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Minimax and bayes estimation in deconvolution problem

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Abstract

We consider the problem of estimation of solution of convolution equation on observations blurred a random noise. The noise is a product of Gaussian stationary process and a weight function $\epsilon h \in L_2(R^1)$ with constant $\epsilon > 0$. The presence of weight function h makes the power of noise finite on R^1. This allows to suppose that the power of solution is also finite. For this model we find asymptotically minimax and Bayes estimators. The solution is supposed infinitely differentiable. The model with solutions having finite number of derivatives was studied in [5].

1 Introduction and main results

1.1 Introduction

Let the convolution equation

$$(a * x)(t) = \int_{-\infty}^{\infty} a(t-s)x(s)ds = f(t)$$

be given with the known kernel $a(t), t \in R^1$ and unknown function f. Instead of the function f we observe a realization of random process $y(t) = f(t) + \epsilon\eta(t)$ where $\epsilon\eta(t)$ is a random noise with constant $\epsilon > 0$. The objective is to estimate the solution x. Such a setting is usually called the deconvolution problem. The deconvolution problem arises in many applications (see e.g. [19], [18], [12], [10] and references therein).

The noise in this model is usually defined as Gaussian stationary process. Such a setting was comprehensively studied from different viewpoints. The statistical properties of Tikhonov regularizing algorithm , the procedure of Wiener filtration, robust and minimax estimators were considered in numerous publications (see, for example, [18], [19], [12], [4] and references therein) Last years interesting adaptive procedures (see [7], [16]), wavelet based estimators (see [2], [11], [16]) were proposed.

If the noise is stationary process, the power of noise on all real line is infinite. The reasonable estimator of solution $x(t)$ exists only if the ratio of power of noise to power of solution is finite. Thus we need to suppose that the power of solution is infinite or tends to infinity (see [18], [4]). Another method to avoid the problem of infinity is to reduce the setting to the deconvolution problem on a circle (see [3],

[11]). In practice the power of solution is usually finite. Thus it seems reasonable to study modification of model admitting finite power of solution on all real line. One such a modification is studied in the paper. Remind that the power of noise on interval (a,b) equals $\epsilon^2 \int_a^b E\eta^2(t)dt$.

In paper the noise has the more complicated structure

$$\epsilon\eta(t) = \epsilon h(t)\zeta(t) \tag{1.1}$$

Here $\zeta(t)$ is Gaussian stationary process, $E\zeta(t) = 0, E[\zeta(t)\zeta(0)] = r(t), t \in R^1$ and h is a weight function $h \in L^2(R^1)$. The assumption $h \in L_2(R^1)$ implies that the power $\epsilon^2 \int_{-\infty}^{\infty} E\eta^2(t)dt = \epsilon^2||h||^2 r(0)$ of noise $\epsilon\eta(t)$ is finite. This allows to suppose that the power of solution is also finite and to consider the problem with such more realistic assumptions.

For modification (1.1) of standard model we find asymptotically minimax and asymptotically Bayes estimators. We suppose that the solution is infinitely differentiable. The case of solutions having finite smoothness in such a setting has been considered in [5]. The problem of estimation with supersmooth solution was already considered in publications (see [8], [9], [17]). This papers are devoted to the signal estimation. The kernel a in the paper can be smooth or supersmooth. Thus in this paper together with [5] we study the both smooth and supersmooth spectrum of behaviour of kernel and solution. It turns out that, if the kernel or the solution is supersmooth, the asymptotically minimax and asymptotically Bayes estimators are the projection estimators. Only if both the solution and the kernel have the finite smoothnes, we are forced to define the estimators with the more complicated structure (see [5]). In this case the Wiener filters are asymptotically Bayes estimators (see [12], [18]) and standard minimax estimators are asymptotically minimax (see [3], [4], [16]). The standard minimax estimators remain asymptotically minimax in the case of supersmooth kernel or solution. However in these cases we can define essentially more simple estimators with the same property.

For any function $z \in L_2(R^1)$ denote by

$$Z(\omega) = \int \exp\{2\pi i\omega t\} z(t)dt$$

the Fourier transform of z and for any $z \in L_2(R^1)$ denote by

$$||z|| = \left(\int z^2(t)dt \right)^2$$

the L_2-norm of z. Hereafter the limits of integration are omitted if integration is taken over all real line.

The kernel a satisfies A0,A1-A3 in the case of finite smoothness and A0,A3,A4 in the supersmooth case.

A0.There holds $A(\omega) = A(-\omega) > 0$ for all $\omega \in R_1$.

A1. There holds

$$\lim_{t\to\infty}\int_0^t A^{-2}(\omega)R(\omega)d\omega = \infty.$$

A2. There exists $\gamma \ge 0$ such that for all $C > 0$

$$\lim_{\omega\to\infty}\frac{A(C\omega)}{A(\omega)} = C^{-\gamma}.$$

A3. There exists $C > 0$ such that for all $\omega, \omega_1 \in R^1$

$$||A(\omega) - A(\omega_1)| < C|\omega - \omega_1|.$$

A4. There exists $\gamma > 0$ such that for all $C > 0$

$$\lim_{\omega\to\infty}\frac{\log A(C\omega)}{\log A(\omega)} = C^{\gamma}.$$

The correlation function r satisfies the following

R. There exists $\alpha > 0$ such that for any $C > 0$

$$\lim_{\omega\to\infty} R(C\omega)/R(\omega) = C^{-\alpha}.$$

If A2,R hold with $\gamma = 0, \alpha = 0$ and $A(\omega) \equiv 1, R(\omega) \equiv 1$, we get the standard setting estimation of signal in weighted Gaussian white noise

$$dy(t) = x(t)dt + \epsilon h(t)dw(t) \tag{1.2}$$

with Gaussian white noise $dw(t)$. Note that the results of Theorems 1.1 and 1.3 in [5] are extended on the model (1.2) as well. The main difference of these theorems from theorems 1 and 3 of the paper is the assumption of finite smoothness of solution x.

1.2 Main Results. Minimax estimation

The assumption about the solution is rather standard (see [3], [5], [17]). We suppose that a priori information is given

$$x \in Q = \left\{x : \int B^2(\omega)|X(\omega)|^2 d\omega < 1, x \in L_2(R^1)\right\}$$

with the function B satisfying the following

B1 The function $B(\omega)$ is even, positive and there exists $\beta > 0$ such that for all $C > 0$

$$\lim_{\omega\to\infty}\frac{\ln B(C\omega)}{\ln B(\omega)} = C^{\beta}.$$

Thus we have a priori information that the solution x belongs ellipsoid in $L_2(R^1)$. B1 implies that the solution $x(t)$ is infinitely differentiable.

The risk of any estimator $x^*(t)$ equals

$$\rho_\epsilon(x^*) = \sup \int E(x^*(t) - x(t))^2 dt.$$

The goal is to find asymptotically minimax estimator x_ϵ^{**}

$$\rho_\epsilon = \rho_\epsilon(x_\epsilon^{**}) = \inf \rho_\epsilon(x*)(1 + o(1)), \quad \epsilon \to 0.$$

Here the infimum is over all estimators x^*.

We suppose that the function h is smoother then the realizations of random process $\zeta(t)$ (see H2 below). Thus all information on noise smoothness is contained in $\zeta(t)$.

H1 The functhion $H(\omega)$ is even, $H(\omega) \in L_2(R^1) \cap L_1(R^1)$ and $h(t) > 0$ for all $t \in R^1$.

