Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

On some estimations of Weyl sums

Lev D. Pustyl'nikov[1] , Jörg Schmeling[2]

submitted: 3rd May 1994

[1] Electric Power Research Institute VNIIF
Kashirskoe shosse 22–3
Moscow 115201
Russia

[2] Institut für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

Preprint No. 98
Berlin 1994

Edited by
Institut für Angewandte Analysis und Stochastik (IAAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e–mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e–mail (Internet): preprint@iaas-berlin.d400.de

Contents

1. Introduction

The problem of finding as good as possible estimations of Weyl sums $S_p = \sum_{x=1}^{p} e^{2\pi i f(x)}$, where $f(x)$–a polynomial of degree n, (i.e. estimations of the absolute value of such sums) is one of the most important, interesting and hard questions in analysis and number theory. In the present time non–trivial estimations are mainly based on applications of methods of Weyl and Winogradov ([1], [2], [3], [4]), which give diminishing factors $\Delta = P^{-\frac{\gamma}{2^n}}$ and $\Delta = P^{-\frac{\gamma}{n^2 \ln n}}$, respectively. The later means that $|S_P| \leq cP\Delta$, where c and γ are constants depending only on n. The disadvantage of these estimations is that they differ less and less from the trivial one as n tends to infinity, because the exponent in Δ tends to zero. Therefore, it is of great importance to find such non–trivial estimations of $|S_P|$ in which the exponent ρ in $\Delta = P^{-\rho}$ $(\rho > 0)$ won't depend on the degree of the polynomial. In [5] and [6] such estimations are found for all $\rho < \frac{1}{2}$ and all n greater some number $n_0(\rho)$ and applied to number–theoretic problems like the estimation of the residual term in the law of the distribution of the fractal parts of the values of a polynomial or their joint distributions.

In theorem 5 (§4, chapter 1) in [5] it is proved that assuming the leading coefficient of the polynomial $f(x) = a_1 x + \cdots + a_n x^n$, a^n to be irrational and the vector $(\{a_1\}, \ldots, \{a_{n-1}\})$ – consisting of the fractal parts of the other coefficients - can be good approximated by some set of full Lebesgue measure in the $(n-1)$–dimensional torus T^{n-1} then $|S_P| \leq cP^{1-\rho}$ for $2 \leq P \leq q$, where c and q depend only on the degree of the approximation and not on P. In particular, if $a = (\{a_1\}, \ldots, \{a_{n-1}\})$ belongs to this set of full measure then the estimation $|S_P| \leq cP^{1-\rho}$ is valid for all natural $P \geq 2$.

The present paper directly attaches to [5]. The aim of it is to investigate some of the topological and measure–theoretic properties of the approximation set and related ones. The main results are summarized in theorem 1 and 2.

In the first theorem we show that $|S_P| < P^{1-\rho}$ if $\rho < \frac{1}{2}$ and $n > n_0(\rho)$ for arbitrary large intervals of values of P as long as a_n is irrational and $a = (\{a_1\}, \ldots, \{a_{n-1}\})$ belongs to an open and dense subset of T^{n-1} which has full Lebesque measure.

The second theorem shows that it is impossible to find an exponent in the diminishing factor which is valid for all polynomials rather than for a subset of full measure. Namely, there is a – in the topological sense - large set of coefficients such that the corresponding Weyl sums have absolute value abitrary close to P as P tends to infinity. This emphasizes the importance of investigations of the nature of the approximation set and the related approximation process.

2. Definitions and notations

(1) n denotes a natural number greater 3.
(2) T^{n-1} is the $(n-1)$–dimensional torus considered as

$$T^{n-1} = \{(\alpha_1, \ldots, \alpha_{n-1}) | 0 \leq \alpha_i < 1 \quad i = 1, \ldots, n-1\}$$

which is the direct product of $(n-1)$ circles of length 1.

(3) mes – Lebesque measure on T^{n-1}

(4) For natural k and t the symbol $\binom{k}{t}$ denotes the corresponding binomical coefficient.

