}_h)$ yields

$$m_\Omega\big((\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n), (\mathbf{0}, -\mathbf{D}\boldsymbol{\Psi}_h)\big) + \triangle t_h^n r_\Omega\big(z_h^n; (\boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n), \boldsymbol{\Phi}_h\big) = m_\Omega\big((\mathbf{v}_h^{n-1}, \boldsymbol{\varepsilon}_h^{n-1}), (\mathbf{0}, -\mathbf{D}\boldsymbol{\Phi}_h)\big)\,,$$

i.e., for all $\boldsymbol{\Psi}_h \in W_h^{\mathrm{dg}}$

$$\begin{aligned}0 &= \triangle t_h^n r_\Omega\big(z_h^n; (\boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n), \boldsymbol{\Psi}_h\big) - \big(\boldsymbol{\varepsilon}_h^n - \boldsymbol{\varepsilon}_h^{n-1}, \mathbf{D}\boldsymbol{\Psi}_h\big)_\Omega\\ &= \triangle t_h^n\big(\boldsymbol{\sigma}_h^n - g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n, \boldsymbol{\Psi}_h\big)_\Omega - \big(\triangle\boldsymbol{\varepsilon}_h^n, \mathbf{D}\boldsymbol{\Psi}_h\big)_\Omega = \triangle t_h^n\big(\boldsymbol{\sigma}_h^n - \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n), \boldsymbol{\Psi}_h\big)_\Omega - \big(\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n, \boldsymbol{\Psi}_h\big)_\Omega\,,\end{aligned}$$

so that we obtain $\boldsymbol{\sigma}_h^n = \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n) + \frac{1}{\triangle t_h^n}\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n$. Next we observe

$$\begin{aligned}m_\Omega\big((\mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n), (\mathbf{v}_h^n, \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n))\big) &= \varrho_0\,\big(\mathbf{v}_h^n, \mathbf{v}_h^n\big)_\Omega + \big(\boldsymbol{\varepsilon}_h^n, \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n)\big)_\Omega\\ &= \varrho_0\,\big(\mathbf{v}_h^n, \mathbf{v}_h^n\big)_\Omega + \big(\boldsymbol{\varepsilon}_h^n, g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n\big)_\Omega = 2\,\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + 2\,\mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^n)\\ m_\Omega\big(\mathbf{v}_h^{n-1}, \boldsymbol{\varepsilon}_h^{n-1}), (\mathbf{v}_h^n, \Pi_h^{\mathrm{dg}}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n))\big) &= \varrho_0\,\big(\mathbf{v}_h^{n-1}, \mathbf{v}_h^n\big)_\Omega + \big(\boldsymbol{\varepsilon}_h^{n-1}, g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n\big)_\Omega\\ &\le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + \mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^{n-1}) + \mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^{n-1})\end{aligned}$$

and

$$\begin{aligned} r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\big) &= \big(\boldsymbol{\sigma}_h^n - g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n,\triangle\boldsymbol{\varepsilon}_h^n\big)_\Omega \\ &= \big(\boldsymbol{\sigma}_h^n - \Pi_h^{\rm dg}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\big)_\Omega = \Big(\frac{1}{\triangle t_h^n}\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n,\triangle\boldsymbol{\varepsilon}_h^n\Big)_\Omega = \frac{2}{\triangle t_h^n}\,\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^n)\,, \end{aligned}$$

which yields together with (14) and testing in (S2) with $\big(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n,(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n\big)$

$$\begin{aligned} 2\mathcal{E}^{\rm kin}(\mathbf{v}_h^n) + 2\mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^n) + \frac{2}{\triangle t_h^n}\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^n) &= m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{v}_h^n,\Pi_h^{\rm dg}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n))\big) + r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),\triangle\boldsymbol{\varepsilon}_h^n\big) \\ &= m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n - (\triangle t_h^n)^{-1}\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n)\big) + \triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n\big) \\ &\le m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n - (\triangle t_h^n)^{-1}\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n))\big) + \triangle t_h^n a_h^{\rm dg}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big) \\ &\qquad + \triangle t_h^n r_\Omega\big(z_h^n;(\boldsymbol{\varepsilon}_h^n,\boldsymbol{\sigma}_h^n),(\triangle t_h^n)^{-1}\triangle\boldsymbol{\varepsilon}_h^n\big) \\ &= m_\Omega\big(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n - (\triangle t_h^n)^{-1}\mathbf{D}\triangle\boldsymbol{\varepsilon}_h^n)\big) + \triangle t_h^n\ell_h^{\rm dg}\big(t_h^n,z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big) \\ &= m_\Omega\big(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{v}_h^n,\Pi_h^{\rm dg}(g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n))\big) + \triangle t_h^n\ell_h^{\rm dg}\big(t_h^n,z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n)\big) \\ &\le \mathcal{E}^{\rm kin}(\mathbf{v}_h^n) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\rm kin}(\mathbf{v}_h^{n-1}) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1}) + \triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega\,, \end{aligned}$$

so that

$$\mathcal{E}^{\rm kin}(\mathbf{v}_h^n) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^n) + \frac{2}{\triangle t_h^n}\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^n) \le \mathcal{E}^{\rm kin}(\mathbf{v}_h^{n-1}) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1}) + \triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega\,. \tag{17}$$

For the solution $z_h^n\in V_h^{\rm cf}$ of (S1) we assume $\mathcal{G}_h^n(z_h^n)\le\mathcal{G}_h^n(z_h^{n-1})$, so that we obtain

$$\frac{1}{\triangle t_h^n}\mathcal{R}_h^{\rm pf}(\triangle z_h^n) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\rm pf}(z_h^n) = \mathcal{G}_h^n(z_h^n) \le \mathcal{G}_h^n(z_h^{n-1}) = \mathcal{E}^{\rm el}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\rm pf}(z_h^{n-1})\,.$$

i.e., $\frac{1}{\triangle t_h^n}\mathcal{R}_h^{\rm pf}(\triangle z_h^n) + \mathcal{E}^{\rm pf}(z_h^n) \le \mathcal{E}^{\rm el}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}) - \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\rm pf}(z_h^{n-1})$. Together with (17) this yields

$$\begin{aligned} &\mathcal{E}^{\rm kin}(\mathbf{v}_h^n) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\rm pf}(z_h^n) + \frac{2}{\triangle t_h^n}\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^n) + \mathcal{R}_h^{\rm pf}(\triangle z_h^n) \\ &\qquad\qquad \le \mathcal{E}^{\rm kin}(\mathbf{v}_h^{n-1}) + \mathcal{E}^{\rm el}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\rm pf}(z_h^{n-1}) + \triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega\,. \end{aligned}$$

For $n>2$ we have

$$\begin{aligned} &\mathcal{E}^{\rm kin}(\mathbf{v}_h^n) + \mathcal{E}^{\rm el}(z_h^n,\boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\rm pf}(z_h^n) + \frac{2}{\triangle t_h^n}\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^n) + \frac{1}{\triangle t_h^n}\mathcal{R}_h^{\rm pf}(\triangle z_h^n) + \frac{2}{\triangle t_h^{n-1}}\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^{n-1}) + \frac{1}{\triangle t_h^{n-1}}\mathcal{R}_h^{\rm pf}(\triangle z_h^{n-1}) \\ &\quad \le \mathcal{E}^{\rm kin}(\mathbf{v}_h^{n-1}) + \mathcal{E}^{\rm el}(z_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}) + \mathcal{E}^{\rm pf}(z_h^{n-1}) + \frac{2}{\triangle t_h^{n-1}}\mathcal{R}^{\rm el}(\triangle\boldsymbol{\varepsilon}_h^{n-1}) + \frac{1}{\triangle t_h^{n-1}}\mathcal{R}_h^{\rm pf}(\triangle z_h^{n-1}) + \triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega \\ &\quad \le \mathcal{E}^{\rm kin}(\mathbf{v}_h^{n-2}) + \mathcal{E}^{\rm el}(z_h^{n-2},\boldsymbol{\varepsilon}_h^{n-2}) + \mathcal{E}^{\rm pf}(z_h^{n-2}) + \triangle t_h^{n-1}\big(\mathbf{f}_h^{n-1},\mathbf{v}_h^{n-1}\big)_\Omega + \triangle t_h^n\big(\mathbf{f}_h^n,\mathbf{v}_h^n\big)_\Omega\,. \end{aligned}$$

