# Weierstraß-Institut
# für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633


# The behaviour of aging functions in one-dimensional Bouchaud's trap model


Jiří Černý



submitted: 26 August 2004



Weierstraß–Institut für Angewandte
Analysis und Stochastik
Mohrenstrasse 39
10117 Germany
E-Mail: cerny@wias-berlin.de



No. 961
Berlin 2004



2000 *Mathematics Subject Classification.* 82D30,60K37,82C41.

*Key words and phrases.* Trap models, aging, Lévy processes, singular diffusions.

Work supported by the DFG Research Center "Matheon".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. Let $\tau_i$ be a collection of i.i.d. positive random variables with distribution in the domain of attraction of $\alpha$-stable law with $\alpha < 1$. The symmetric Bouchaud's trap model on $\mathbb{Z}$ is a Markov chain $X(t)$ whose transition rates are given by $w_{xy} = (2\tau_x)^{-1}$ if $x$, $y$ are neighbours in $\mathbb{Z}$. We study the behaviour of two correlation functions: $\mathbb{P}[X(t_w + t) = X(t_w)]$ and $\mathbb{P}\big[X(t') = X(t_w) \forall t' \in [t_w, t_w + t]\big]$. It is well known that for any of these correlation functions a time-scale $t = f(t_w)$ such that aging occurs can be found. We study these correlation functions on time-scales different from $f(t_w)$, and we describe more precisely the behaviour of a singular diffusion obtained as the scaling limit of Bouchaud's trap model.

## 1. Introduction and results

Bouchaud's trap model (BTM) was introduced in [Bou92] as a phenomenological model for studying the dynamics of complex disordered systems like spin glasses. Although the model is quite simple, it manifests some major features observed in real physical systems, aging in low temperature regime being one of them. For physical motivation behind the model see the original paper [Bou92] or [BCKM98] and their references.

The aim of this paper is to obtain a finer description of the behaviour of some time-correlation functions that are known to manifest aging in the one-dimensional BTM. Our attention is concentrated mainly on the behaviour of these functions at time scales that are much shorter or longer than the scale at which aging occurs. The description of this behaviour is important in systems where multiple aging scales exist, since even if such a system is observed only at one time-scale where aging occurs, the other time-scales can produce corrections that should be controlled. We further describe asymptotic behaviour of "aging limits" whose existence was proved in [FIN02, BČ04]. Our work is mainly motivated by the paper [BB03] where similar results are obtained using computer simulations and physical arguments.

The methods that we use to control the behaviour of the time-correlation functions allow us to get a more precise description of a singular diffusion obtained as the scaling limit of the one-dimensional BTM, and which was for the first time introduced in [FIN02]. In particular, we get sub-diffusive upper bounds on the transition kernel of this diffusion, which we believe to be (except for multiplicative constants) optimal.

We will study the following, so called symmetric, one-dimensional version of BTM. Let $\boldsymbol{\tau} = \{\tau_x\}_{x\in\mathbb{Z}}$ be a collection of i.i.d. positive random variables on some probability space $(\Omega, \mathbb{P}, \mathcal{F})$. The distribution of $\tau_i$ will be specified later. The symmetric BTM in dimension one is a continuous time Markov chain on $\mathbb{Z}$ satisfying $X(0) = 0$ and

$$\mathbb{P}[X(t+dt) = y | X(t) = x] = w_{xy} dt \tag{1}$$

with transition rates $w_{xy}$ given by

$$w_{xy} = (2\tau_x)^{-1} \qquad \text{if } |x-y| = 1, \tag{2}$$

and zero otherwise. In words, the process $X$ waits at site $x$ for an exponentially distributed time with mean value $\tau_x$ and then jumps with equal probability to one of the neighbouring sites. For this reason we call $\tau_x$ the depth of the trap at $x$.