(5) If $a_n \in \mathbb{R}$ ($\mathbb{R}$ – the real numbers) we define the transformation $\hat{A}_{n-1} = \hat{A}_{n-1}(a_n)$ of T^{n-1} by:

$$\hat{A}_{n-1} : \alpha \; = (\alpha_1, \dots, \alpha_{n-1}) \to \alpha' = (\alpha'_1, \dots, \alpha'_{n-1})$$

$$\alpha'_s \; = \sum_{v=0}^{n-s-1} \binom{s+v}{v} (-1)^v \alpha_{s+v} + \binom{n}{n-s} (-1)^{n-s} \alpha_n \bmod 1$$

$$1 \leq s \leq n-1$$

(6) $f(x)$ denotes the polynomial $f(x) = a_1 x + \dots + a_n x^n$ over the reals with vanishing constant term and irrational leading coefficient a_n. The vector $a = (a_1, \dots, a_{n-1})$ belongs to T^{n-1}.

(7) For a natural number P we write S_P for the Weyl sum

$$S_P = \sum_{x=1}^{P} e^{2\pi i f(x)}$$

(8) ρ is a positive real number less than $\frac{1}{2}$.

(9) $n_0 = n_0(\rho) = 2 + \frac{2}{1-2\rho}$.

3. Main theorems

Theorem 1. *Suppose that $n > n_0(\rho)$ and C is an arbitrary positive number. Then there exist an open subset $\Omega = \Omega(C)$ of the torus T^{n-1} with properties:*

(1) *The complement $T^{n-1} \backslash \Omega$ is nowhere dense and has zero Lebesgue measure in T^{n-1}.*

(2) *For all $a = (a_1, \dots, a_{n-1}) \in \Omega$ there exist a constant $P_0 = P_0(a)$ such that for $P_0 \leq P \leq P_0 + C$ the inequality*

$$|S_P| \leq P^{1-\rho}$$

holds.

Proof. Let ρ_1 be the constant defined by

$$\rho_1 = \frac{1}{2} - \frac{1}{n_0 - 1}.$$

The definition of n_0 in (9) implies

$$\rho < \rho_1 < \frac{1}{2} \tag{3.1}$$

In the following we use the results of theorem 5, §4, chapter 1 in [1]. For $n > n_0 = 1 + \frac{2}{1-2\rho_1}$ this theorem states the existence of a Lebesgue measurable set $\Gamma_{n-1} = \Gamma_{n-1}(\rho_1) \subset T^{n-1}$ having the following properties:

(I) mes $(T^{n-1} \backslash \bigcup\limits_{k=0}^{\infty} \hat{A}^k(\Gamma_{n-1})) = 0$ where $\hat{A}^k$ is the k–th iteration of the transformation $\hat{A}$ and A^0 is the identity.

(II) If for some $\beta \in \Gamma_{n-1}$, integers $k \geq 0$ and $q \geq 0$ $\quad a = (a_1, \ldots, a_{n-1})$ can be written as

$$a = \hat{A}^k \beta + z$$

where $z = (z_1, \ldots, z_{n-1})$ is a vector with coordinates fulfilling the inequalities

$$|z_s| \leq \frac{q^{-s-\rho_1}}{2\pi(n-1)} \quad , s = 1, \ldots, n-1$$

then for all integer P from the interval $2 \leq P \leq q$ the inequality

$$|S_P| \leq (k + 1 + ((n-1)!)^{\frac{1}{2(n-1)}}) P^{1-\rho_1} \tag{3.2}$$

holds.

For $k = 0, 1, \ldots$ we define two sequences P_k and q_k of natural numbers and a sequence Ω_k of subsets of T^{n-1} in the following manner.

We set

$$P_k = (k + 1 + ((n-1)!)^{\frac{1}{2(n-1)}})^{\frac{1}{\rho_1 - \rho}} \tag{3.3}$$

and let q_k be an arbitrary number fitting the inequality

$$P_k + C < q_k \tag{3.4}$$

and Ω_k is the union of all open parallelepipedes

$$\prod = \{\alpha_1, \ldots, \alpha_{n-1} | a'_s - \frac{q_k^{-s-\rho_1}}{2\pi(n-1)} < \alpha_1 < a'_s + \frac{g_k^{-s-\rho_1}}{2\pi(n-1)}, s = 1, \ldots, n-1\}$$

where a'_s $(s = 1, \ldots, n-1)$ are the coordinates of the vector $a' = (a'_1, \ldots, a'_{n-1}) = \hat{A}^k \beta$ and β is an arbitrary element of the set Γ_{n-1}.