H2 There exists $\delta > 0$ such that

$$\lim_{\omega \to \infty} R^{-1}(\omega) H^2(\omega) \omega^{1+\delta} = 0.$$

H3. $\int |th(t)| dt < \infty$.

Define the functions

$$\Psi_\epsilon(\mu) = \epsilon^2 ||h||^2 \int A^{-2}(\omega)(1 - \mu B(\omega))_+ R(\omega) d\omega,$$

$$\bar{\Psi}(\theta) = \epsilon^2 ||h||^2 \int_{-\theta}^{\theta} A^{-2}(\omega) R(\omega) d\omega.$$

Hereafter $(u)_+ = \max\{u, 0\}$ for all $u \in R^1$. We put $\omega_{1\epsilon} = \sup\{\omega : \bar{\Psi}_\epsilon(\omega) \le B^{-2}(\omega)\}$ and $\omega_\epsilon = \omega_{1\epsilon}(1 + \delta_\epsilon)$ where $\delta_\epsilon > 0$ is such that $B(\omega_{1\epsilon}) = o(B(\omega_\epsilon)), A^{-1}(\omega_{1\epsilon}) = A^{-1}(\omega_\epsilon)(1 + o(1))$ and $\delta_\epsilon \to 0$ as $\epsilon \to 0$. Denote $\mu_\epsilon = \sup\{\mu : B(\omega_\epsilon) > \mu\}$.

Define the kernels

$$K_{\mu_\epsilon}(\omega) = A^{-1}(\omega)(1 - \mu_\epsilon B(\omega))_+,$$
$$K_{\omega_\epsilon}(\omega) = A^{-1}(\omega)\chi(|\omega| < \omega_\epsilon).$$

Hereafter $\chi(U)$ denotes the indicator of events U.

Define also the kernels k_{α_ϵ} of Tikhonov regularization algorithms

$$K_\alpha(\omega) = \frac{A(\omega)}{A^2(\omega) + \alpha M(\omega)}$$

where the function $M(\omega)$ satisfies the following

M1. The function $M(\omega)$ is even, nonnegative and increasing in R^1_+.

M2. The function $M(\omega)|B^{-1}(\omega)A^{-2}(\omega)|$ is nondecreasing in R^1_+.

M2 implies that the function $M(\omega)$ has the exponential growth. The exponential growth of function $M(\omega)$ in Tikhonov regularizing algorithm is supposed usually in the case of a priori information on supersmooth solution (see [18]).

Define the parameters of regularization $\alpha_\epsilon = A^2(\omega_\epsilon)M^{-1}(\omega_\epsilon)$.

Theorem 1. *Assume* A0,A1,A2,A3,B1,H1-H3,R *or* A0,A3,A4,B1,H1-H3,R *with* $\beta > \gamma$. *Then the estimators* $x^{**}_{\mu_\epsilon} = (k_{\mu_\epsilon} * y)(t)$ *and* $x^{**}_{\omega_\epsilon}(t,y) = (k_{\omega_\epsilon} * y)(t)$ *are asymptotically minimax. The asymptotically minimax risks equal*

$$\rho_\epsilon(x^{**}_{\mu_\epsilon}) = \rho_\epsilon(x^{**}_{\omega_\epsilon})(1+o(1)) = \Psi_\epsilon(\mu_\epsilon)(1+o(1)) = \bar{\Psi}_\epsilon(\omega_\epsilon)(1+o(1)). \tag{1.3}$$

If M1,M2 *hold also, the Tikhonov regularization algorithm* $x^{**}_{\alpha_\epsilon} = k_{\alpha_\epsilon} * y$ *is asymptotically minimax.*

Remark 1. Denote Λ the set of functions h satisfying H1,H3 and such that the convergence in H2 is uniform w.r.t. all $h \in \Lambda$. Suppose also that there exist functions $h_{01}, h_0 \in \Lambda$ such that $h_{01} \ge h(t) \ge h_0(t) > 0, t \in R^1$ for all $h \in \Lambda$. Then the asymptotics of minimax risks $\rho_{\epsilon h}(x^{**}_{\mu_\epsilon}) = \rho_{\epsilon h}(x^{**}_{\omega_\epsilon})(1+o(1))$ are uniform with respect to $h \in \Lambda$. Moreover all information on h in the estimators $x^{**}_{\omega_\epsilon}, x^{**}_{\mu_\epsilon}$ is contained in $||h||$. Thus we can consider h as unknown in the model supposing only $h \in \Lambda$. Naturally, since $||h||$ is unknown, the problems of the choice of regularization parameters μ_ϵ and ω_ϵ arise. However, since other parameters of models are usually unknown as well (for example $R(\omega), B(\omega)$), these problems arise almost always in practice. Note that similar statements on uniform risks convergence hold for the other theorems of this paper and theorems of [5] as well.

Remark 2. In practice another model of deconvolution (see [2], [7], [14], [16]) often arises. Let a sample $Z_1, \dots, Z_n$ of independent random observations be given. Let it be known that $Z_1 = X_1 + Y_1, \dots, Z_n = X_n + Y_n$ where $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are independent identically distributed random variables from R^1. The density $a(t)$ of $Y_1, \dots, Y_n$ is known. One needs to estimate the density $x(s), s \in R^1$ of $X_1, \dots, X_n$. To distinguish this model we call such a setting the problem of density deconvolution.

As almost all nonparametric statistical problem the model of density deconvolution admits asymptotic version in terms of Gaussian white noise $dw(t)$

$$dy(t) = (a * x)(t)dt + \frac{1}{\sqrt{n}}((a * x)(t))^{1/2}dw(t). \tag{1.4}$$

A wide class of linear estimators have the same asymptotic distributions in these two models.

The relation of the model of density deconvolution with the weighted white noise model almost has not been studied. However such a relation is wellknown in a particular case of nonparametric estimation of density (see [15]). In this case the observations $Y_1,\dots,Y_n$ are absent, $A(\omega)\equiv 1$ and stochastic equation (1.4) is the following

$$dy(t) = x(t)dt + \frac{1}{\sqrt{n}} x^{1/2}(t)dw(t).$$

In the model (1.4) the function $h(t) = ((a*x)(t))^{1/2}$ is unknown. However the estimators $k_{\mu_\epsilon} * y$ and $k_{\omega_\epsilon} * y$ depend only on $||h|| = ||((a*x)(t))^{1/2}|| = 1$ and, by Remark 1, the asymptotics of minimax risks are uniform w.r.t. unknown $h \in \Lambda$. Thus if $h = (a*x)^{1/2} \in \Lambda$ these estimators have the same risk asymptotics in the model (1.4) and in the paper model.

If the kernel a is smoother then the solution x and $\gamma > \beta$ in A3,B1, the projection estimator $x^{**}_{\omega_\epsilon} = k_{\omega_\epsilon} * y$ and the Tikhonov regularizing algorithm $x^{**}_{\alpha_\epsilon} = k_{\alpha_\epsilon} * y$ remain asymptotically minimax. However the asymptotic of minimax risks is defined differently.

Let $\omega_{1\epsilon}$ satisfy the equation

$$B^{-2}(\omega_{1\epsilon}) = \epsilon^2 ||h||^2 \int_{-\omega_{1\epsilon}}^{\omega_{1\epsilon}} A^{-2}(\omega)R(\omega)d\omega. \tag{1.5}$$

Define δ_ϵ such that $\delta_\epsilon = o(\omega_{1\epsilon}^{-\beta})$ and $\omega_{1\epsilon}^{-\gamma} = o(\delta_\epsilon)$. Denote $\omega_\epsilon = (1-\delta_\epsilon)\omega_{1\epsilon}$ and $\alpha_\epsilon = A^2(\omega_\epsilon)M^{-1}(\omega_\epsilon)$.