Let us recall briefly the main results about the aging in the BTM which are relevant to our paper. Usually, proving an aging result consists in finding a two-point function $F(t_w, t_w + t)$, that is a quantity measuring the behaviour of the system at time $t + t_w$ after it has aged for the time $t_w$, such that a nontrivial limit

$$\lim_{t_w\to\infty} F(t_w, t_w + f(t_w)) = C_f \tag{3}$$

exists for an increasing function $f(t)$, satisfying $f(t) \to \infty$ as $t \to \infty$. In BTM such two-point functions may be found if for some $0 < \alpha < 1$ the random variables $\tau_x$ satisfy[1]

$$\mathbb{P}[\tau_x > u] = u^{-\alpha}(1 + o(1)) \qquad \text{as } u \to \infty. \tag{4}$$

It was observed in [RMB00] that if this condition is satisfied, then the two-point function

$$R(t_w, t_w + t) = \mathbb{E}\mathbb{P}[X(t + t_w) = X(t_w)|\boldsymbol{\tau}], \tag{5}$$

i.e. the probability that at the end of the observation period (at time $t + t_w$) the system is in the same state as it was in the beginning (at time $t_w$) averaged over the random environment $\boldsymbol{\tau}$, has aging behaviour. This was proved in [FIN02]. There it is shown that under (4) there is a non-trivial function $R(\theta)$ such that

$$\lim_{t_w\to\infty} R(t_w, t_w + \theta t_w) = R(\theta). \tag{6}$$

This limiting function $R(\theta)$ depends on the law of $\tau_x$ only through the index $\alpha$.

Another two-point function that is often considered in BTM is

$$\Pi(t_w, t_w + t) = \mathbb{E}\mathbb{P}[X(t') = X(t_w) \forall t' \in [t_w, t_w + t]|\boldsymbol{\tau}], \tag{7}$$

---

[1] Some of the cited results were actually proved under weaker condition, namely that $\tau_x$ are in the domain of attraction of an $\alpha$-stable law. We prefer, however, the condition (4) since the more general setting unnecessarily complicates the reasoning.

giving the probability that the system does not change its state between $t_w$ and $t_w + t$ again averaged over $\boldsymbol{\tau}$. In [BČ04] it was proved that if (4) holds, then the two-point function $\Pi$ satisfies

$$\lim_{t_w\to\infty} \Pi(t_w, t_w + \theta t_w^\gamma) = \Pi(\theta) \tag{8}$$

for some non-trivial function $\Pi(\theta)$ and

$$\gamma = \frac{1}{1+\alpha} < 1. \tag{9}$$

Since the time-scale $t_w^\gamma$ is much smaller than $t_w$, such behaviour may be referred to as sub-aging.

The proofs of the these results are based on the fact that it is possible to find a scaling limit of the one-dimensional BTM. This limit was for the first time identified in [FIN02] as a singular diffusion $Z$ with speed measure $\rho$ given as Lebesgue-Stieltjes measure associated to an $\alpha$-stable subordinator. (The definition of the singular diffusion is recalled in Section 2 of this paper.) The fixed time distributions of this diffusion are purely atomic, and the functions $R(\theta)$ and $\Pi(\theta)$ can be expressed using $Z$ as

$$R(\theta) = \mathbb{P}[Z(1+\theta) = Z(1)], \tag{10}$$

$$\Pi(\theta) = \int_0^\infty \mathbb{P}[\rho(Z(1)) \in du] e^{-\theta/u}. \tag{11}$$

It is a rather direct consequence of the methods of [FIN02, BČ04] that the functions $R(\theta)$ and $\Pi(\theta)$ both tend to zero as $\theta \to \infty$ and to one as $\theta \to 0$. Similarly, it is not difficult to show that

$$\lim_{t_w\to\infty} \Pi\big(t_w, t_w + f(t_w)\big) = \begin{cases} 0 & \text{if } t_w^\gamma = o(f(t_w)) \\ 1 & \text{if } f(t_w) = o(t_w^\gamma), \end{cases} \tag{12}$$

and

$$\lim_{t_w\to\infty} R\big(t_w, t_w + f(t_w)\big) = \begin{cases} 0 & \text{if } t_w = o(f(t_w)) \\ 1 & \text{if } f(t_w) = o(t_w). \end{cases} \tag{13}$$

To conclude the overview we would like to point out some results about aging in trap models on different state spaces. The BTM on a large complete graph was initially proposed in physics literature as an ansatz for the dynamics of the Random Energy Model [MB96, BM97]. The relation between these two models was justified rigorously in [BBG03a, BBG03b]. The BTM on the lattice $\mathbb{Z}^2$ was studied in [BČM03], where (sub)aging was proved for both two-point functions $R$ and $\Pi$. This result was further generalised to $\mathbb{Z}^d$, $d \geq 3$, in [Čer03].