Now we set

$$\Omega = \bigcup_{k=0}^{\infty} \Omega_k.$$

We will show that for this set Ω theorem 1 is true. Obviously Ω is open and Ω contains $\bigcup\limits_{k=0}^{\infty} \hat{A}^k(\Gamma_{n-1})$. Therefore, using (I), we have

$$\text{mes}\ (T^{n-1} \backslash \Omega) = 0.$$

Moreover this yields that $T^{n-1} \backslash \Omega$ is nowhere dense in T^{n-1}. Let us assume that for some k $\quad a = (a_1, \ldots, a_{n-1}) \in \Omega_k$. For such an a we set $P_0 = P_k$. The definition of Ω_k then tells us that a can be written as $a = a' + z$, with

$$a' = \hat{A}^k \beta, \quad \beta \in \Gamma_{n-1}$$

and $z = (z_1, \dots, z_{n-1})$ respects the inequalities

$$|z_s| < \frac{q_k^{-s-\rho_1}}{2\pi(n-1)}, \quad s = 1, \dots, n-1.$$

Now (II) implies inequality (3.2) for the polynomial $f(x) = a_1x + \cdots + a_nx^n$ with $a = (a_1, \dots, a_{n-1}) \in \Omega_k$ and for $2 \le P \le q_k$. From (3.3) and (3.1) we can derive

$$|S_P| \le (k + 1 + ((n-1)!)^{\frac{1}{2(n-1)}})P^{1-\rho_1} \le P^{1-\rho} \tag{3.5}$$

for $q_k \ge P \ge P_0 = P_k$.

Now (3.4) implies the theorem . □

Next we want to show that the estimates of [5] and [6] of theorem 1 can't be improved. Namely, there is no diminishing factor $\Delta = P^{-\rho}, \rho > 0$, estimating Weyl sums for all polynomials with irrational main coefficient.

Definition 3.1. A set is called residual if it contains a countable intersection of open and dense sets.

Remark. Residual subsets of T^n are always dense and the intersection of a countable number of residual sets is residual (see f.i. [7]).

Theorem 2. *Let $n \ge 2$. then there exists a residual subset W in T^n such that for $a = (a_1, \dots, a_n) \in W$, $f(x) = a_1x + \cdots + a_nx^n$ and arbitrary $\rho > 0$ the inequality*

$$|S_P| \le P^{1-\rho}$$

is violated infinitely often.

The proof is based on some well–known facts about Weyl–sums (Lemma 1–3) which one can find f.i. in [4].

Lemma 1. *Assume that $f(x) = f_1(x) + \cdots + f_s(x)$ and the fractal parts of $f_i(x), i = 1, \dots, s$ are periodic with mutually relatively prime periods $\tau_1, \dots, \tau_s$. Then*

$$\sum_{x=1}^{\tau_1 \cdot \ldots \cdot \tau_s} e^{2\pi i f(x)} = \prod_{\nu=1}^{s} \sum_{x_\nu=1}^{\tau_\nu} e^{2\pi i f_\nu(x_\nu)} \quad .$$

Lemma 2. *Assume that $2a_2$ and q are relatively prime then:*

$$\left| \sum_{x=1}^{q} e^{2\pi i \frac{a_1x + a_2x^2}{q}} \right| = \sqrt{q}.$$

Lemma 3. *Assume that a and p are mutually prime and $n \geq 3$ then:*

$$\sum_{x=1}^{p^n} e^{2\pi i \frac{ax^n}{p^n}} = p^{n-1}.$$

Let us fix $n \geq 2$ and consider the sets

$$W_m = \{a \in T^n | \exists P \geq m \text{ such that } |S_P| > P^{1-\frac{1}{m}}\}$$

where $S_P = S_P(a) = \sum_{x=1}^{P} e^{2\pi i(a_1 x + \cdots + a_n x^n)}$, $m \in \mathbb{N}$. Because finite Weyl sums depend continuously on $a \in T^n$, all the sets $W_m, m \in \mathbb{N}$ are open.

Lemma 4. *$W_m, m \in \mathbb{N}$, is dense in T^n.*

Proof. We fix $\varepsilon > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Subject to lemma 2 we select integers a_1, a_2 and q_2 with $|a_1| < q_2, |a_2| < q_2, q_2$ – an odd prime number,

$$\begin{aligned} |\frac{a_1}{q_2} - \alpha_1| &< \varepsilon \qquad \text{and} \\ |\frac{a_2}{q_2} - \alpha_2| &< \varepsilon. \end{aligned}$$