Theorem 2. *Assume* A0,A4,B1,H1,H2 *and* R. *Let $\gamma > \beta$. Then the family of projection estimators $x^{**}_{\omega_\epsilon}(t,y) = (k_{\omega_\epsilon} * y)(t)$ is asymptotically minimax. There holds*

$$\rho_\epsilon(x^{**}_{\omega_\epsilon}) = B^{-2}(\omega_\epsilon)(1+o(1)). \tag{1.6}$$

If M1 *holds, the Tikhonov regularizing algorithm $x^{**}_{\alpha_\epsilon} = k_{\alpha_\epsilon} * y$ is also asymptotically minimax.*

Example 1 Let

$$A(\omega) = A_1(\omega)|\omega|^{-\gamma}, \tag{1.7}$$

$$B(\omega) = C\exp\{-B_1(\omega)|\omega|^{\beta}\}, \tag{1.8}$$

$$R(\omega) = R_1(\omega)|\omega|^{-\alpha}. \tag{1.9}$$

Then

$$\omega_\epsilon = \frac{|\ln\epsilon|^{1/\beta}}{|B_1(|\ln\epsilon|^{1/\beta})|^{1/\beta}}(1+o(1)), \tag{1.10}$$

$$\rho_\epsilon = \frac{2||h||^2}{2\gamma-\alpha+1}\epsilon^2\omega_\epsilon^{2\gamma-\alpha+1}A_1^{-2}(\omega_\epsilon)R_1(\omega_\epsilon)(1+o(1)).$$

Example 2. Let

$$A(\omega) = C\exp\{-A_1(\omega)|\omega|^{\gamma}\} \tag{1.11}$$

and (1.8), (1.9) hold with $\gamma < \beta$. Suppose that $A_1(\omega) = A_1, B_1(\omega) = B_1$ are constants if $2\gamma > \beta$. Then the values of ω_ϵ is defined (1.10) and

$$\rho_\epsilon = \frac{2}{\gamma} C^{-2} ||h||^2 \epsilon^2 \omega_\epsilon^{1-\gamma-\alpha} R_1(\omega_\epsilon) \exp\left\{ 2A_1(\omega_\epsilon)|\omega_\epsilon|^\gamma \left(1 - \frac{\gamma}{\phi} \frac{A_1(\omega_\epsilon)}{B_1(\omega_\epsilon)} \omega_\epsilon^{\gamma-\beta} \right) \right\} (1+o(1)).$$

Example 3. Let (1.7),(1.8),(1.11) hold with $\gamma > \beta$. Suppose that $A_1(\omega) = A_1, B_1(\omega) = B_1$ are constants if $2\beta > \gamma$. Then

$$\omega_\epsilon = \frac{|\ln \epsilon|^{1/\gamma}}{|A_1(|\ln \epsilon|^{1/\gamma})|^{1/\gamma}} (1 + o(1)), \tag{1.12}$$

$$\rho_\epsilon = \exp\left\{ -2B_1(\omega_\epsilon)\omega_\epsilon^\beta \left(1 - \frac{\beta}{\gamma} B_1(\omega_\epsilon) A_1^{-1}(\omega_\epsilon) \omega_\epsilon^{\beta-\gamma} \right) \right\} (1 + o(1)).$$

1.3 Main Results. Bayes Approach.

In Bayes setting we suppose that the solution x is a realization of random process

$$x(t) = h_1(t)\zeta(t)$$

where $h_1(t) \in L_2(R^1)$ and ζ is Gaussian stationary random process, $E\zeta(t) = 0, E[\zeta(t)\zeta(0)] = v(t), t \in R^1$.

As follows from assumption V1 given bellow the realizations of random process $\zeta(t)$ are infinitely differentiable.

V1. There exists $\beta > 0$ such that for all $C > 0$ there holds

$$\lim_{\omega \to \infty} \frac{\ln V(C\omega)}{\ln V(\omega)} = C^\beta.$$

The function $h_1(t)$ satisfies the following assumptions.

H4 The function $h_1(t)$ is even,bounded and $h_1(t) > ch(t) > 0$ with constant $c > 0$ for all $t \in R^1$.

H5. There holds

$$\lim_{\omega \to \infty} \frac{\ln H_1(\omega)}{\ln V(\omega)} = \infty.$$

H5 implies that the main information on smoothness x is contained in the random process $\zeta(t)$.

For any estimator x^* define the Bayes risk

$$\hat\rho_\epsilon(x^*) = E_\zeta E_\xi ||x^* - h_1\zeta||^2$$

We say that the estimator $\bar x^*_\epsilon$ is asymptotically Bayes if

$$\hat\rho_\epsilon = \hat\rho_\epsilon(\bar x^*_\epsilon) = \inf \hat\rho_\epsilon(x^*)(1 + o(1)), \qquad \epsilon \to 0.$$

Here the infimum is over all estimators.

In this setting we could not prove that the Wiener filters

$$K_\epsilon(\omega) = ||h_1||^2 A(\omega)V(\omega)(||h_1||^2 A^2(\omega)V(\omega) + \epsilon^2||h||^2 R(\omega))^{-1}$$

are asymptotically Bayes. At the same time we show that more simple projection estimator $x^{**}_{\omega_\epsilon}$ are asymptotically Bayes. The value of ω_ϵ is defined the equation $\omega_\epsilon = \omega_{1\epsilon}(1+\delta_\epsilon)$ where

$$A^2(\omega_{1\epsilon})||h_1||^2 V(\omega_{1\epsilon}) = \epsilon^2||h||^2 R(\omega_{1\epsilon}), \tag{1.13}$$

$$V(\omega_\epsilon) = o(V(\omega_{1\epsilon})), \quad H_1^2\left(\frac{1}{2}\delta_\epsilon\omega_{1\epsilon}\right) = o(\epsilon^2 A^{-2}(\omega_\epsilon)R(\omega_\epsilon)) \tag{1.14}$$

and $\delta_\epsilon > 0, \delta_\epsilon \to 0$ as $\epsilon \to 0$.

Theorem 3. *Assume* **A0,A1,A2,H1,H2,H4,H5,R,V1**. *Then the family of estimators* $x^{**}_{\omega_\epsilon} = (k_{\omega_\epsilon} * y)(t)$ *is asymptotically Bayes. There holds*

$$\hat{\rho}_\epsilon(x^{**}_{\omega_\epsilon}) = \epsilon^2||h||^2 \int_{-\omega_\epsilon}^{\omega_\epsilon} A^{-2}(\omega)R(\omega)d\omega(1+o(1)). \tag{1.15}$$

Example 4. Let

$$V(\omega) = \exp\{-V_1(\omega)|\omega|^\beta\}$$

and (1.7),(1.9) hold. Then

$$\omega_\epsilon = \frac{|\ln\epsilon|^{1/\beta}}{|V_1(|\ln\epsilon|^{1/\beta})|^{1/\beta}}(1+o(1)),$$

$$\hat{\rho}_\epsilon = \frac{2||h||^2}{2\gamma-\alpha+1}\epsilon^2\omega_\epsilon^{2\gamma-\alpha+1}A_1^{-2}(\omega_\epsilon)R_1(\omega_\epsilon)(1+o(1)).$$

Remark 3. In practice an information on a kernel a can be often obtained only from statistical experiment (see [2], [14]). The arising estimator $\hat{a}$ of kernel a is known with a random error. Usually it is supposed that the error admits the Gaussian approximation. If we study the quality of estimation of solution one needs to find the influence of this random error on a risk function.