This continues for $n-2, n-3, ..., 1$ and thus proves the assertion. □

We define $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h) \in \mathrm{L}_2(0,T; V_h^{\mathrm{cf}} \times V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ by $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h)(t) = (z_h^n, \mathbf{v}_h^n, \boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n)$ in (t_h^{n-1}, t_h^n) and $(\dot z_h, \dot{\boldsymbol{\varepsilon}}_h) \in \mathrm{L}_2(0,T; V_h^{\mathrm{cf}} \times W_h^{\mathrm{dg}})$ by $(\dot z_h, \dot{\boldsymbol{\varepsilon}}_h)(t) = \frac{1}{\triangle t_h^n}(\triangle z_h^n, \triangle \boldsymbol{\varepsilon}_h^n)$ for $t \in (t_h^{n-1}, t_h^n)$. We have $\boldsymbol{\sigma}_h = \Pi_h^{\mathrm{dg}}(g(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h) + \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h$. The following analysis holds for both cases, visco-elasticity with positive definite $\mathbf{D}$, and the elastodynamics without viscosity with $\mathbf{D} = \mathbf{0}$.

Lemma 4. *The discrete solution $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \dot z_h, \mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h)$ is uniformly bounded in $Q = (0,T) \times \Omega$ by*

$$\frac{\varrho_0}{4}\big\|\mathbf{v}_h\big\|_Q^2 + \frac{1}{2}\big\|g(z_h)^{1/2}\mathbf{C}^{1/2}\boldsymbol{\varepsilon}_h\big\|_Q^2 + \frac{G_{\mathrm{c}}}{2}\Big(\big\|1 - z_h\big\|_Q^2 + l_{\mathrm{c}}^2\,\big\|\nabla z_h\big\|_Q^2\Big) + \big\|\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2 + \frac{\tau_r}{2}\big\|\dot z_h\big\|_Q^2 + \frac{\theta_h}{2}\big\|\max\{\dot z_h, 0\}\big\|_Q^2$$
$$\le \max\{T,1\}\Big(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0)\Big) + \frac{\max\{T,1\}^2}{\varrho_0}\big\|\mathbf{f}\big\|_Q^2\,.$$

Proof. We observe for the total energy

$$\frac{\varrho_0}{2}\big\|\mathbf{v}_h\big\|_Q^2 + \frac{1}{2}\big\|g(z_h)^{1/2}\mathbf{C}^{1/2}\boldsymbol{\varepsilon}_h\big\|_Q^2 + \frac{G_{\mathrm{c}}}{2}\Big(\big\|1 - z_h\big\|_Q^2 + l_{\mathrm{c}}^2\,\big\|\nabla z_h\big\|_Q^2\Big) = \sum_{n=1}^N \triangle t_h^n\Big(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + \mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\mathrm{pf}}(z_h^n)\Big)$$

and for the dissipation

$$\begin{aligned}
&\big\|\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2 + \frac{\tau_{\mathrm{r}}}{2}\big\|\dot z_h\big\|_Q^2 + \frac{\theta_h}{2}\big\|\max\{\dot z_h, 0\}\big\|_Q^2\\
&\quad= \sum_{n=1}^N\bigg(\big\|\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_{(t_h^{n-1},t_h^n)\times\Omega}^2 + \frac{\tau_{\mathrm{r}}}{2}\big\|\dot z_h\big\|_{(t_h^{n-1},t_h^n)\times\Omega}^2 + \frac{\theta_h}{2}\big\|\max\{\dot z_h, 0\}\big\|_{(t_h^{n-1},t_h^n)\times\Omega}^2\bigg)\\
&\quad= \sum_{n=1}^N\frac{1}{\triangle t_h^n}\bigg(\big\|\mathbf{D}^{1/2}\triangle\boldsymbol{\varepsilon}_h^n\big\|_\Omega^2 + \frac{\tau_{\mathrm{r}}}{2}\big\|\triangle z_h^n\big\|_\Omega^2 + \frac{\theta_h}{2}\big\|\max\{\triangle z_h^n, 0\}\big\|_\Omega^2\bigg) = \sum_{n=1}^N\bigg(\frac{2}{\triangle t_k}\mathcal{R}^{\mathrm{el}}(\triangle\boldsymbol{\varepsilon}_h^k) + \frac{1}{\triangle t_k}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^k)\bigg)
\end{aligned}$$

Using (13), we get $\sum_{n=1}^N \triangle t_h^n\big(\mathbf{f}_h^n, \mathbf{v}_h^n\big)_\Omega = \sum_{n=1}^N\big(\mathbf{f}, \mathbf{v}_h\big)_{(t_h^{n-1},t_h^n)\times\Omega} = \big(\mathbf{f}, \mathbf{v}_h\big)_Q$.

Together, the estimate (15) for the energy ($n = 1, \dots, N$) and for the dissipation ($n = N$) yields the assertion by

$$\begin{aligned}
&\frac{\varrho_0}{2}\big\|\mathbf{v}_h\big\|_Q^2 + \frac{1}{2}\big\|g(z_h)^{1/2}\mathbf{C}^{1/2}\boldsymbol{\varepsilon}_h\big\|_Q^2 + \frac{G_{\mathrm{c}}}{2}\Big(\big\|1 - z_h\big\|_Q^2 + l_{\mathrm{c}}^2\,\big\|\nabla z_h\big\|_Q^2\Big) + \big\|\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2 + \frac{\tau_{\mathrm{r}}}{2}\big\|\dot z_h\big\|_Q^2 + \frac{\theta_h}{2}\big\|\max\{\dot z_h, 0\}\big\|_Q^2\\
&\quad= \sum_{n=1}^N \triangle t_h^n\left(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + \mathcal{E}^{\mathrm{el}}(z_h^n, \boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\mathrm{pf}}(z_h^n)\right) + \sum_{n=1}^N\bigg(\frac{2}{\triangle t_k}\mathcal{R}^{\mathrm{el}}(\triangle\boldsymbol{\varepsilon}_h^k) + \frac{1}{\triangle t_k}\mathcal{R}_h^{\mathrm{pf}}(\triangle z_h^k)\bigg)\\
&\quad\le \max\Big\{\sum_{n=1}^N\triangle t_h^n, 1\Big\}\left(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0) + \big(\mathbf{f}, \mathbf{v}_h\big)_Q\right)\\
&\quad\le \max\{T,1\}\Big(\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0, \boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0)\Big) + \frac{\max\{T,1\}^2}{\varrho_0}\big\|\mathbf{f}\big\|_Q^2 + \frac{\varrho_0}{4}\big\|\mathbf{v}_h\big\|_Q^2\,.
\end{aligned}$$

□

7. Weak limit of the discrete solutions

We consider a shape-regular family $\big(\Omega_h\big)_{h\in\mathcal{H}}$ of meshes with $0\in\overline{\mathcal{H}}$, e.g., obtained by uniform refinement of a coarse mesh. For simplicity, we may assume for this limit analysis uniform time steps $\triangle t_h^n = \triangle t_h = T/N_h$ with $N_h\in\mathbb{N}$ such that $c_{\mathrm{ws}}\triangle t_h \approx h$ with respect to a reference wave speed $c_{\mathrm{ws}}>0$. We set $t_h^n = n\triangle t_h$ and $t_h^{n-1/2} = \frac12(t_h^{n-1}+t_h^n)$.