For $2 < s \leq n$ we choose an integer a_s and a – prime to a_s – prime number $q_s, \quad q_s > q_{s-1}$ with

$$|\frac{a_s}{q_s^s} - \alpha_s| < \varepsilon.$$

Hence the polynomial $\hat{f}(x) = \frac{a_1}{q_2}x + \frac{a_2}{q_2}x^2 + \frac{a_3}{q_3^3}x^3 + \cdots + \frac{a_n}{q_n^n}x^n$ is ε–close in the topology induced from T^n to $f(x) = \alpha_1 x + \cdots + \alpha_n x^n$ and according to lemma 1–3

$$\left|S_{q_2 q_3^3 \cdots q_n^n}\right| = \left|\prod_{\nu=3}^{n} \sum_{x_\nu=1}^{q_\nu^\nu} e^{2\pi i \frac{a_\nu}{q_\nu^\nu} x^\nu}\right| \left|\sum_{x=1}^{q_2} e^{2\pi i \frac{a_1 x + a_2 x^2}{q_2}}\right| = q_3^2 \cdot q_4^3 \cdot \ldots \cdot q_n^{n-1} \cdot \sqrt{q_2} = \delta > 0.$$

It is easy to see that for $k \in \mathbb{N}$

$$\left|S_{k \cdot q_2 q_3^3 \cdot \ldots \cdot q_n^n}\right| = k \left|S_{q_2 q_3^3 \cdot \ldots \cdot q_n^n}\right| = k \cdot \delta.$$

Setting $q_2 \cdot q_3^3 \cdot \ldots \cdot q_n^n = q$ and $P_k = kq$ we can derive

$$|S_{P_k}| = |S_{kq}| = k \cdot \delta > (kq)^{1-\frac{1}{m}} = P_k^{1-\frac{1}{m}}$$

if $k > \frac{q^{m-1}}{\delta^m}$. Moreover, if $P = kq + r, 0 \leq r < q$ and $k > \frac{q^{m-1}}{\delta^m} + \frac{2q}{\delta}$ then

$$|S_P| = |S_{kq+r}| > k \cdot \delta - r > (kq + r)^{1-\frac{1}{m}} = P^{1-\frac{1}{m}}.$$

This means $\hat{a} = (\frac{a_1}{q_2}, \frac{a_2}{q_2}, \frac{a_3}{q_3^3}, \ldots, \frac{a_n}{q_n^n})$ belongs to W_m for all $m \in \mathbb{N}$, and consequently all W_m are dense, because of the arbitrary choice of ε. $\square$

Proof of the theorem: We define

$$W = \bigcap_{m=0}^{\infty} W_m.$$

If a point $a \in T^n$ belongs to W then it has the property: there are two sequences $\{P_l\}$ and $\{m_l\}$ of natural numbers, such that

$$|S_{p_l}| > P_l^{(1-\frac{1}{m_l})}$$

But this gives the statement of theorem 2. □

Corollary 3.1. *The set of vectors $a = (a_1, \ldots, a_n) \in T^n$ with all a_i $(i = 1, \ldots, n)$ irrational and violating*

$$|S_p| < P^{1-\rho}$$

for infinitely many P and arbitrary $\rho > 0$ is residual.

Proof. The set

$$\widetilde{W} = W \cap \{\underbrace{(S^1 \backslash Q) \times \cdots \times (S^1 \backslash Q)}_{n-\text{times}}\}$$

fits the theorem. the set $\{(S^1\backslash Q) \times \cdots \times (S^1\backslash Q)\}$ is residual and hence by the remark following definition 3.1. $\tilde{W}$ is residual itself. □

4. Concluding remarks

In [4] it is shown, that we can get uniform estimates (not depending on the degree of the polynomial) of the Weyl sum $|S_P|$ for all large enough P for a set of coefficients Ξ in T^{n-1} having full Lebesque measure. Moreover, these estimates are depending on the approximability of the vector $(a_1, \ldots, a_{n-1})$ by a certain approximation process (see point II). On the other hand the corollary to theorem 2 shows that the complement of Ξ in T^{n-1} is residual. These facts suggest that a more precise description of that approximation process together with a measure–theoretical and topological analysis of sets related to that process could help to get a much better understanding of Weyl sums and related problems.

5. Acknowledgements

This work was prepared during a visit of L.D. Pustyl'nikov to the IAAS. L.D. Pustyl'nikov thanks the IAAS for supporting his stay in Berlin.