We consider the following model

$$\hat{a}(t) = a(t)dt + \kappa_\epsilon\bar{h}(t)dw(t)$$

with $\bar{h} \in L_2(R^1)$. Assume the following

$$\int t^2\bar{h}^2(t)dt < \infty, \tag{1.16}$$

$$\lim_{\omega\to\infty} \bar{H}^2(\omega)|\omega|^{1+\delta} = 0 \tag{1.17}$$

with $\delta > 0$.

The Fourier transform of $\hat{a}$ can be written in the following form

$$\hat{A}(\omega) = A(\omega) + \kappa_\epsilon \hat{\tau}(\omega)$$

where

$$\hat{\tau}(\omega) = \int \bar{H}(\omega - \omega_1) dw(\omega_1)$$

is Gaussian stationary process.

If $M_\epsilon = \sup\{|\hat{\tau}(\omega), |\omega| \le \omega_\epsilon\} = o_P(A(\omega_\epsilon)/\kappa_\epsilon)$, the random error $\kappa_\epsilon \hat{\tau}(\omega)$ does not influence on minimax and Bayes estimators $x^{**}_{\omega_\epsilon}, x^{**}_{\mu_\epsilon}$.

If (1.16),(1.17) holds, by Theorem 12.3.5 in [13]

$$\lim_{\epsilon \to 0} P\left(||h||^{-1}(2\ln(2\omega_\epsilon))^{-1/2}|M_\epsilon - ||h||(2\ln(2\omega_\epsilon))^{1/2}| > u\right) = \exp\{-C\exp\{-u\}\}.$$

Hence, if $\kappa_\epsilon = o\left(A(\omega_\epsilon)|\ln \omega_\epsilon|^{1/2}\right)$, the influence of noise $\kappa_\epsilon \bar{h}(t)dw(t)$ on the asymptotics of risk functions of estimator $x^{**}_{\omega_\epsilon}$ is negligible both in minimax and Bayes settings. This is proved by straightforward calculations. A similar statement hold also for minimax estimators of Theorems 1.1 and 1.2 in [6]. Note that $\epsilon = o\left(A(\omega_\epsilon)(\ln \omega_\epsilon)^{1/2}\right)$. Thus we can estimate the kernel with essentially larger error then ϵ. The same remark can be made also about the systematic error of kernel estimator.

2 Proofs of Theorems

2.1 Proof of lower bound in Theorem 1.1

The proof of lower bounds in minimax setting is based traditionally on the fact that the Bayes risk does not exceed the minimax one. We define such Bayes a priori distributions that the powers of realizations $\int_\omega^{\omega+\Delta\omega} X^2(u)du$ on each interval $(\omega, \omega+\Delta\omega) \subset \Omega_\epsilon = (-\omega_{1\epsilon}(1-\delta_\epsilon), \omega_{1\epsilon}(1-\delta_\epsilon))$ have the large order then corresponding power of noise $\epsilon^2||h||^2 \int_\omega^{\omega+\Delta\omega} A^{-2}(u)R(u)du$. This allows to choose $A^{-1}(\omega)Y(\omega)$ as asymptotically Bayes estimators $X(\omega)$ on the intervals Ω_ϵ and to get the required lower bound (1.3) for minimax risks.

In what follows, we put $\gamma = 0$ if A2 holds.

We put $\omega_{2\epsilon} = (1 - \delta_\epsilon)\omega_{1\epsilon}$ where $\delta_\epsilon > 0$ is such that $A(\omega_{2\epsilon}) = A(\omega_\epsilon)(1 + o(1))$ and

$$B^{-1}(\omega_{1\epsilon}) = o(B^{-1}(\omega_{2\epsilon})), \quad \epsilon^2 A^{-2}(\omega_{2\epsilon})(\omega_\epsilon^{1+\gamma+\beta} R^{-1}(\omega_\epsilon) + \omega_\epsilon^3) = o(B^{-2}(\omega_{2\epsilon})) \quad (2.1)$$

and $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

Define D_ϵ such that

$$D_\epsilon = o(B^{-1}(\omega_{2\epsilon})\omega_\epsilon^{(1-\beta)/2}), \quad \epsilon^2 A^{-2}(\omega_{2\epsilon})(\omega_\epsilon^{1+\gamma+\beta} R^{-1}(\omega_\epsilon) + \omega_\epsilon^3) = o(D_\epsilon^2). \quad (2.2)$$

The Bayes a priori measures λ_ϵ are defined as conditional probability measures of Gaussian random processes ζ_ϵ under the condition $\zeta_\epsilon \in Q$. The Fourier transform $\hat{\zeta}_\epsilon(\omega) = \hat{\zeta}_{\epsilon\Delta_\epsilon}(\omega)$ of random process $\zeta_\epsilon = \zeta_{\epsilon\Delta_\epsilon}$ equals

$$\hat{\zeta}_\epsilon(\omega) = D_\epsilon \sum_{i=-l}^{l} \eta_i \chi(\omega \in I_i)$$

where $l = l_\epsilon = [\omega_{2\epsilon}/\Delta_\epsilon], I_i = ((i-1/2)\Delta_\epsilon, (i+1/2)\Delta_\epsilon)$ and η_i are independent Gaussian random variables, $E\eta_i = 0, E\eta_i^2 = 1, -l \le i \le l$. The value of parameter $\Delta = \Delta_\epsilon$ is such that

$$\Delta = \Delta_\epsilon = o(\epsilon^2 D_\epsilon^{-2} R(\omega_\epsilon)\omega_\epsilon^{-\gamma} + \omega_\epsilon^{-3/2-\gamma} R(\omega_\epsilon) + \omega_\epsilon^{1-\beta}). \tag{2.3}$$

Hereafter $[\omega_\epsilon/\Delta]$ denotes the whole part of ω_ϵ/Δ.

Denote ν_ϵ the probability measure of ζ_ϵ.

Lemma 1. *There holds*

$$\lim_{\epsilon\to 0} P(\zeta_\epsilon \in Q) = 1 \tag{2.4}$$

Proof. By straightforward calculations, we get

$$E\left[\int B^2(\omega)\hat{\zeta}_\epsilon^2(\omega)d\omega\right] = O(D_\epsilon^2 B^2(\omega_{2\epsilon})\omega_\epsilon^{1-\beta}),$$

$$\mathrm{Var}\left[\int B^2(\omega)\hat{\zeta}_\epsilon^2(\omega)d\omega\right] = O(D_\epsilon^4 B^4(\omega_{2\epsilon})\omega_\epsilon^{1-\beta})$$

as $\epsilon \to 0$. Hence, by Chebyshov inequality, using (2.2), we get (2.4). This completes the proof of Lemma 1.

For any estimator x_ϵ^* define the Bayes risks

$$\hat{\rho}_{\epsilon\lambda}(x_\epsilon^*) = \int_Q d\lambda_\epsilon(x) E||x_\epsilon^* - x||^2,$$

$$\hat{\rho}_{\epsilon\nu}(x_\epsilon^*) = \int_Q d\nu_\epsilon(x) E||x_\epsilon^* - x||^2.$$

Denote $\bar{x}_\epsilon^*$ and $\bar{x}_\epsilon$ the Bayes estimators corresponding to a priori measures λ_ϵ and ν_ϵ respectively.