We will now formulate our results assuming always that (4) is satisfied. To avoid technical complications we further assume that there exists $c > 0$ such that

$$\mathbb{P}[\tau_i > c] = 1. \tag{14}$$

This assumption is harmless since very shallow traps have little influence on the dynamics. In the following theorem we give the rates of convergence in (12) and describe the asymptotic behaviour of $\Pi(\theta)$.

**Theorem 1.1.** *Assume that conditions* (4) *and* (14) *hold.*

*(a)* Short time behaviour. *Let $f(t)$ be an increasing function satisfying $t^\kappa \ge f(t) \ge t^\mu$ for all $t$ large and for some $\gamma > \kappa \ge \mu > 0$. Then*

$$\lim_{t\to\infty} \Big(\frac{f(t)}{t^\gamma}\Big)^{\alpha-1} \big(1 - \Pi(t, t + f(t))\big) = K_1, \tag{15}$$

*with* $0 < K_1 < \infty$.

*(b)* Long time behaviour. *Let $g(t)$ be such that $t^\gamma = o(g(t))$. Then*

$$\lim_{t\to\infty} \Big(\frac{g(t)}{t^\gamma}\Big)^{\alpha} \Pi\big(t, t + g(t)\big) = K_2, \tag{16}$$

*with* $0 < K_2 < \infty$.

*(c)* Behaviour of $\Pi(\theta)$. *The function $\Pi(\theta)$ defined in* (8) *satisfies*

$$\lim_{\theta\to 0} \theta^{\alpha-1}(1 - \Pi(\theta)) = K_1, \tag{17}$$

$$\lim_{\theta\to\infty} \theta^{\alpha}\Pi(\theta) = K_2. \tag{18}$$

Results that we present here for the two-point function $R$ are relatively weaker. The reason for this is that the process $X$ usually makes a lot of excursions from $X(t_w)$ between the times $t_w$ and $t_w + t$ that we want to consider. The behaviour of the function $R$ is therefore influenced by the random environment in the neighbourhood of $X(t_w)$ which we cannot control precisely.

**Theorem 1.2.** *Assume that conditions* (4)*and* (14) *hold.*

*(a)* Long time behaviour. *Let $f(t)$ be such that $t = o(f(t))$. Then*

$$0 < \liminf_{t\to\infty} \Big(\frac{f(t)}{t}\Big)^{\alpha\gamma} R\big(t, t + f(t)\big) \le \limsup_{t\to\infty} \Big(\frac{f(t)}{t}\Big)^{\alpha\gamma} R\big(t, t + f(t)\big) < \infty. \tag{19}$$

*(b)* Behaviour of $R(\theta)$. *The function $R(\theta)$ defined in* (6) *satisfies*

$$0 < \liminf_{\theta\to\infty} \theta^{\alpha\gamma} R(\theta) \le \limsup_{\theta\to\infty} \theta^{\alpha\gamma} R(\theta) < \infty. \tag{20}$$

*Remarks.* 1. We will give quite explicit formulas for the constants $K_1$ and $K_2$ appearing in Theorem 1.1. As these formulas require some additional notation, we prefer to postpone their presentation to Sections 4 and 6.

2. Note also that, as in [FIN02] and [BČ04], we take in (5) and (7) the average over the random environment $\boldsymbol{\tau}$. The averaging is necessary for the existence of a limit in (6) and (8). For fixed $\boldsymbol{\tau}$ these limits do not exist. However, from the proofs we present for averaged case we can

get quenched results in the short time regime without a major effort. Namely, the following quenched version of Theorem 1.1(a) holds true.

**Theorem 1.3.** *Let the conditions of Theorem 1.1(a) be fulfilled and let*

$$\Pi(t_w, t_w + t|\boldsymbol{\tau}) = \mathbb{P}[X(t') = X(t_w) \forall t' \in [t_w, t_w + t]|\boldsymbol{\tau}]. \tag{21}$$

*Then*

$$\Big(\frac{f(t)}{t^\gamma}\Big)^{\alpha-1}\big(1 - \Pi(t, t + f(t)\big|\boldsymbol{\tau})\big) \xrightarrow{law} \mathcal{Z} \qquad \text{as } t \to \infty \tag{22}$$