By Lem. 4 the discrete solutions $(z_h,\mathbf{v}_h,\boldsymbol{\varepsilon}_h,\dot z_h,\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}}$ are uniformly bounded.

Lemma 5. *A weakly converging subsequence $(z_h,\mathbf{v}_h,\boldsymbol{\varepsilon}_h,\dot z_h,\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ with $\mathcal{H}_0\subset\mathcal{H}$ and $0\in\overline{\mathcal{H}}_0$ exists. For the limit*

$$(z,\mathbf{v},\boldsymbol{\varepsilon},\dot z,\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}) \in \mathrm{L}_2\big(0,T;\mathrm{H}^1(\Omega)\big)\times \mathrm{L}_2(Q;\mathbb{R}^d)\times \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\times \mathrm{L}_2(Q)\times \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\times \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,, \tag{18}$$

the weak derivative $\partial_t z$ exists, and we have $z\in \mathrm{H}^1\big(0,T;\mathrm{L}_2(\Omega)\big)$ with $z(0)=z_0$, $\partial_t z = \dot z\le 0$ a.e. in Q.
If, in addition, $\mathbf{D}$ is positive definite, also the weak derivatives $\partial_t\boldsymbol{\varepsilon}$ and $\mathrm{sym}(\mathrm{D}\mathbf{v})$ *exist, and we have $\partial_t\boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}} = \mathrm{sym}(\mathrm{D}\mathbf{v})$.*

Proof. By Lem. 4, the discrete solutions $(z_h,\mathbf{v}_h,\boldsymbol{\varepsilon}_h,\dot z_h,\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}}$ are uniformly bounded by

$$\big\|\mathbf{v}_h\big\|_Q^2 + \big\|\boldsymbol{\varepsilon}_h\big\|_Q^2 + \big\|z_h\big\|_Q^2 + \big\|\nabla z_h\big\|_Q^2 + \big\|\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h\big\|_Q^2 + \big\|\dot z_h\big\|_Q^2 + \theta_h\big\|\max\{\dot z_h,0\}\big\|_Q^2 \le C$$

with a constant $C>0$ independent of $h\in\mathcal{H}$ but depending on the initial data $\mathbf{v}_0$, z_0 $\boldsymbol{\varepsilon}_0$, the load $\mathbf{f}$, the lower bound $g(z_h)\ge g_*>0$, and the material parameters. Thus, a weakly converging subsequence $(z_h)_{h\in\mathrm{H}_0}\subset \mathrm{L}_2\big(0,T;\mathrm{H}^1(\Omega)\big)$ and $(\mathbf{v}_h,\boldsymbol{\varepsilon}_h,\dot z_h,\mathbf{D}^{1/2}\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ in L_2 exists.

Since $\theta_h\longrightarrow\infty$ for $h\longrightarrow 0$, we obtain for the limit $\big\|\max\{\dot z,0\}\big\|_Q \le \lim\limits_{h\in\mathcal{H}_0}\big\|\max\{\dot z_h,0\}\big\|_Q \le \lim\limits_{h\in\mathcal{H}_0}\dfrac{C}{\theta_h} = 0$ and thus $\dot z\le 0$ a.e. in Q. Moreover, we observe for smooth test functions $\phi\in \mathrm{C}^1(Q)$ with $\phi(T)=0$

$$\begin{aligned}\big(z_h,\partial_t\phi\big)_Q &= \sum_{n=1}^{N_h}\big(z_h^n,\partial_t\phi\big)_{(t_h^{n-1},t_h^n)\times\Omega} = \sum_{n=1}^{N_h}\big(z_h^n,\phi(t_h^n)-\phi(t_h^{n-1})\big)_\Omega = -\big(z_h^0,\phi(0)\big)_\Omega + \sum_{n=1}^{N_h}\big(z_h^{n-1}-z_h^n,\phi(t_h^{n-1})\big)_\Omega\\ &= -\big(z_h^0,\phi(0)\big)_\Omega - \sum_{n=1}^{N_h}\big(\triangle z_h^n,\phi(t_h^{n-1})\big)_\Omega = -\big(z_h^0,\phi(0)\big)_\Omega - \sum_{n=1}^{N_h}\big(\dot z_h,\phi(t_h^{n-1})\big)_{(t_h^{n-1},t_h^n)\times\Omega}\,,\end{aligned}$$

so that $\lim_{h\in\mathcal{H}_0}\big\|\phi(t_h^{n-1})-\phi\big\|_{(t_h^{n-1},t_h^n)\times\Omega} = 0$ and $z_0 = z_h^0 = 1$ gives

$$\begin{aligned}\big(z_0,\phi(0)\big)_\Omega + \big(z,\partial_t\phi\big)_Q &= \lim_{h\in\mathcal{H}_0}\Big(\big(z_h^0,\phi(0)\big)_\Omega + \big(z_h,\partial_t\phi\big)_Q\Big)\\ &= -\lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\big(\dot z_h,\phi(t_h^{n-1})\big)_{(t_h^{n-1},t_h^n)\times\Omega} = -\lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\big(\dot z_h,\phi\big)_{(t_h^{n-1},t_h^n)\times\Omega} = -\big(\dot z,\phi\big)_Q\,.\end{aligned}$$

Testing with $\phi\in \mathrm{C}^1_{\mathrm{c}}(Q)$ shows that the weak derivative in time of z exists with $\partial_t z = \dot z$, so that $z\in \mathrm{H}^1\big(0,T;\mathrm{L}_2(\Omega)\big)$ and thus continuous in time; testing with $\phi(0)\ne 0$ and $\phi(T)=0$ shows $z(0)=z_0$.

If, in addition, $\mathbf{D}$ is positive definite, also $(\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ is weakly converging to $\dot{\boldsymbol{\varepsilon}}$, and one shows in the same way that the weak derivative in time of $\boldsymbol{\varepsilon}$ exists and that $\partial_t\boldsymbol{\varepsilon} = \dot{\boldsymbol{\varepsilon}}$.

Moreover, we select a smooth test functions $\boldsymbol{\Phi}\in \mathrm{C}^1_{\mathrm{c}}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, and let $\boldsymbol{\Phi}_h^n\in W_h^{\mathrm{dg}}\cap \mathrm{H}^1_0(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be the an approximation of $\boldsymbol{\Phi}$ in $(t_h^{n-1},t_h^n)\times\Omega$ with $\lim\limits_{h\longrightarrow 0}\Big(\big\|\boldsymbol{\Phi}_h^n-\boldsymbol{\Phi}\big\|_{(t_h^{n-1},t_h^n)\times\Omega} + \big\|\operatorname{div}(\boldsymbol{\Phi}_h^n-\boldsymbol{\Phi})\big\|_{(t_h^{n-1},t_h^n)\times\Omega}\Big) = 0$.