Lemma 2. *There holds*

$$\hat{\rho}_{\epsilon\lambda}(\bar{x}_\epsilon^*) \ge \hat{\rho}_{\epsilon\nu}(\bar{x}_\epsilon)(1+o(1)), \quad \epsilon \to 0.$$

The proof of Lemma 2 is akin to that of Lemma 2.3 in [6] and is omitted.

It follows from Lemmas 1 and 2 that the main term of asymptotic of minimax risks ρ_ϵ does not exceed the corresponding term for the Bayes risks of random processes ζ_ϵ. Thus it suffices to find the asymptotic of Bayes risks $\hat{\rho}_{\epsilon\nu}(\bar{x}_\epsilon)$.

Since the random process ζ_ϵ is Gaussian the Bayes estimator is linear

$$\bar{x}_\epsilon = \int k_\epsilon(t,s)y(s)ds$$

with the kernel k_ϵ satisfying the equation

$$V_\Delta(\omega,\omega_0)A(\omega_0) = \int K_\epsilon(\omega,\omega_1)A(\omega_1)V_\Delta(\omega_1,\omega_0)d\omega_1 A(\omega_0)+$$

$$\epsilon^2 \int K_\epsilon(\omega,\omega_2)\int H(\omega_2-\omega_1)R(\omega_1)H(\omega_1-\omega_0)d\omega_1 d\omega_2, \quad \omega,\omega_0 \in R^1. \tag{2.5}$$

Hereafter $V_\Delta = V_{\Delta_\epsilon}(\omega_1,\omega_0) = E[\zeta_\epsilon(\omega_1)\zeta_\epsilon(\omega_0)] = D_\epsilon^2 \sum_{i=-l}^{i=l} \chi(\omega_1 \in I_i, \omega_0 \in I_i)$ with $\omega_0,\omega_1 \in R^1$.

In what follows, we shall make use of operator symbolic in the problem of risk minimization. Denote A, R the multiplication operators with the kernels $A(\omega), R(\omega)$. Denote K_ϵ, V_Δ, H the integral operators $L_2(R^1) \to L_2(R^1)$ with the kernels $K_\epsilon(\omega,\omega_0)$, $V_\Delta(\omega,\omega_0), H(\omega-\omega_0)$ with $\omega,\omega_0 \in R^1$ respectively and denote I the unit operator. In this notation the Bayes estimator minimizes the Bayes risk

$$\rho_\epsilon(K) = \mathrm{Sp}\left[(K^TA-I)V_\Delta(AK-I)+\epsilon^2K^THRHK\right] \tag{2.6}$$

among all estimators $x_\epsilon^* = k * y$. Here $Sp[U]$ denotes the trace of operator U.

Denote $R_\epsilon(\omega) = R(\omega)\chi(|\omega| < \omega_\epsilon)$ and

$$\bar{R}_\epsilon(\omega,\omega_0) = \int H(\omega-\omega_1)R_\epsilon(\omega_1)H(\omega_1-\omega_0)d\omega_1\chi(\omega \in (-\omega_\epsilon,\omega_\epsilon),\omega_0 \in (-\omega_\epsilon,\omega_\epsilon)).$$

Since $R_\epsilon \le R$, we get

$$\bar{\rho}_\epsilon(K) = \mathrm{Sp}[(K^TA-I)V_\Delta(AK-I)+\epsilon^2K^T\bar{R}_\epsilon K] \le$$

$$\mathrm{Sp}[(K^TA-I)V_\Delta(AK-I)+\epsilon^2K^THR_\epsilon HK] \le \rho_\epsilon(K). \tag{2.7}$$

Hence it suffices to consider the problem of minimization of $\bar{\rho}_\epsilon(K)$. Define operator $\bar{K}_\epsilon$ such that $\bar{\rho}_\epsilon(\bar{K}_\epsilon) = \inf \bar{\rho}_\epsilon(K)$ where the infimum is over all operators K such that $\bar{\rho}_\epsilon(K) < \infty$.

If we write a version of equation (2.5) for the operator $\bar{K}_\epsilon$, we get easily

$$\bar{K}_\epsilon = V_\Delta A(AV_\Delta A + \epsilon^2 HR_\epsilon H)^{-1}. \tag{2.8}$$

It follows from definition of $\bar{\rho}_\epsilon(K)$ that in the problem of minimization of $\bar{\rho}_\epsilon(K)$ it suffices to consider the operators $A, V_\Delta, \bar{R}_\epsilon$ as operators $L_2((-\omega_\epsilon,\omega_\epsilon)) \to L_2((-\omega_\epsilon,\omega_\epsilon))$. Thus, in what follows, we suppose that such a contruction of operators takes place.

In this setting it holds

$$||\bar{K}_\epsilon|| \le ||A^{-1}||\,||AK_\epsilon|| \le A^{-1}(\omega_{2\epsilon}). \tag{2.9}$$

Substituting (2.8) in the left hand side of (2.7), we get

$$\bar\rho_\epsilon(\bar K_\epsilon)=\epsilon^4\mathrm{Sp}[(AV_\Delta A+\epsilon^2\bar R_\epsilon)^{-1}\bar R_\epsilon V_\Delta \bar R_\epsilon(AV_\Delta A+\epsilon^2\bar R_\epsilon)^{-1}]+$$
$$\epsilon^2\mathrm{Sp}[(AV_\Delta A+\epsilon^2\bar R_\epsilon)^{-1}AV_\Delta \bar R_\epsilon V_\Delta A(AV_\Delta A+\epsilon^2\bar R_\epsilon)^{-1}]. \qquad (2.10)$$

Now we define some discrete version $\bar\rho_{\epsilon\Delta}(\bar K_\epsilon)$ of $\bar\rho_\epsilon(\bar K_\epsilon)$ and show that $|\bar\rho_{\epsilon\Delta}(\bar K_\epsilon)-\bar\rho_\epsilon(K_\epsilon)|$ is negligible. After that the lower bound of $\bar\rho_{\epsilon\Delta}(\bar K_\epsilon)$ will be found.

For each $i, |i|\le l$ we fix $\omega_i\in I_i$ and define the functions

$$A_\Delta(\omega)=\sum_{i=-l}^{l}A(\omega_i)\chi(\omega\in I_i),$$

$$H_\Delta(\omega,\omega_1)=\sum_{i=-l}^{l}H(\omega-\omega_1)\chi(\omega_1\in I_i).$$

$$R_\Delta(\omega_0,\omega_1)=(H_\Delta^T R_\epsilon H_\Delta)(\omega_0,\omega_1).$$

Denote

$$\bar\rho_{\epsilon\Delta}(\bar K_\epsilon)=\mathrm{Sp}\left[(\bar K_\epsilon^T A_\Delta-I)V_\Delta(A_\Delta\bar K_\epsilon-I)+\epsilon^2\bar K_\epsilon^T R_\Delta\bar K_\epsilon\right]. \qquad (2.11)$$

We have

$$\bar\rho_\epsilon(\bar K_\epsilon)-\bar\rho_{\epsilon\Delta}(\bar K_\epsilon)=\mathrm{Sp}[\bar K_\epsilon^T(A-A_\Delta)V_\Delta(A\bar K_\epsilon-I)]+$$
$$\mathrm{Sp}[\bar K_\epsilon^T A_\Delta V_\Delta(A-A_\Delta)\bar K_\epsilon]+\epsilon^2\mathrm{Sp}[\bar K_\epsilon^T(\bar R_\epsilon-R_\Delta)\bar K_\epsilon]\doteq J_1+J_2+\epsilon^2 J_3. \qquad (2.12)$$

Since $||A\bar K_\epsilon-I||\le 1$ and $||\bar K_\epsilon A||\le 1$, by A4, (2.9), we get

$$|J_1|\le A^{-1}(\omega_{2\epsilon})||A\bar K_\epsilon||||A-A_\Delta||\,\mathrm{Sp}[V_\Delta]\,||A\bar K_\epsilon-I||\le CA^{-1}(\omega_{2\epsilon})\Delta\mathrm{Sp}[V_\Delta]. \qquad (2.13)$$

Hereafter C stands for positive constants.