*for some non-degenerate random variable $\mathcal{Z}$.*

Before we proceed to the proofs of these theorems, let us explain at the heuristic level the behaviour of the process $X$ at large times. After the first $n$ jumps the process typically visits $O(n^{1/2})$ sites. The deepest trap that it finds during $n$ jumps has therefore the depth $O(n^{1/2\alpha})$ as can be verified from (4). This trap is typically visited $O(n^{1/2})$ times. Since the depths are in the domain of attraction of an $\alpha$-stable law with $\alpha < 1$, the time needed for $n$ jumps is essentially determined by the time spent in the deepest trap. This time is $O(n^{(1+\alpha)/2\alpha})$. Inverting this expression we get that before time $t$ the process visits typically $O(t^{\alpha\gamma})$ sites and the deepest traps it finds during this time have the depth of order $t^\gamma$. Moreover, the process is usually located in one of these deep traps at the time $t$. More precisely, it was proved in [BČ04] that the distribution of the random variable $\tau_{X(t)}/t^\gamma$ converges to a non-degenerate distribution as $t \to \infty$. The sub-aging (8) is then an almost direct consequence of this claim.

Being in the trap with the depth $t^\gamma$, the process needs typically a time of the same order to jump out. In Theorem 1.1(a) we are interested in $1 - \Pi(t, t + f(t))$ with $f(t) \ll t^\gamma$, that is in the probability that a jump occurs in a time much shorter than $t^\gamma$. There are essentially two possible strategies which leads to such an event:

(i) $\tau_{X(t)}$ has the typical order $t^\gamma$ but the jump occurs in an exceptionally short time.

(ii) $X(t)$ is in a non-typically shallow trap and stays there a typical time.

We prove in Section 4 that the second strategy dominates. Therefore, we will need to study the probability of being in a very shallow trap or, equivalently, to describe the tail of $\mathbb{P}[\tau_{X(t)}/t^\gamma \le u]$ for $u$ close to 0. This description can be found in Proposition 4.2. We will use the fact that although the BTM never reaches equilibrium in a finite time, it is nearby equilibrium if we observe only traps that are much shallower than the typical depth $t^\gamma$ on intervals that are small with respect to the typical size of $X(t)$. This puts on rigorous basis the concept of *local equilibrium* that was introduced in [RMB00]. The concept does not give the right predictions for the values of the limiting functions

$R(\theta)$ and $\Pi(\theta)$ but it is useful to describe their asymptotic behaviour as was already observed in this paper.

In Theorem 1.1(b) we are interested in the possibility that the system does not jump for an exceptionally large time $g(t)$. It should be not surprising that this event is related to the event of being in an unusually deep trap with the depth of order $g(t)$. We will see that the process $X$ can reach such trap only if it is sufficiently close to its starting point. Since the process visits usually $O(t^{\alpha\gamma})$ sites before time $t$, it is not difficult to argue heuristically that the probability that $X$ hits a trap with depth larger than $g(t)$ before $t$ decreases as $(g(t)/t^{\gamma})^{-\alpha}$. We will give precise arguments that leads to this claim and to the proof of Theorem 1.1(b) in Section 6.

For the study of the two-point function $R$ in Section 7 we need to know how behaves the quenched probability $\mathbb{P}[X(t) = X(0)|\boldsymbol{\tau}]$ for large times $t$. In Section 3 an upper bound for this probability is given together with quenched sub-diffusive bounds on the decay of the probability to get far from the starting point. These bounds are used frequently within the paper.

One of the main tools in proving aging in one-dimensional BTM is coupling between different time scales which was for the first time introduced in [FIN02]. As we make frequent use of it, we recall it in Section 2.

## 2. Coupling between different time scales

Let $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a probability space. On this space we define (as in [FIN02]) a two-sided $\alpha$-stable Lévy process $V(x)$, $x \in \mathbb{R}$, with cadlag paths given by $V(0) = 0$ and

$$\begin{aligned}\bar{\mathbb{E}}\big[\exp\big(-\lambda(V(x+y)-V(x))\big)\big] &= \exp\Big[\alpha y\int_0^\infty \big(e^{-\lambda w}-1\big)w^{-1-\alpha}\,dw\Big]\\ &= \exp\big[-y\lambda^{\alpha}\Gamma(1-\alpha)\big].\end{aligned} \tag{23}$$

We use $\rho$ to denote the Lebesgue-Stieltjes measure associated to $V$, that is $\rho(x,y] = V(y) - V(x)$. It is known that

$$\rho(dx) = \sum_i v_i \delta_{x_i}(dx), \tag{24}$$

where $(x_i, v_i)$ yields an inhomogeneous Poisson point process on $\mathbb{R} \times (0,\infty)$ with intensity $dx\,\alpha v^{-1-\alpha}\,dv$.