Then, testing (S2) with $(\mathbf{0},\boldsymbol{\Phi}_h^n,\mathbf{0})$ yields

$$m_\Omega\big((\mathbf{v}_h^n,\boldsymbol{\varepsilon}_h^n),(\mathbf{0},\boldsymbol{\Phi}_h^n)\big) + \triangle t_h^n a_h^{\mathrm{dg}}\big(z_h^n;(\mathbf{v}_h^n,\boldsymbol{\sigma}_h^n),(\mathbf{0},\boldsymbol{\Phi}_h^n)\big) = m_\Omega\big(\mathbf{v}_h^{n-1},\boldsymbol{\varepsilon}_h^{n-1}),(\mathbf{0},\boldsymbol{\Phi}_h^n)\big)\,,$$

i.e.,

$$\begin{aligned}\frac{1}{\triangle t_h}\big(\triangle\boldsymbol{\varepsilon}_h^n,\boldsymbol{\Phi}_h^n\big)_\Omega + \big(\mathbf{v}_h,\operatorname{div}\boldsymbol{\Phi}_h^n\big)_{\Omega_h} = \frac12\sum_{K\in\mathcal{K}_h}\sum_{f\in\mathcal{F}_K}\Bigg(&\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathrm{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\cdot\big(Z_{\mathrm{P}}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K\Big)_f\\ &+\Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{\mathrm{P}}(z_h)\mathbf{v}_{h,K}\big),\mathbf{n}_K\times\big(Z_{\mathrm{P}}(z_h)^{-1}[\boldsymbol{\Phi}_h]_{K,f}\mathbf{n}_K\Big)_f\Bigg)\,.\end{aligned}$$

Since the approximations $\mathbf{\Phi}_h^n \in W_h^{\rm dg} \cap \mathrm{H}_0^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ satisfy $[\mathbf{\Phi}_h^n]_{K,f} = \mathbf{0}$, we obtain

$$\big(\dot{\boldsymbol{\varepsilon}},\mathbf{\Phi}\big)_Q + \big(\mathbf{v},\operatorname{div}\mathbf{\Phi}\big)_Q = \lim_{h\in\mathcal{H}_0}\Big(\big(\dot{\boldsymbol{\varepsilon}}_h,\mathbf{\Phi}_h\big)_Q + \big(\mathbf{v}_h,\operatorname{div}\mathbf{\Phi}_h\big)_Q\Big) = \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\Big(\big(\triangle\boldsymbol{\varepsilon}_h^n,\mathbf{\Phi}_h^n\big)_\Omega + \triangle t_h\big(\mathbf{v}_h,\operatorname{div}\mathbf{\Phi}_h^n\big)_{\Omega_h}\Big) = 0\,, \tag{19}$$

so that, in case of positive viscosity, for $\mathbf{v}$ a weak symmetric gradient in space exists satisfying $\dot{\boldsymbol{\varepsilon}} = \operatorname{sym}(\mathrm{D}\mathbf{v})$. □

By the Aubin-Lions Lemma [Roubíček, 2013, Lem. 7.7], the embedding

$$\mathrm{H}^1\big(0,T;\mathrm{L}_2(\Omega)\big)\cap\mathrm{L}_2\big(0,T;\mathrm{H}^1(\Omega)\big)\longrightarrow\mathrm{L}_2(Q)$$

is compact. This yields strong convergence of the discrete phase field approximations in L_2.

Lemma 6. *We have strong convergence of $(z_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_2(Q)$, i.e., $\lim_{h\in\mathcal{H}_0}\|z_h - z\|_Q = 0$, and weak convergence of $(\boldsymbol{\sigma}_h)_{h\in\mathcal{H}_0}$ to $\boldsymbol{\sigma} = g(z)\mathbf{C}\boldsymbol{\varepsilon} + \mathbf{D}\partial_t\boldsymbol{\varepsilon} \in \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\rm sym})$.*

Proof. Since z_h is discontinuous in time, the Aubin-Lions Lemma cannot be applied directly. Thus we define

$$\hat{z}_h(t) = z^0 + \int_0^t \dot{z}_h(s)\,\mathrm{d}s \in V_h^{\rm cf}\,,\qquad t\in[0,T]\,,$$

so that $\hat{z}_h \in \mathrm{H}^1(0,T;V_h^{\rm cf})$ and $\partial_t\hat{z}_h = \dot{z}_h$; from $\dot{z}_h^n = \frac{1}{\triangle t_h^n}(z_h^n - z_h^{n-1})$ we get $\hat{z}_h(t_h^n) = z_h^n$ for $n = 0,\dots,N_h$, and using uniform time step sizes $\triangle t_h^n = \triangle t_h$ we obtain

$$\begin{aligned}\|z_h - \hat{z}_h\|_Q^2 &= \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\Big\|z_h^n - z_h^{n-1} - \frac{t - t_h^{n-1}}{\triangle t_h}(z_h^n - z_h^{n-1})\Big\|_\Omega^2\,\mathrm{d}t = \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\frac{(t_h^n - t)^2}{(\triangle t_h)^2}\big\|z_h^n - z_h^{n-1}\big\|_\Omega^2\,\mathrm{d}t\\ &= \sum_{n=1}^{N_h}\frac{\triangle t_h}{3}\|z_h^{n-1} - z_h^n\|_\Omega^2 = \sum_{n=1}^{N_h}\frac{(\triangle t_h)^3}{3}\|\dot{z}_h^n\|_\Omega^2 = \frac{(\triangle t_h)^2}{3}\|\dot{z}_h\|_Q^2\,. \end{aligned}\tag{20}$$

Since $(z_h)_{h\in\mathcal{H}_0}$ is converging weakly to $z\in\mathrm{L}_2(Q)$ and $(\dot{z}_h)_{h\in\mathcal{H}_0}$ is uniformly bounded in $\mathrm{L}_2(Q)$, also $(\hat{z}_h)_{h\in\mathcal{H}_0}$ is converging weakly to $z\in\mathrm{L}_2(Q)$. Then, we obtain

$$0 = \lim_{h\in\mathcal{H}_0}\big(\nabla z - \nabla z_h,\varphi\big)_Q = -\lim_{h\in\mathcal{H}_0}\big(z - z_h,\nabla\varphi\big)_Q = -\lim_{h\in\mathcal{H}_0}\big(z - \hat{z}_h,\nabla\varphi\big)_Q = \lim_{h\in\mathcal{H}_0}\big(\nabla z - \nabla\hat{z}_h,\varphi\big)_Q\,,\qquad \varphi\in\mathrm{C}_{\rm c}^1(Q)\,,$$

so that also $(\nabla\hat{z}_h)_{h\in\mathcal{H}_0}$ is converging weakly to ∇z, i.e., $(\hat{z}_h)_{h\in\mathcal{H}_0}$ is converging weakly to z in $\mathrm{L}_2\big(0,T;\mathrm{H}^1(\Omega)\big)$. Since in addition $(\dot{z}_h)_{h\in\mathcal{H}_0}$ is converging weakly to $\partial_t z\in\mathrm{L}_2(Q)$, we conclude that together $(\hat{z}_h)_{h\in\mathcal{H}_0}$ is converging to z weakly in $\mathrm{H}^1\big(0,T;\mathrm{L}_2(\Omega)\big)\cap\mathrm{L}_2\big(0,T;\mathrm{H}^1(\Omega)\big)$. Since the embedding to $\mathrm{L}_2(Q)$ is compact, we obtain strong convergence of $(\hat{z}_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_2(Q)$, and by (20) also $\lim\limits_{h\in\mathcal{H}_0}\|z_h - z\|_Q = 0$.