Arguing similarly, we get

$$|J_2|\le A^{-1}(\omega_{2\epsilon})||A\bar K_\epsilon||\,||A_\Delta||\mathrm{Sp}[V_\Delta]||A-A_\Delta||\,A^{-1}(\omega_{2\epsilon})\,||A\bar K_\epsilon||\le C\Delta A^{-2}(\omega_{2\epsilon})\mathrm{Sp}[V_\Delta]. \qquad (2.14)$$

We have

$$J_3=\mathrm{Sp}[\bar K_\epsilon\bar K_\epsilon^T(H-H_\Delta)R_\epsilon H]+\mathrm{Sp}[\bar K_\epsilon\bar K_\epsilon^T H_\Delta R_\epsilon(H-H_\Delta)]\doteq J_{31}+J_{32}. \qquad (2.15)$$

Since, by H3,

$$||H-H_\Delta||\le\left(\int_{-\omega_\epsilon}^{\omega_\epsilon}d\omega_0\int_{-\omega_\epsilon}^{\omega_\epsilon}d\omega\,|H(\omega_0-\omega)-\right.$$
$$\left.\sum_{i=-l}^{l}H(\omega_i-\omega)\chi(\omega_0\in I_i)\right|^2\Biggr)^{1/2}\le C\Delta\omega_\epsilon,$$

$$||H_\Delta||\le\left(\int_{-\omega_\epsilon}^{\omega_\epsilon}d\omega_0\int_{-\omega_\epsilon}^{\omega_\epsilon}d\omega\left|\sum_{i=-l}^{l}H(\omega_i-\omega)\chi(\omega_0\in I_i)\right|^2\right)^{1/2}\le\omega_\epsilon^{1/2}||H|| \qquad (2.16)$$

we get

$$J_{31} \le ||K_\epsilon||^2||H - H_\Delta||\mathrm{Sp}[R_\epsilon]||H|| \le C\Delta\omega_\epsilon A^{-2}(\omega_{2\epsilon})\mathrm{Sp}[R_\epsilon], \tag{2.17}$$

$$I_{32} \le ||K_\epsilon||^2||H_\Delta||\mathrm{Sp}[R_\epsilon]||H - H_\Delta|| \le C\Delta\omega_\epsilon^{3/2}A^{-2}(\omega_{2\epsilon})\mathrm{Sp}[R_\epsilon]. \tag{2.18}$$

By (2.12)-(2.15), (2.17),(2.18), we get

$$|\bar\rho_\epsilon(\bar K_\epsilon) - \bar\rho_{\epsilon\Delta}(\bar K_\epsilon)| \le C\Delta A^{-2}(\omega_\epsilon)\mathrm{Sp}[V_\Delta] + C\epsilon^2\Delta\omega_\epsilon^{3/2}A^{-2}(\omega_\epsilon)\mathrm{Sp}[R_\epsilon] \le$$

$$C\Delta A^{-2}(\omega_{2\epsilon})B^{-2}(\omega_{2\epsilon})\omega_\epsilon + C\epsilon^2\Delta\omega_\epsilon^{5/2}A^{-2}(\omega_{2\epsilon}). \tag{2.19}$$

By (2.3),(2.19), we get

$$\bar\rho_\epsilon(\bar K_\epsilon) - \rho_{\epsilon\Delta}(\bar K_\epsilon) = o(\bar\Psi(\omega_\epsilon)). \tag{2.20}$$

It remains to study the problem of minimization of $\rho_{\epsilon\Delta}(K)$. In discretized setting,$V_\Delta = D_\epsilon^2 I$. This simplifies essentially estimates.

The next two estimates are auxilliary. By (2.16), we get

$$||R_\Delta|| \le ||H_\Delta||^2||R_\epsilon|| \le C\omega_\epsilon, \tag{2.21}$$

$$||A_\Delta^{-1}R_\Delta A_\Delta^{-1}|| \le CA^{-2}(\omega_\epsilon)\omega_\epsilon. \tag{2.22}$$

Now we utilize (2.21),(2.22), to estimate

$$\hat\rho_{\epsilon\Delta} = \epsilon^4\mathrm{Sp}[(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}R_\Delta V_\Delta R_\Delta(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}]+$$

$$\epsilon^2\mathrm{Sp}[(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}A_\Delta V_\Delta R_\Delta V_\Delta A_\Delta(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}] \doteq U_1 + U_2. \tag{2.23}$$

We have

$$U_1 \le \epsilon^4 D_\epsilon^{-4}A^{-4}(\omega_{2\epsilon})\mathrm{Sp}[(I + \epsilon^2D_\epsilon^{-1}A_\Delta^{-1}R_\Delta A_\Delta^{-1})^{-1}\times$$

$$R_\Delta V_\Delta R_\Delta(I + \epsilon^2D_\epsilon^{-1}A_\Delta^{-1}R_\Delta A_\Delta^{-1})^{-1}] \le$$

$$\le C\epsilon^4D_\epsilon^{-4}A^{-4}(\omega_{2\epsilon})||R_\Delta||^2\mathrm{Sp}[V_\Delta] \le C\epsilon^4D_\epsilon^{-2}A^{-4}(\omega_\epsilon)||R_\Delta||^2\omega_\epsilon \le$$

$$C\epsilon^4D_\epsilon^{-2}A^{-4}(\omega_{2\epsilon})\omega_\epsilon^3 = o(\epsilon^2\mathrm{Sp}[A^{-2}R]) \tag{2.24}$$

where the last equality follows from (2.1),(2.2).