For each $\varepsilon > 0$ we define a sequence of random variables $\tau_i^{\varepsilon}$ on $\bar\Omega$ in a way that the family $\{\tau_i^{\varepsilon}, i \in \mathbb{Z}\}$ has the same distribution as $\{\tau_i, i \in \mathbb{Z}\}$. The construction is as follows.

Let $G : [0,\infty) \mapsto [0,\infty)$ be such that

$$\bar{\mathbb{P}}[V(1) > G(u)] = \mathbb{P}[\tau_0 > u]. \tag{25}$$

The function $G$ is well defined since $V(1)$ has a continuous distribution function, $G$ is nondecreasing, right continuous, and hence has a nondecreasing, right continuous generalised inverse $G^{-1}(s) = \inf\{t : G(t) \geq s\}$. The random variables $\tau_i^\varepsilon$ are then defined by

$$\tau_i^\varepsilon = G^{-1}\big(\varepsilon^{-1/\alpha}\rho(\varepsilon i, \varepsilon(i+1)]\big). \tag{26}$$

The family $\tau_i^\varepsilon$ has the required properties, that is for fixed value of $\varepsilon$ the random variables $\tau_i^\varepsilon$ are i.i.d. and have the same distribution as $\tau_0$. An easy proof of this fact can be found in [FIN02] or [BČ04].

To couple the different time scales of BTM we introduce a collection of measures $\mu_\varepsilon$,

$$\mu_\varepsilon(dx) = \sum_{i\in\mathbb{Z}} \varepsilon^{1/\alpha}\tau_i^\varepsilon \delta_{\varepsilon i}(dx). \tag{27}$$

Let $W$ be a Brownian motion defined also on $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ that is independent of $V$. For any $\varepsilon > 0$ we use $X_\varepsilon(t)$ to denote a process defined as the time change of $W$ using the speed measure $\mu_\varepsilon$. For sake of completeness and in order to introduce some notation we give here a definition of the time change.

**Definition 2.1.** Let $\nu$ be a positive measure on $\mathbb{R}$ and let $\ell(t,x)$ be a local time of Brownian motion $W$. We define $\phi(t) = \int \ell(t,x)\nu(dx)$. Let $\psi(s)$ be a generalised right-continuous inverse of $\phi(t)$, $\psi(s) = \inf\{t; \phi(t) \geq s\}$. Then the process $Y$ defined by $Y(t) = W(\psi(t))$ is called time-changed Brownian motion with speed measure $\nu$.

The relevance of the processes $X_\varepsilon$ can be seen from the following lemma that was proved in [FIN02] and [BČ04].

**Lemma 2.2.** *For all $\varepsilon > 0$ the process $\big(X_\varepsilon(t), \mu_\varepsilon(X_\varepsilon(t))\big)$, $t \geq 0$, has the same distribution as the process $\big(\varepsilon X(t\varepsilon^{-1/\alpha\gamma}), \varepsilon^{1/\alpha}\tau_{X(t\varepsilon^{-1/\alpha\gamma})}\big)$, $t \geq 0$. In particular, $X_\varepsilon$ is a nearest-neighbour random walk on $\varepsilon\mathbb{Z}$.*

We see from this lemma that the behaviour of $X$ at a large time $t_\varepsilon = \varepsilon^{-1/\alpha\gamma}$ can be obtained from the behaviour of $X_\varepsilon$ at time $t = 1$. In particular,

$$R(t_\varepsilon, t_\varepsilon + f(t_\varepsilon)) = \bar{\mathbb{P}}\big[X_\varepsilon(1) = X_\varepsilon(1 + f(t_\varepsilon)/t_\varepsilon)\big], \tag{28}$$

$$\begin{aligned}\Pi(t_\varepsilon, t_\varepsilon + f(t_\varepsilon)) &= \int_0^\infty \mathbb{P}\big[\tau_{X(t_\varepsilon)} \in du\big] e^{-f(t_\varepsilon)/u} \\ &= \int_0^\infty \bar{\mathbb{P}}\big[\mu_\varepsilon[X_\varepsilon(1)] \in du\big] \exp\Big(-\frac{f(t_\varepsilon)\varepsilon^{1/\alpha}}{u}\Big).\end{aligned} \tag{29}$$

The first step in proving aging in BTM (see [FIN02, BČ04]) is the observation that the family $\mu_\varepsilon$ converges to $\rho$ $\bar{\mathbb{P}}$-a.s. vaguely and also in so called *point process* sense. The point process convergence is defined as follows.