This implies also strong convergence of $\big(g(z_h)\big)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_2(Q)$. In addition, we have $g(z_h)\in\mathrm{L}_\infty(Q)$ for all $h\in\mathcal{H}_0$. Together with the weak convergence of $(\boldsymbol{\varepsilon}_h,\dot{\boldsymbol{\varepsilon}}_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\rm sym},\mathbb{R}^{d\times d}_{\rm sym})$ this yields for all $\mathbf{\Psi}\in\mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\rm sym})$

$$\begin{aligned}\lim_{h\in\mathcal{H}_0}\big(\boldsymbol{\sigma}_h,\mathbf{\Psi}\big)_Q &= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\boldsymbol{\sigma}_h^n,\Pi_h^{\rm dg}\mathbf{\Psi}^n\big)_\Omega = \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(\Pi_h^{\rm dg}(g(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h^n) + \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h^n,\Pi_h^{\rm dg}\mathbf{\Psi}^n\big)_\Omega\\ &= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\big(g(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h^n + \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h^n,\Pi_h^{\rm dg}\mathbf{\Psi}^n\big)_\Omega = \lim_{h\in\mathcal{H}_0}\big(g(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h + \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h,\mathbf{\Psi}\big)_Q\\ &= \lim_{h\in\mathcal{H}_0}\big(\mathbf{C}\boldsymbol{\varepsilon}_h,g(z_h)\mathbf{\Psi}\big)_Q + \lim_{h\in\mathcal{H}_0}\big(\mathbf{D}\dot{\boldsymbol{\varepsilon}}_h,\mathbf{\Psi}\big)_Q\\ &= \big(\mathbf{C}\boldsymbol{\varepsilon},g(z)\mathbf{\Psi}\big)_Q + \big(\mathbf{D}\partial_t\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q = \big(g(z)\mathbf{C}\boldsymbol{\varepsilon} + \mathbf{D}\partial_t\boldsymbol{\varepsilon},\mathbf{\Psi}\big)_Q\end{aligned}$$

with $\mathbf{\Psi}^n = \frac{1}{\triangle t_h^n}\int_{t_{n-1}}^{t_n}\mathbf{\Psi}(t)\,\mathrm{d}t$, so that $(\boldsymbol{\sigma}_h)_{h\in\mathcal{H}_0}$ is converging weakly. □

Using a lower semicontinuity result for Carathéodory functions we can now show that the limit solves the variational inequality for the phase field evolution.

Lemma 7. *The weak limit* $(z,\boldsymbol{\varepsilon}) \in \mathrm{H}^1\big(0,T;\mathrm{L}_2(\Omega)\big) \cap \mathrm{L}_2\big(0,T;\mathrm{H}^1(\Omega)\big) \times \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *of* $(z_h,\boldsymbol{\varepsilon}_h)_{h\in\mathcal{H}_0}$ *solves* (8b).

Proof. For a test function $\varphi \in \mathcal{Z}$ we define the approximation $\varphi_h \in \mathrm{H}^1(0,T;V_h^{\mathrm{cf}})$ by nodal interpolation in space defined by $\varphi_h(t_n,\mathbf{x}) = \varphi(t_n,\mathbf{x})$ for $\mathbf{x}\in\mathcal{N}_h$, and $n=0,\dots,N_h$, and by linear interpolation in time

$$\varphi_h(t) = \frac{1}{\triangle t_h^n}\Big((t_n-t)\varphi_h(t_{n-1}) + (t-t_{n-1})\varphi_h(t_n)\Big)\,, \qquad t\in(t_{n-1},t_n)\,,\ n=1,\dots,N_h\,. \tag{21}$$

Since $\varphi\le 0$ and we use lowest order finite elements, we also get $\varphi_h\le 0$ in Q. By construction, since φ is smooth, we have also strong convergence of the interpolation $(\varphi_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_\infty(Q)$.

Now we define $f(y,\boldsymbol{\xi}) = y\mathbf{C}\boldsymbol{\xi}:\boldsymbol{\xi}$ for $(y,\boldsymbol{\xi})\in\mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}$, and we observe that $f(\cdot,\cdot)$ is a Carathéodory function which is convex in $\boldsymbol{\xi}$. This is now used to show a lower semicontinuity of the functional

$$J\colon \mathrm{L}_2(Q)\times\mathrm{L}_\infty(Q)\times\mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \longrightarrow \mathbb{R}\,, \qquad J(z,\varphi,\boldsymbol{\varepsilon}) := \int_Q f(-g'(z)\varphi,\boldsymbol{\varepsilon})\,\mathrm{d}(t,\mathbf{x}) = \big(g'(z)\mathbf{C}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon},-\varphi\big)_Q\,.$$

The strong convergence of $(z_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_2(Q)$ by Lem. 6 and $(\varphi_h)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_\infty(Q)$ by construction yields strong convergence of $\big(g'(z_h)\varphi_h\big)_{h\in\mathcal{H}_0}$ in $\mathrm{L}_2(Q)$. Together with the weak convergence of $(\boldsymbol{\varepsilon})_{h\in\mathcal{H}_0}$ established in Lem. 5 this yields by [Dacorogna, 2008, Thm. 3.23] lower semicontinuity $\liminf_{h\in\mathcal{H}_0} J(z_h,\varphi_h,\boldsymbol{\varepsilon}_h) \ge J(z,\varphi,\boldsymbol{\varepsilon})$, i.e.,

$$\liminf_{h\in\mathcal{H}_0}\big(g'(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h:\boldsymbol{\varepsilon}_h,-\varphi_h\big)_Q \ge \big(g'(z)\mathbf{C}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon},-\varphi\big)_Q\,. \tag{22}$$

Inserting $\varphi_h^n = \varphi_h^n(t_h^{n-1/2})$ we observe $\triangle t_h^n\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n:\boldsymbol{\varepsilon}_h^n,-\varphi_h^n\big)_\Omega = \big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n:\boldsymbol{\varepsilon}_h^n,-\varphi_h\big)_{(t_h^{n-1},t_h^n)\times\Omega}$, since φ_h is linear and z_h and $\boldsymbol{\varepsilon}_h$ are constant in time in every interval (t_h^{n-1},t_h^n), so that we have

$$\big(g'(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h:\boldsymbol{\varepsilon}_h,-\varphi_h\big)_Q = \sum_{n=1}^{N_h}\triangle t_h^n\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n:\boldsymbol{\varepsilon}_h^n,-\varphi_h^n\big)_\Omega\,.$$

From (S1) we obtain

$$\begin{aligned}0 &= \sum_{n=1}^{N_h}\triangle t_h^n\Big(\frac{\tau_{\mathrm{r}}}{\triangle t_h^n}\big(\triangle z_h^n,\varphi_h^n\big)_\Omega + \frac{\theta_h}{\triangle t_h^n}\big(\partial M_+^2(\triangle z_h^n),\varphi_h^n\big)_\Omega + \frac{1}{2}\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1}:\boldsymbol{\varepsilon}_h^{n-1},\varphi_h^n\big)_\Omega + b_\Omega(z_h^n,\varphi_h^n)\Big)\\ &= \tau_{\mathrm{r}}\big(\dot z_h,\varphi_h\big)_Q + \theta_h\big(\partial M_+^2(\dot z_h),\varphi_h\big)_Q + \frac{1}{2}\sum_{n=1}^{N_h}\triangle t_h^n\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1}:\boldsymbol{\varepsilon}_h^{n-1},\varphi_h^n\big)_\Omega + b_Q(z_h,\varphi_h)\,,\end{aligned}$$

so that, using $-\varphi_h\ge 0$ and $\partial M_+^2(\dot z_h)\ge 0$,

$$\tau_{\mathrm{r}}\big(\dot z_h,\varphi_h\big)_Q + \frac{1}{2}\sum_{n=1}^{N_h}\triangle t_h^n\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1}:\boldsymbol{\varepsilon}_h^{n-1},\varphi_h^n\big)_\Omega + b_Q(z_h,\varphi_h) = \theta_h\big(\partial M_+^2(\dot z_h),-\varphi_h\big)_Q \ge 0\,. \tag{23}$$