We have

$$U_2 = U_{21} - U_{22} - U_{23} \tag{2.25}$$

where

$$U_{21} = \epsilon^2\mathrm{Sp}[A_\Delta^{-1}R_\Delta A_\Delta^{-1}],$$

$$U_{22} = \epsilon^4\mathrm{Sp}[(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}R_\Delta A_\Delta^{-1}R_\Delta A_\Delta^{-1}A_\Delta V_\Delta A_\Delta(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}],$$

$$U_{23} = \epsilon^4\mathrm{Sp}[A^{-1}R_\Delta A^{-1}R_\Delta(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}].$$

The estimates U_{22}, U_{23} are akin to (2.13)

$$U_{22} \le \epsilon^4D_\epsilon^{-2}A^{-2}(\omega_{2\epsilon})||R_\Delta||\mathrm{Sp}[A_\Delta^{-1}R_\Delta A_\Delta^{-1}]||A_\Delta V_\Delta A_\Delta(A_\Delta V_\Delta A_\Delta + \epsilon^2R_\Delta)^{-1}|| \le$$

$$C\epsilon^4D_\epsilon^{-2}A^{-2}(\omega_{2\epsilon})\omega_\epsilon\mathrm{Sp}[A_\Delta^{-2}R_\epsilon](1 + o(1)) = o(\epsilon^2\mathrm{Sp}[A_\Delta^{-2}R_\epsilon]), \tag{2.26}$$

$$U_{23} \le \epsilon^4 ||R_\Delta (A_\Delta V_\Delta A_\Delta + \epsilon^2 R_\Delta)^{-1}|| \,\mathrm{Sp}[A_\Delta^{-2} R_\Delta] \le$$
$$C\epsilon^4 ||R_\Delta|| A^{-2}(\omega_{2\epsilon}) D_\epsilon^{-2} \mathrm{Sp}[A_\Delta^{-2} R_\Delta] \le$$
$$C\epsilon^4 \omega_\epsilon A^{-2}(\omega_{2\epsilon}) D_\epsilon^{-2} \mathrm{Sp}[A_\Delta^{-2} R_\Delta] = o(\epsilon^2 \mathrm{Sp}[A_\Delta^{-2} R_\Delta]) \tag{2.27}$$

where the last equalities in (2.26),(2.27) follows from (2.1),(2.2).

Now (2.23)-(2.26) together imply the lower bound in (1.3).

2.2 Proof of Theorem 2

We begin with the proof of lower bound. The arguments are akin to the proof of Theorem 1.2 in [6] and are based on the method proposed in [4]. Define the parametric family of functions

$$G_\theta(\omega) = \frac{1}{\sqrt{2}} \tilde{\delta}_\epsilon^{-1/2} \theta \chi(\omega_\epsilon < |\omega|, (1+\tilde{\delta}_\epsilon)\omega_\epsilon) = \theta \tilde{H}_\epsilon$$

where $\tilde{\delta}_\epsilon \asymp \omega_\epsilon^{1-\beta-\kappa}, 0 < \kappa < \gamma - \beta$.

Consider the problem of estimation of parameter θ if θ has the Binomial distribution

$$P(\theta = \pm\theta_\epsilon) = \frac{1}{2}, \quad \theta_\epsilon = B^{-1}(\omega_\epsilon).$$

Since the noise is Gaussian it is not difficult to find the sufficient statistics in this problem. As a result the problem is reduced to estimation of θ on observation

$$y_\epsilon = \theta + \epsilon ||\tilde{H}_\epsilon||^{-1} \int \tilde{H}_\epsilon(\omega) A^{-1}(\omega) \int H(\omega - \omega_1) R^{1/2}(\omega_1) dw(\omega_1)$$

that can be written in a more simple form

$$y_\epsilon = \theta + d_\epsilon \zeta$$

where ζ is Gaussian random variable, $E\zeta = 0, E\zeta^2 = 1$ and

$$d_\epsilon^2 = \epsilon^2 ||\tilde{H}_\epsilon||^{-2} \int R(\omega_1) \left(\int H(\omega_1 - \omega) A^{-1}(\omega) \tilde{H}_\epsilon(\omega) d\omega \right)^2 \ge C ||H||^2 \tilde{\delta}_\epsilon R(\omega_\epsilon) A^{-2}(\omega_\epsilon).$$

Since $\theta_\epsilon = o(d_\epsilon)$, the Bayes risks equals

$$\theta_\epsilon^2 (1 + o(1)) = B^{-2}(\omega_\epsilon)(1 + o(1))$$

and this is the lower bound for minimax risks.

The proof of upper bounds is also akin to that of Theorem 1.2 in [6] and is omitted.

2.3 Proof of Theorem 3. Lower Bound.

We consider the auxillary problem of Bayes estimation with spectral density $\bar{V}_\epsilon(\omega) \le V(\omega), \omega \in R^1$ such that, for the spectral densities $\bar{V}_\epsilon$, the reasoning the proof of Theorem 1.3 in [6] can be applied. As a result we get the required lower bound in Theorem 3.

We put $\bar{V}_\epsilon(\omega) = 0$ if $|\omega| > (1 - \bar{\delta}_\epsilon)\omega_{1\epsilon} = \bar{\omega}_\epsilon$ and $\bar{V}_\epsilon(\omega) = V(\bar{\omega}_\epsilon)$ if $|\omega| < \bar{\omega}_\epsilon$ where $\bar{\omega}_\epsilon$ is such that

$$\epsilon^2 A^{-2}(\omega_\epsilon) R(\omega_\epsilon) = o(V(\bar{\omega}_\epsilon)) \tag{2.28}$$

and $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

Since

$$\rho_\epsilon(K) = \mathrm{Sp}[(K^T A - I)HVH(AK - I) + \epsilon^2 K^T HRHK] \ge$$
$$\mathrm{Sp}[(K^T A - I)H\bar{V}_\epsilon H(AK_I) + \epsilon^2 K^T HRHK] \doteq \hat{\rho}_\epsilon(K)$$

it suffices to find lower bound for the asymptotic of $\hat{\rho}_\epsilon(K)$. Such a lower bound can be found if we repeat all estimates in the proof of lower bound for the asymptotic of Bayes risks in Theorem 1.3 in [6]. This lower bound equals

$$\epsilon^2 ||h||^2 ||h_1||^2 \int \frac{R(\omega)\bar{V}_\epsilon(\omega)}{||h_1||^2 A^2(\omega)\bar{V}_\epsilon(\omega) + \epsilon^2 ||h||^2 R(\omega)} d\omega(1 + o(1)) =$$

$$\epsilon^2 ||h||^2 \int_{-\omega_\epsilon}^{\omega_\epsilon} A^{-2}(\omega) R(\omega) d\omega (1 + o(1))$$

where the last relation follows from (2.28). This completes the proof of lower bound in Theorem 3.

2.4 Proof of Theorem 1. Upper Bound.

We have

$$\rho_\epsilon(x^{**}_{\omega_\epsilon}) = \sup_{x\in Q} \int X^2(\omega)\chi(|\omega| > \omega_\epsilon)d\omega + \epsilon^2 \int_{-\omega_\epsilon}^{\omega_\epsilon} A^{-2}(\omega) \int H^2(\omega - \omega_1)R(\omega_1)d\omega_1 d\omega =$$

$$o(B^{-2}(\omega_\epsilon)) + \bar{\Psi}_\epsilon(\omega_\epsilon)(1 + o(1)) = \bar{\Psi}_\epsilon(\omega_\epsilon)(1 + o(1)).$$

This implies the upper bound for the estimator $x^{**}_{\omega_\epsilon}$.

By definition of μ_ϵ it is easy to see that $\Psi_\epsilon(\mu_\epsilon) = \bar{\Psi}_\epsilon(\omega_\epsilon)(1 + o(1))$.