**Definition 2.3.** [FIN02] Given a family $\nu$, $\nu_\varepsilon$, $\varepsilon > 0$ of locally finite measures on $\mathbb{R}$, we say that $\nu_\varepsilon$ converges *in the point process sense* to $\nu$, and write $\nu_\varepsilon \overset{pp}{\to} \nu$, as $\varepsilon \to 0$, provided the following holds: if the atoms of $\nu$, $\nu_\varepsilon$ are, respectively, at the distinct locations $y_i$, $y^\varepsilon_{i'}$ with weights $w_i$, $w^\varepsilon_{i'}$, then the subsets of $V_\varepsilon := \cup_{i'}\{(y^\varepsilon_{i'}, w^\varepsilon_{i'})\}$ of $\mathbb{R} \times (0,\infty)$ converge to $V := \cup_i\{(y_i, w_i)\}$ as $\varepsilon \to 0$ in the sense that for any open $U$, whose closure $\bar{U}$ is a compact subset of $\mathbb{R} \times (0,\infty)$ such that its boundary contains no points of $V$, the number of points $|V_\varepsilon \cap U|$ in $V_\varepsilon \cap U$ is finite and equals $|V \cap U|$ for all $\varepsilon$ small enough.

Remark that, unlike vague or weak convergence, the point process convergence is sensitive to the event being exactly in one specified trap. That is why this convergence plays a decisive role in studying the behaviour of the two-point functions $R$ and $\Pi$.

The convergence of the speed measures $\mu_\varepsilon$ implies convergence of the processes $X_\varepsilon$ as was proved for the weak convergence in [Sto63] and for the point process convergence in [FIN02].

**Proposition 2.4.** *For $\bar{\mathbb{P}}$-a.e. realisation of the measure $\rho$ and for any $t > 0$, the distribution of $X_\varepsilon(t)$ converges weakly and in the point process sense to the distribution of a singular diffusion $Z$ at time $t$, where the singular diffusion $Z$ is defined as the time change of $W$ using $\rho$ as the speed measure. Further, the distribution of the random variable $\mu_\varepsilon[X_\varepsilon(t)]$ converges to the distribution of $\rho[Z(t)]$ for a.e. $\rho$ weakly and in the point process sense.*

Using this proposition and formulas (28), (29) it is possible to get the expressions for $\Pi(\theta)$ and $R(\theta)$ as they appear in [FIN02, BČ04],

$$R(\theta) = \bar{\mathbb{P}}[Z(1+\theta) = Z(1)], \tag{30}$$

$$\Pi(\theta) = \int_0^\infty \bar{\mathbb{P}}\big[\rho[Z(1)] \in du\big] e^{-\theta/u}. \tag{31}$$

From this point we consider only the processes $X_\varepsilon$ defined on the space $(\bar{\Omega}, \bar{\mathbb{P}})$. Therefore we simplify the notation by omitting the bars.

We will often use scaling arguments based on the fact that the following equalities in distribution hold. For any $t \geq 0$, $\lambda > 0$ and $x \in \mathbb{R}$

$$\begin{aligned} V(x) &\overset{d}{=} \lambda^{-1/\alpha} V(\lambda x) \qquad & \ell(t,x) &\overset{d}{=} \lambda^{-1}\ell(\lambda^2 t, \lambda x) \\ W(t) &\overset{d}{=} \lambda^{-1} W(\lambda^2 t) \qquad & Z(t) &\overset{d}{=} \lambda^{-1} Z(t\lambda^{(1+\alpha)/\alpha}). \end{aligned} \tag{32}$$