For the next step, we consider the difference

$$\begin{aligned}&\big(g'(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h:\boldsymbol{\varepsilon}_h,-\varphi_h\big)_Q - \sum_{n=1}^{N_h}\triangle t_h^n\big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1}:\boldsymbol{\varepsilon}_h^{n-1},-\varphi_h^n\big)_\Omega - \triangle t_h^{N_h}\big(g'(z_h^{N_h})\mathbf{C}\boldsymbol{\varepsilon}_h^{N_h}:\boldsymbol{\varepsilon}_h^{N_h},-\varphi_h^{N_h}\big)_\Omega\\ &\quad= \triangle t_h^1\big(g'(z_h^1)\mathbf{C}\boldsymbol{\varepsilon}_h^0:\boldsymbol{\varepsilon}_h^0,-\varphi_h^1\big)_\Omega + \sum_{n=1}^{N_h-1}\triangle t_h^n\Big(\big(g'(z_h^{n+1})\mathbf{C}\boldsymbol{\varepsilon}_h^n:\boldsymbol{\varepsilon}_h^n,-\varphi_h^{n+1}\big)_\Omega - \big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n:\boldsymbol{\varepsilon}_h^n,-\varphi_h^n\big)_\Omega\Big)\,.\end{aligned}$$

Since φ is smooth, we obtain for the interpolation $\lim_{h\in\mathcal{H}_0}\big(\varphi_h^{n+1}-\varphi_h^n\big) = 0$ in $\mathrm{L}_\infty(\Omega)$, and since $z\in\mathrm{H}^1(0,T;\Omega)$ and thus continuous in time, and g' is continuous and bounded, we also observe $\lim_{h\in\mathcal{H}_0}\big(g'(z_h^{n+1})-g'(z_h^n),\psi\big)_\Omega = 0$ for all

$\psi \in \mathrm{L}_2(\Omega)$. Moreover, $\big(\mathbf{C}\boldsymbol{\varepsilon}_h^n, \boldsymbol{\varepsilon}_h^n\big)_\Omega$ is uniformly bounded, so that together

$$\lim_{h\in\mathcal{H}_0} \left(\big(g'(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_Q - \sum_{n=1}^{N_h} \triangle t_h^n \big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1}, -\varphi_h^n\big)_\Omega \right) = 0\,.$$

Combining this with (22) and inserting (23) yields

$$\begin{aligned}\big(g'(z)\mathbf{C}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, -\varphi\big)_Q &\le \liminf_{h\in\mathcal{H}_0} \big(g'(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h : \boldsymbol{\varepsilon}_h, -\varphi_h\big)_Q = \liminf_{h\in\mathcal{H}_0} \sum_{n=1}^{N_h} \triangle t_h^n \big(g'(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^{n-1} : \boldsymbol{\varepsilon}_h^{n-1}, -\varphi_h^n\big)_\Omega \\ &\le \liminf_{h\in\mathcal{H}_0} \Big(\tau_\mathrm{r}\big(\dot z_h, \varphi_h\big)_Q + b_Q(z_h,\varphi_h)\Big) = \lim_{h\in\mathcal{H}_0} \Big(\tau_\mathrm{r}\big(\dot z_h, \varphi_h\big)_Q + b_Q(z_h,\varphi_h)\Big) = \tau_\mathrm{r}\big(\partial_t z, \varphi\big)_Q + b_Q(z,\varphi)\end{aligned}$$

since $(z_h, \dot z_h)_{h\in\mathcal{H}_0}$ is converging weakly to $(z, \partial_t z)$ and $(\varphi_h)_{h\in\mathcal{H}_0}$ is converging strongly to φ. This proves (8b). $\square$

8. Convergence to a weak solution

Finally we show that the weak limit of the discrete solutions in Lem. 5 also solves (8a), so that together we obtain a weak solution of the elastodynamic phase field model.

Theorem 8. *The weak limit* $(z, \mathbf{v}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) \in \mathrm{H}^1(0,T;\mathrm{L}_2(\Omega)) \cap \mathrm{L}_2(0,T;\mathrm{H}^1(\Omega)) \times \mathrm{L}_2(Q;\mathbb{R}^d) \times \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathrm{L}_2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *of the sequence* $(z_h, \mathbf{v}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h)_{h\in\mathcal{H}_0}$ *is a weak solution of the variational system* (8).

Proof. For the limit, the variational inequality (8b) is established in Lem. 7.

For $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi}) \in \mathcal{V}_{T,\mathrm{D}} \times \mathcal{W}_T \times \mathcal{W}_{\mathrm{N}}$ let $(\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n, \boldsymbol{\Psi}_h^n) \in (V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}}) \cap \mathrm{C}^0(\Omega;\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$ be the nodal interpolation in space of $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})(t_h^n)$ defined by $(\mathbf{w}_h^n, \boldsymbol{\Phi}_h^n, \boldsymbol{\Psi}_h^n)(t_n, \mathbf{x}) = (\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})(t_h^n, \mathbf{x})$ for $\mathbf{x} \in \mathcal{N}_h$ and $n = 0, \dots, N_h$, and let $(\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h) \in \mathrm{H}^1(0,T;V_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}} \times W_h^{\mathrm{dg}})$ be the linear interpolation in time, cf. (21), so that we have strong convergence of $(\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h)_{h\in\mathcal{H}_0}$ to $(\mathbf{w}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$.

We set $(\mathbf{w}_h^{n-1/2}, \boldsymbol{\Phi}_h^{n-1/2}, \boldsymbol{\Psi}_h^{n-1/2}) = (\mathbf{w}_h, \boldsymbol{\Phi}_h, \boldsymbol{\Psi}_h)(t_h^{n-1/2})$ and observe $\partial_t \mathbf{w}_h(t) = \frac{1}{\triangle t_h^n} \triangle \mathbf{w}_h^n$ for $\triangle \mathbf{w}_h^n = \mathbf{w}_h^n - \mathbf{w}_h^{n-1}$ and $t \in (t_{n-1}, t_n)$, $n = 1, \dots, N_h$. Using $\mathbf{w}_h^{N_h} = \mathbf{0}$, we obtain

$$
\begin{aligned}
-\big(\varrho_0 \mathbf{v}_h, \partial_t \mathbf{w}_h\big)_Q = -\sum_{n=1}^{N_h} \big(\varrho_0 \mathbf{v}_h^n, \triangle \mathbf{w}_h^n\big)_\Omega &= -\sum_{n=1}^{N_h} \big(\varrho_0 \mathbf{v}_h^n, \mathbf{w}_h^n\big)_\Omega + \sum_{n=1}^{N_h} \big(\varrho_0 \mathbf{v}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega \\
&= \big(\varrho_0 \mathbf{v}_h^0, \mathbf{w}_h^0\big)_\Omega + \sum_{n=1}^{N_h} \big(\varrho_0 \triangle \mathbf{v}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega
\end{aligned}
$$

and for $\triangle \boldsymbol{\Phi}_h^n = \boldsymbol{\Phi}_h^n - \boldsymbol{\Phi}_h^{n-1}$ analogously, i.e., $-\big(\boldsymbol{\varepsilon}_h, \partial_t \boldsymbol{\Phi}_h\big)_Q = \big(\boldsymbol{\varepsilon}_h^0, \boldsymbol{\Phi}^0\big)_\Omega + \sum_{n=1}^{N_h} \big(\triangle \boldsymbol{\varepsilon}_h^n, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega$.