References

1. H. Weyl: *Über die Gleichverteilung der Zahlen mod 1.*, Math. Ann. 77 (1916), 313–352.
2. K. Chandrasekharan: *Arithmetical Functions*, Springer, 1970.
3. I.M. Vinogradov: *The Method of Exponential Sums in the Theory of Numbers*, Nauka, Moscow, 1971.
4. N.M. Korobov: *Exponential Sums and their applications*, Nauka, Moscow, 1989.
5. L.D. Pustyl'nikov: *The distribution of the fractional parts of a polynomial's values, Weyl sums and ergodic theory*, Uspekhi Mat. Nauk 4 (1993), 131–166.
6. L.D. Pustyl'nikov: *New estimates of Weyl sums and the remainder term in the law of distribution of the fractional part of a polynomial*, Ergodic Theory and Dynamical Systems, 11 (1991), 515–534.
7. J.C. Oxtoby: *Measure and Category*, Springer 1980

Recent publications of the Institut für Angewandte Analysis und Stochastik

Preprints 1993

69. John W. Barrett, Peter Knabner: Finite element approximation of transport of reactive solutes in porous media. Part 2: Error estimates for equilibrium adsorption processes.

70. Herbert Gajewski, Willi Jäger, Alexander Koshelev: About loss of regularity and "blow up" of solutions for quasilinear parabolic systems.

71. Friedrich Grund: Numerical solution of hierarchically structured systems of algebraic–differential equations.

72. Henri Schurz: Mean square stability for discrete linear stochastic systems.

73. Roger Tribe: A travelling wave solution to the Kolmogorov equation with noise.

74. Roger Tribe: The long term behavior of a Stochastic PDE.

75. Annegret Glitzky, Konrad Gröger, Rolf Hünlich: Rothe's method for equations modelling transport of dopants in semiconductors.

76. Wolfgang Dahmen, Bernd Kleemann, Siegfried Prößdorf, Reinhold Schneider: A multiscale method for the double layer potential equation on a polyhedron.

77. Hans–Günter Bothe: Attractors of non invertible maps.

78. Gregori Milstein, Michael Nussbaum: Autoregression approximation of a nonparametric diffusion model.

Preprints 1994

79. Anton Bovier, Véronique Gayrard, Pierre Picco: Gibbs states of the Hopfield model in the regime of perfect memory.

80. Roland Duduchava, Siegfried Prößdorf: On the approximation of singular integral equations by equations with smooth kernels.

81. Klaus Fleischmann, Jean–François Le Gall: A new approach to the single point catalytic super–Brownian motion.

82. Anton Bovier, Jean–Michel Ghez: Remarks on the spectral properties of tight binding and Kronig–Penney models with substitution sequences.

83. Klaus Matthes, Rainer Siegmund–Schultze, Anton Wakolbinger: Recurrence of ancestral lines and offspring trees in time stationary branching populations.

84. Karmeshu, Henri Schurz: Moment evolution of the outflow–rate from nonlinear conceptual reservoirs.

85. Wolfdietrich Müller, Klaus R. Schneider: Feedback stabilization of nonlinear discrete–time systems.

86. Gennadii A. Leonov: A method of constructing of dynamical systems with bounded nonperiodic trajectories.

87. Gennadii A. Leonov: Pendulum with positive and negative dry friction. Continuum of homoclinic orbits.

88. Reiner Lauterbach, Jan A. Sanders: Bifurcation analysis for spherically symmetric systems using invariant theory.

89. Milan Kučera: Stability of bifurcating periodic solutions of differential inequalities in $\mathbb{R}^3$.

90. Peter Knabner, Cornelius J. van Duijn, Sabine Hengst: An analysis of crystal dissolution fronts in flows through porous media Part I: Homogeneous charge distribution.

91. Werner Horn, Philippe Laurençot, Jürgen Sprekels: Global solutions to a Penrose–Fife phase–field model under flux boundary conditions for the inverse temperature.

92. Oleg V. Lepskii, Vladimir G. Spokoiny: Local adaptivity to inhomogeneous smoothness. 1. Resolution level.

93. Wolfgang Wagner: A functional law of large numbers for Boltzmann type stochastic particle systems.

94. Hermann Haaf: Existence of periodic travelling waves to reaction–diffusion equations with excitable–oscillatory kinetics.

95. Anton Bovier, Véronique Gayrard, Pierre Picco: Large deviation principles for the Hopfield model and the Kac–Hopfield model.

96. Wolfgang Wagner: Approximation of the Boltzmann equation by discrete velocity models.

97. Anton Bovier, Véronique Gayrard, Pierre Picco: Gibbs states of the Hopfield model with extensively many patterns.