The equalities in the left column are well known. The first equality on the right can be derived easily using properties of the local time. The forth equality follows from the definition of $Z$ and the previous three equalities. Indeed, let $V^\lambda(x)$, $W^\lambda(t)$, and $\ell^\lambda(t,x)$ be the right-hand sides of first three equalities in (32). We define $Z^\lambda$ as the time change of $W^\lambda$ using the Lebesgue-Stieltjes measure associated to $V^\lambda$ as the speed measure. Then using the notation from Definition 2.1 we get $\phi^\lambda(t) := \int \ell^\lambda(t,x)\, dV^\lambda(x) = \lambda^{-(1+\alpha)/\alpha}\phi(\lambda t^2)$. Therefore, its inverse $\psi^\lambda(s) = \lambda^{-2}\psi(\lambda^{(1+\alpha)/\alpha}s)$ and $Z^\lambda(t) := W^\lambda(\psi^\lambda(t)) = \lambda Z(\lambda^{(1+\alpha)/\alpha}t)$.

Finally, we introduce some additional analytical objects related to time changing. Let $Y$ be the time change of Brownian motion with locally finite speed measure $\nu$. Then there exists a function $p_\nu(x,y;t)$ such that for any Borel $B \subset \mathbb{R}$, $t > 0$ (see for example [RW00])

$$\mathbb{P}_x[Y(t) \in B] = \int_B p_\nu(x,y;t)\nu(dy), \tag{33}$$

where we use the symbol $\mathbb{P}_x$ for distribution of the process started at $x$, that is defined naturally by time changing of Brownian motion started at $x$. The function $p_\nu$ is given by (33) only for $x, y$ in the support of $\nu$. Therefore, we define $p_{\mu_\varepsilon}(x,y,t) =: p_\varepsilon(x,y,t)$ for $x, y \notin \varepsilon\mathbb{Z}$ by linear interpolation. The set of atoms of $\rho$ is a dense subset of $\mathbb{R}$. It is a result of the theory of quasi-diffusions that this function is continuous on this set if $t > 0$, and therefore has unique continuous extension on $\mathbb{R}$ that we call also $p_\rho$. We summarise some known properties of $p_\varepsilon$, $p_\rho$ in the following lemma.

**Lemma 2.5.** *Let $\nu \in \{\rho, \mu_\varepsilon\}$ and let $x$ be an atom of $\nu$. Then*

*(i) The function $p_\nu$ is a solution of the system*

$$\begin{aligned} \frac{\partial p_\nu}{\partial t}(x,y;t) &= \mathcal{L}_\nu p_\nu(x,y;t) \\ p_\nu(x,y;0) &= \nu(x)^{-1}\delta_x(y). \end{aligned} \tag{34}$$

*The generator $\mathcal{L}_{\mu_\varepsilon} =: \mathcal{L}_\varepsilon$ of the process $X_\varepsilon$ is weighted discrete Laplace operator, that is for $y \in \varepsilon\mathbb{Z}$*

$$\mathcal{L}_\varepsilon f(y) = \frac{\partial^2 f(y)}{\mu_\varepsilon(\partial y)\partial y} = \frac{1}{2\tau^\varepsilon_{y\varepsilon^{-1}}\varepsilon^{1/\alpha\gamma}}\big\{f(y+\varepsilon) + f(y-\varepsilon) - 2f(y)\big\}. \tag{35}$$

*For the definition of the generator $\mathcal{L}_\rho f = d^2 f/d\rho dx$ (whose properties we do not use) we refer to* [DM76, KW82].

*(ii) $p_\nu$ is continuous on the set $\{(x,y,t) : x, y \in \mathbb{R}, t > 0\}$, and $p_\nu(x,y;t) = p(y,x;t)$.*

*(iii) For all $t > 0$, $x \in \mathbb{R}$, the function $p_\nu(x,\cdot,t)$ has one maximum in $\mathbb{R}$ and there exists an interval $I(x,t)$ such that it is concave on $I(x,t)$ and convex outside of it.*

*(iv) The Dirichlet form $\mathcal{E}_\varepsilon$ associated to $X_\varepsilon$ defined on $L^2(\mu_\varepsilon)$ is given by*

$$\mathcal{E}_\varepsilon(f,g) = \int f\mathcal{L}_\varepsilon g d\mu_\varepsilon = \frac{1}{2\varepsilon}\sum_{x\in\varepsilon\mathbb{Z}}\big(f(x+\varepsilon) - f(x)\big)\big(g(x+\varepsilon) - g(x)\big). \tag{36}$$

We will use $p_\varepsilon(x)$ as a shorthand for $p_{\mu_\varepsilon}(0,x;1)$, similarly we write $p_\rho(x)$ for $p_\rho(0,x;1)$. We use the letters $C$, $c$ to denote positive constants that have no particular importance. The value of these constants can change during computations.