The minimax risk of Tikhonov regularizing procedure equals

$$\rho_\epsilon(x^{**}_{\alpha_\epsilon}) = \alpha_\epsilon^2 \max_\omega M^2(\omega)B^{-2}(\omega)(A^2(\omega) + \alpha_\epsilon M(\omega))^{-2} +$$

$$\epsilon^2 \int A^2(\omega)(A^2(\omega) + \alpha_\epsilon M(\omega))^{-2} \int H^2(\omega - \omega_1)R(\omega_1)d\omega_1 d\omega \doteq I_1 + I_2. \tag{2.29}$$

By H2,R, we get

$$I_2 = 2\epsilon^2 ||h||^2 \int_{\delta\omega_\epsilon}^{\infty} A^2(\omega) R(\omega) (A^2(\omega) + \alpha_\epsilon M(\omega))^{-2} d\omega + O(\epsilon^2 \delta\omega_\epsilon A^{-2}(\delta\omega_\epsilon) R(\delta\omega_\epsilon)). \tag{2.30}$$

Define δ_ϵ such that $A^2(\omega_\epsilon) = A^2(\omega_\epsilon(1-\delta_\epsilon))(1+o(1))$ and $M(\omega_\epsilon(1-\delta_\epsilon)) = o(M(\omega_\epsilon))$. Then

$$I_2 = I_{21} + I_{22} + I_{23} + O(\epsilon^2 \delta\omega_\epsilon A^{-2}(\delta\omega_\epsilon) R(\delta\omega_\epsilon)) \tag{2.31}$$

where

$$I_{21} = 2\epsilon^2 ||h||^2 \int_{\delta\omega_\epsilon}^{(1-\delta_\epsilon)\omega_\epsilon} A^2(\omega) R(\omega) (A^2(\omega) + \alpha_\epsilon M(\omega))^{-2} d\omega =$$

$$2\epsilon^2 ||h||^2 \int_{\delta\omega_\epsilon}^{(1-\delta_\epsilon)\omega_\epsilon} A^{-2}(\omega) R(\omega) d\omega (1 + o(1)) = \bar\Psi_\epsilon(\omega_\epsilon)(1 + o(1)), \tag{2.32}$$

$$I_{22} = 2\epsilon^2 ||h||^2 \int_{(1-\delta_\epsilon)\omega_\epsilon}^{(1+\delta_\epsilon)\omega_\epsilon} A^2(\omega) R(\omega) (A^2(\omega) + \alpha_\epsilon M(\omega))^{-2} d\omega,$$

$$I_{23} = 2\epsilon^2 ||h||^2 \int_{(1+\delta_\epsilon)\omega_\epsilon}^{\infty} A^2(\omega) R(\omega) (A^2(\omega) + \alpha_\epsilon M(\omega))^{-2} d\omega.$$

By M2 and definitions of $\delta_\epsilon, \omega_\epsilon$, we get

$$I_{22} = o(I_{21}), I_{23} = o(I_{21}). \tag{2.33}$$

By (2.29)-(2.33), we get

$$I_2 = I_{21}(1 + o(1)) = \bar\Psi_\epsilon(\omega_\epsilon)(1 + o(1)). \tag{2.34}$$

By M2 and definition of α_ϵ, we get

$$I_1 < CB^{-2}(\omega_\epsilon) = o(\bar\Psi_\epsilon(\omega_\epsilon)). \tag{2.35}$$

Now (2.29),(2.34),(2.35) together imply the asymptotic minimaxity of Tikhonov regularizing procedure.

2.4 Proof of Theorem 3. Upper bound

The Bayes risk equals

$$2\int_{\omega_\epsilon}^{\infty} \int H_1^2(\omega - \omega_1) V(\omega_1) d\omega d\omega_1 + \epsilon^2 \int_{-\omega_\epsilon}^{\omega_\epsilon} A^{-2}(\omega) \int H^2(\omega - \omega_1) R(\omega_1) d\omega_1 \doteq 2I_1 + I_2 \tag{2.36}$$

By straightforward calculations, we get

$$I_2 = \epsilon^2 ||H||^2 \int_{-\omega_\epsilon}^{\omega_\epsilon} A^{-2}(\omega) R(\omega) d\omega (1 + o(1)). \tag{2.37}$$

Denote $\omega_{2\epsilon} = \frac{1}{2}\omega_{1\epsilon} + \frac{1}{2}\omega_\epsilon$.

We have

$$I_1 = I_{11} + I_{12} \tag{2.38}$$

where

$$I_{11} = \int_{\omega_\epsilon}^{\infty} \int_{\omega_{2\epsilon}}^{\infty} H_1^2(\omega - \omega_1) V(\omega_1) d\omega d\omega_1,$$

$$I_{12} = \int_{\omega_\epsilon}^{\infty} \int_{-\infty}^{\omega_{2\epsilon}} H_1^2(\omega - \omega_1) V(\omega_1) d\omega d\omega_1.$$

By (2.13),(1.14), we get

$$I_{11} \le ||H_1||^2 \int_{\omega_{2\epsilon}}^{\infty} V(\omega_1) d\omega_1 = o(I_2), \tag{2.39}$$

$$I_{12} \le \int_{\omega_{2\epsilon}-\omega_\epsilon}^{\infty} H_1^2(\omega) d\omega \int V(\omega) d\omega = o(I_1). \tag{2.40}$$

Now the upper bound follows from (2.36)-(2.40).

References

[1] Carrol R J and Hall P 1988 Optimal rates of convergence for deconvolving a density *J. Amer. Statist.Assoc.* **83** 1184-6

[2] Donoho D L 1992 Nonlinear solution of linear inverse problems by wavelet-vaguelette decomposition *Appl.Comput. Harmon. Anal* **2** 101-26

[3] Efroimovich S 1999 *Nonparametric Curve Estimation: Methods, Theory and Applications* (New York: Springer)

[4] Ermakov M S 1992 Minimax estimation in a deconvolution problem *J.Phys.A: Math.Gen.* **25** 1273-82

[5] Ermakov M S 2003 Asymptotically minimax and Bayes estimation in a deconvolution problem *Inverse Problems* **19** 1339-59

[6] Fan J 1991 Asymptotric normality for deconvolution kernel estimators *Sankhia Ser.* A **53** 97-110

[7] Goldenshluger A 1999 On pointwise adaptive nonparametric deconvolution *Bernoulli* **5** 907-25.

[8] Golubev Yu.K., Levit B Y and Tsybakov A B Asymptotically efficient estimation of Analitic functions in Gaussian noise

[9] Ibragimov I A and Hasminski R Z 1982 Estimation of distribution density belonging to a class of entire functions. Theory Probab Appl 27 551-562

[10] Jansson P A 1984 *Deconvolution, with application to Spectroscopy* (New York: Academic)

[11] Kalifa J and Mallat S 2003 Threshholding estimators for linear inverse problems and deconvolutions *Ann. Stat.* **31** 58-109

[12] Kassam S and Poor H 1985 Robust techniques for signal processing. A survey *Proc.IEEE* **73** 433-81

[13] Leadbetter M R, Lindgren G and Rootzen H 1986 *Extremes and Related Properties of Random sequences and Processes* Springer-Verlag NY

[14] Nussbaum M 1996 Asymptotic equivalence of density estimation and Gaussian white noise *Ann.Stat.* **24** 2399-430

[15] Pensky M and Vidakovic B 1999 Adaptive wavelet estimator for nonparametric density deconvolution *Ann. Statist.* **27** 2033-53

[16] Pinsker M S 1980 Optimal filtration of square-integral signal in Gaussian noise. *Problems Inform. Transm.* **16** 52-68

[17] Smola A J, Scholkopf B and Miller K 1998 The connection between regularization operators and support vector kernels *Newral Networks* **11** 637-649

[18] Tikhonov A and Arsenin V 1977 *Solution of Ill-Posed Problems* (New-York: Wiley)

[19] Wiener N 1950 *Extrapolation, Interpolation and Smoothing of Stationary Time Series* (New York: Wiley)