Since for $(\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})$ all jump terms and boundary terms vanish, we obtain consistency (10) for the DG bilinear form

$$
a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{v}_h^n, \boldsymbol{\sigma}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) = \big(\boldsymbol{\sigma}_h^n, \mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega + \big(\mathbf{v}_h^n, \mathrm{div}\, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega .
$$

Thus we obtain from (S2), since we assume homogenous boundary conditions $\mathbf{v}_{\mathrm{D}} = \mathbf{0}$ and $\mathbf{g}_{\mathrm{N}} = \mathbf{0}$,

$$
\begin{aligned}
&m_\Omega\big((\triangle \mathbf{v}_h^n, \triangle \boldsymbol{\varepsilon}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1} - \mathbf{D}\boldsymbol{\Psi}_h^{n-1})\big) \\
&\quad = \triangle t_h^n \Big(\ell_h^{\mathrm{dg}}\big(t_h^n, z_h^n; (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) - a_h^{\mathrm{dg}}\big(z_h^n; (\mathbf{v}_h^n, \boldsymbol{\sigma}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) - r_\Omega\big(z_h^n; (\boldsymbol{\varepsilon}_h^n, \boldsymbol{\sigma}_h^n), \boldsymbol{\Psi}_h^{n-1}\big)\Big) \\
&\quad = \triangle t_h^n \Big(\big(\mathbf{f}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega - \big(\boldsymbol{\sigma}_h^n, \mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega - \big(\mathbf{v}_h^n, \mathrm{div}\, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega - \big(\boldsymbol{\sigma}_h^n - g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n, \boldsymbol{\Psi}_h^{n-1}\big)_\Omega\Big) .
\end{aligned}
$$

This yields together with $m_\Omega\big((\triangle \mathbf{v}_h^n, \triangle \boldsymbol{\varepsilon}_h^n), (\mathbf{0}, \mathbf{D}\boldsymbol{\Psi}_h^{n-1})\big) = \big(\triangle \boldsymbol{\varepsilon}_h^n, \mathbf{D}\boldsymbol{\Psi}_h^{n-1}\big)_\Omega = \triangle t_h^n \big(\mathbf{D}\dot{\boldsymbol{\varepsilon}}_h^n, \boldsymbol{\Psi}_h^{n-1}\big)_\Omega$

$$
\begin{aligned}
&\big(\varrho_0 \mathbf{v}_h, \partial_t \mathbf{w}_h\big)_Q + \big(\boldsymbol{\varepsilon}_h, \partial_t \boldsymbol{\Phi}_h\big)_Q + \big(\varrho_0 \mathbf{v}_h^0, \mathbf{w}_h^0\big)_\Omega + \big(\boldsymbol{\varepsilon}_h^0, \boldsymbol{\Phi}^0\big)_\Omega \\
&\quad = -\sum_{n=1}^{N_h} \Big(\big(\varrho_0 \triangle \mathbf{v}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega + \big(\triangle \boldsymbol{\varepsilon}_h^n, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega\Big) = -\sum_{n=1}^{N_h} m_\Omega\big((\triangle \mathbf{v}_h^n, \triangle \boldsymbol{\varepsilon}_h^n), (\mathbf{w}_h^{n-1}, \boldsymbol{\Phi}_h^{n-1})\big) \\
&\quad = \sum_{n=1}^{N_h} \triangle t_h^n \Big(\big(\boldsymbol{\sigma}_h^n, \mathrm{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\big)_\Omega + \big(\mathbf{v}_h^n, \mathrm{div}\, \boldsymbol{\Phi}_h^{n-1}\big)_\Omega + \big(\boldsymbol{\sigma}_h^n - g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n - \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h^n, \boldsymbol{\Psi}_h^{n-1}\big)_\Omega - \big(\mathbf{f}_h^n, \mathbf{w}_h^{n-1}\big)_\Omega\Big)
\end{aligned}
$$

and thus in the limit, using strong convergence of the test functions and of $\big(g(z_h)\big)_{h\in\mathcal{H}_0}$,

$$
\begin{aligned}
&\varrho_0\left(\mathbf{v},\partial_t\mathbf{w}\right)_Q + \left(\boldsymbol{\varepsilon},\partial_t\boldsymbol{\Phi}\right)_Q + \varrho_0\left(\mathbf{v}_0,\mathbf{w}(0)\right)_\Omega + \left(\boldsymbol{\varepsilon}_0,\boldsymbol{\Phi}(0)\right)_\Omega\\
&\quad= \lim_{h\in\mathcal{H}_0}\Big(\left(\varrho_0\mathbf{v}_h,\partial_t\mathbf{w}_h\right)_Q + \left(\boldsymbol{\varepsilon}_h,\partial_t\boldsymbol{\Phi}_h\right)_Q + \left(\varrho_0\mathbf{v}_h^0,\mathbf{w}_h^0\right)_\Omega + \left(\boldsymbol{\varepsilon}_h^0,\boldsymbol{\Phi}^0\right)_\Omega\Big)\\
&\quad= \lim_{h\in\mathcal{H}_0}\sum_{n=1}^{N_h}\triangle t_h^n\Big(\left(\boldsymbol{\sigma}_h^n,\operatorname{sym}(\mathrm{D}\mathbf{w}_h^{n-1})\right)_\Omega + \left(\mathbf{v}_h^n,\operatorname{div}\boldsymbol{\Phi}_h^{n-1}\right)_\Omega + \left(\boldsymbol{\sigma}_h^n - g(z_h^n)\mathbf{C}\boldsymbol{\varepsilon}_h^n - \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h^n,\boldsymbol{\Psi}_h^{n-1}\right)_\Omega - \left(\mathbf{f}_h^n,\mathbf{w}_h^{n-1}\right)_\Omega\Big)\\
&\quad= \lim_{h\in\mathcal{H}_0}\Big(\left(\boldsymbol{\sigma}_h,\operatorname{sym}(\mathrm{D}\mathbf{w}_h)\right)_Q + \left(\mathbf{v}_h,\operatorname{div}\boldsymbol{\Phi}_h\right)_Q + \left(\boldsymbol{\sigma}_h - g(z_h)\mathbf{C}\boldsymbol{\varepsilon}_h - \mathbf{D}\dot{\boldsymbol{\varepsilon}}_h,\boldsymbol{\Psi}_h\right)_Q - \left(\mathbf{f}_h,\mathbf{w}_h\right)_Q\Big)\\
&\quad= \left(\boldsymbol{\sigma},\operatorname{sym}(\mathrm{D}\mathbf{w})\right)_Q + \left(\mathbf{v},\operatorname{div}\boldsymbol{\Phi}\right)_Q + \left(\boldsymbol{\sigma} - g(z)\mathbf{C}\boldsymbol{\varepsilon} - \mathbf{D}\dot{\boldsymbol{\varepsilon}},\boldsymbol{\Psi}\right)_Q - \left(\mathbf{f},\mathbf{w}\right)_Q.
\end{aligned}
$$

Thus, the weak limit solves (8). □

9. The energy-dissipation estimate

The energy-dissipation balance [Thomas and Tornquist, 2021, Def. 1.3]

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(t)) + \mathcal{E}^{\mathrm{el}}(z(t),\boldsymbol{\varepsilon}(t)) + \mathcal{E}^{\mathrm{pf}}(z(t)) - \mathcal{E}^{\mathrm{ext}}(t,\mathbf{u}(t)) + \int_0^t \big(\mathcal{R}^{\mathrm{el}}(\dot{\boldsymbol{\varepsilon}}(s)) + \mathcal{R}^{\mathrm{pf}}(\dot z(s))\big)\,\mathrm{d}s \\ = \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_0) + \mathcal{E}^{\mathrm{el}}(z_0,\boldsymbol{\varepsilon}_0) - \mathcal{E}^{\mathrm{ext}}(0,\mathbf{u}(0)) - \int_0^t \dot{\mathcal{E}}^{\mathrm{ext}}(s,\mathbf{u}(s))\,\mathrm{d}s$$

with

$$\dot{\mathcal{E}}^{\mathrm{ext}}(s,\mathbf{u}(s)) = \big(\partial_t \mathbf{f}(s),\mathbf{u}(s)\big)_\Omega\,\mathrm{d}\mathbf{x} + \big(\partial_t \mathbf{g}_{\mathrm{N}}(s),\mathbf{u}(s)\big)_{\partial_{\mathrm{N}}\Omega}$$

can be established for sufficiently regular solutions [Thomas and Tornquist, 2021, Thm. 5.1]; integration by parts yields

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(t)) + \mathcal{E}^{\mathrm{el}}(z(t),\boldsymbol{\varepsilon}(t)) + \mathcal{E}^{\mathrm{pf}}(z(t)) + \int_0^t \big(\mathcal{R}^{\mathrm{el}}(\dot{\boldsymbol{\varepsilon}}(s)) + \mathcal{R}^{\mathrm{pf}}(\dot z(s))\big)\,\mathrm{d}s = \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_0) + \mathcal{E}^{\mathrm{el}}(z_0,\boldsymbol{\varepsilon}_0) + \int_0^t \mathcal{E}^{\mathrm{ext}}(s,\mathbf{v}(s))\,\mathrm{d}s\,.$$

Here, with less regularity this is relaxed.

Lemma 9. *A subsequence $\mathcal{H}_1 \subset \mathcal{H}_0$ with $0 \in \overline{\mathcal{H}}_1$ exists, so that $(z_h(T),\mathbf{v}_h(T),\boldsymbol{\varepsilon}_h(T))_{h\in\mathcal{H}_1}$ is weakly converging to*

$$(z_T,\mathbf{v}_T,\boldsymbol{\varepsilon}_T) \in \mathrm{H}^1(\Omega)\times \mathrm{L}_2(\Omega;\mathbb{R}^d)\times \mathrm{L}_2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,. \tag{24}$$

Proof. For $h \in \mathcal{H}$ and $\mathbf{g}_{\mathrm{N}} = \mathbf{0}$, the discrete energy-dissipation inequality (15) takes the form

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^n) + \mathcal{E}^{\mathrm{el}}(z_h^n,\boldsymbol{\varepsilon}_h^n) + \mathcal{E}^{\mathrm{pf}}(z_h^n) + \int_0^{t_h^n} \Big(2\,\mathcal{R}^{\mathrm{el}}(\dot{\boldsymbol{\varepsilon}}_h(s)) + \mathcal{R}_h^{\mathrm{pf}}(\dot z_h(s))\Big)\,\mathrm{d}s \\ \le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0,\boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0) + \big(\mathbf{f}_h,\mathbf{v}_h\big)_{(0,t_h^n)\times\Omega}\,, \tag{25}$$

so that we obtain for $n = N_h$

$$\frac{\varrho_0}{2}\big\|\mathbf{v}_h(T)\big\|_\Omega^2 + \frac12\big\|g(z_h(T))^{1/2}\mathbf{C}^{1/2}\boldsymbol{\varepsilon}_h(T)\big\|_\Omega^2 + \frac{G_{\mathrm{c}}}{2}\Big(\big\|1-z_h(T)\big\|_\Omega^2 + l_{\mathrm{c}}^2\big\|\nabla z_h(T)\big\|_\Omega^2\Big) \\ \le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h(T)) + \mathcal{E}^{\mathrm{el}}(z_h(T),\boldsymbol{\varepsilon}_h(T)) + \mathcal{E}^{\mathrm{pf}}(z_h(T)) + \int_0^T \Big(2\,\mathcal{R}^{\mathrm{el}}(\dot{\boldsymbol{\varepsilon}}_h(s)) + \mathcal{R}_h^{\mathrm{pf}}(\dot z_h(s))\Big)\,\mathrm{d}s \\ \le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h^0) + \mathcal{E}^{\mathrm{el}}(z_h^0,\boldsymbol{\varepsilon}_h^0) + \mathcal{E}^{\mathrm{pf}}(z_h^0) + \big(\mathbf{f}_h,\mathbf{v}_h\big)_Q\,.$$

Thus, $(z_h(T),\mathbf{v}_h(T),\boldsymbol{\varepsilon}_h(T))_{h\in\mathcal{H}_0}$ is uniformly bounded in $\mathrm{H}^1(\Omega)\times\mathrm{L}_2(\Omega;\mathbb{R}^d)\times\mathrm{L}_2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, so that a weakly converging subsequence exists. □

In particular, this shows that for the weak solution $(z,\mathbf{v},\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ in $Q = (0,T)\times\Omega$ the evaluation at $t = T$ is well-defined with $(z(T),\mathbf{v}(T),\boldsymbol{\varepsilon}(T)) = (z_T,\mathbf{v}_T,\boldsymbol{\varepsilon}_T)$.

Lemma 10. *The weak limit (18) satisfies the energy-dissipation estimate*

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(t)) + \mathcal{E}^{\mathrm{el}}(z(t),\boldsymbol{\varepsilon}(t)) + \mathcal{E}^{\mathrm{pf}}(z(t)) + \int_0^t \big(\mathcal{R}^{\mathrm{el}}(\dot{\boldsymbol{\varepsilon}}(s)) + \mathcal{R}^{\mathrm{pf}}(\dot z(s))\big)\,\mathrm{d}s \le \mathcal{E}^{\mathrm{kin}}(\mathbf{v}_0) + \mathcal{E}^{\mathrm{el}}(z_0,\boldsymbol{\varepsilon}_0) + \int_0^t \mathcal{E}^{\mathrm{ext}}(s,\mathbf{v}(s))\,\mathrm{d}s \tag{26}$$

for all $t \in \mathcal{I}_{\mathcal{H}_0} = \{t_h^n \colon n = 0,\dots,N_h,\ h\in\mathcal{H}_0\}$.

Note that $\mathcal{I}_{\mathcal{H}_0} \subset [0,T]$ is dense.

Proof. We show the result for $t = T$ (the general case is open). For the weak limit we obtain the estimates

$$\mathcal{E}^{\mathrm{kin}}(\mathbf{v}(T)) \le \liminf_{h\in\mathcal{H}_1}\mathcal{E}^{\mathrm{kin}}(\mathbf{v}_h(T))\,,\quad \mathcal{E}^{\mathrm{el}}(z(T),\boldsymbol{\varepsilon}(T)) \le \liminf_{h\in\mathcal{H}_1}\mathcal{E}^{\mathrm{el}}(z_h(T),\boldsymbol{\varepsilon}_h(T))\,,\quad \mathcal{E}^{\mathrm{pf}}(z(T)) \le \liminf_{h\in\mathcal{H}_1}\mathcal{E}^{\mathrm{pf}}(z_h(T))\,, \\ \int_0^T \mathcal{R}^{\mathrm{pf}}(\dot z(s))\,\mathrm{d}s \le \liminf_{h\in\mathcal{H}_0}\int_0^T \mathcal{R}^{\mathrm{pf}}(\dot z_h(s))\,\mathrm{d}s \le \liminf_{h\in\mathcal{H}_0}\int_0^T \mathcal{R}_h^{\mathrm{pf}}(\dot z_h(s))\,\mathrm{d}s\,,$$

so that together we obtain (26) from (25) for the case $\mathbf{g}_{\mathrm{N}} = \mathbf{0}$. □

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