## 3. Some estimates on $p_\varepsilon$

In this section we give some estimates on $p_\varepsilon$, which can be of their own interest. In the averaged case they correspond up to multiplicative constants to the numerical results obtained in [BB03]. Some of the averaged results proved here can be obtained more easily using scaling arguments. However, as we will need quenched results later, we use more robust techniques.

**Lemma 3.1.** *(Diagonal upper bound) Let $\mathcal{B}(x,r)$ denotes the closed ball around $x$ with radius $r$, $\mathcal{B}(x,r)=[x-r,x+r]$. We use $\mathcal{V}_\varepsilon(x,r)$ to denote its volume with respect to $\mu_\varepsilon$, $\mathcal{V}_\varepsilon(x,r)=\mu_\varepsilon(\mathcal{B}(x,r))$. Then for any $x\in\mathbb{R}$ and $r>0$*

$$p_\varepsilon\big(x,x;4r\mathcal{V}_\varepsilon(x,r)\big)\le\lim_{s\uparrow r}\frac{2}{\mathcal{V}_\varepsilon(x,s)}. \tag{37}$$

*Proof.* We use a similar method as in [BCK04]. Without loss of generality we assume that $x=0$. To simplify the notation we define $\mathcal{V}_\varepsilon(r)=\mathcal{V}_\varepsilon(0,r)$, and $\mathcal{B}(r)=\mathcal{B}(0,r)=[-r,r]$. Since all arguments are independent of $\varepsilon$ we omit it from the notation. Let $\delta>0$ and let $\mathcal{U}_\delta(r)$ be an increasing smooth function satisfying

$$\mathcal{V}(r-\delta)\le\mathcal{U}_\delta(r)\le\mathcal{V}(r). \tag{38}$$

Set $f_t(y)=p_\varepsilon(0,y;t)$ and $\psi(t)=\|f_t\|^2_{L^2(\mu_\varepsilon)}$. Markov property and Lemma 2.5(ii) imply that $\psi(t)=f_{2t}(0)$. Since, by (33),

$$\int_{[-r,r]}f_t(y)\mu_\varepsilon(dy)<1, \tag{39}$$

there exists $y=y(r)\in[-r,r]$ with $f_t(y)\le\mathcal{V}(r)^{-1}\le\mathcal{U}_\delta(r)^{-1}$. To estimate $p(0,0;t)=f_t(0)=\psi(t/2)$ we write

$$\frac{1}{2}f_t(0)^2\le f_t(y)^2+|f_t(y)-f_t(0)|^2. \tag{40}$$

Let $I=[0\wedge y,0\vee y]$ and let $\bar f_t(x)=f_t(x)\mathbb{1}\{x\in I\}$. Further, let $h(x)=xy^{-1}\mathbb{1}\{x\in I\}$. Then $h$ is a harmonic function for $\mathcal{L}_\varepsilon$ on $I$ as can be checked easily. From well known properties of Dirichlet forms (or using the Cauchy-Schwarz inequality) it follows that

$$\begin{aligned}\mathcal{E}(f_t,f_t)&\ge\mathcal{E}(\bar f_t,\bar f_t)\\&\ge\mathcal{E}(h,h)(\bar f_t(y)-\bar f_t(0))^2=(2|y|)^{-1}(f_t(y)-f_t(0))^2.\end{aligned} \tag{41}$$

Putting the last two displays together we get

$$2r\mathcal{E}(f_t,f_t)\ge-\frac{1}{\mathcal{U}_\delta(r)^{-2}}+\frac{1}{2}\psi(t/2)^2. \tag{42}$$

From (36)

$$\psi'(t)=-2\mathcal{E}(f_t,f_t)\le\frac{2\mathcal{U}_\delta(r)^{-2}-\psi(t/2)^2}{2r}, \tag{43}$$

and $\psi''(t) = 4\int(\mathcal{L}f_t)^2\,d\mu \geq 0$, so $\psi'(t/2) \leq \psi'(t)$. Hence,

$$\psi'(t) \leq \psi'(2t) \leq \frac{2\mathcal{U}_\delta(r)^{-2} - \psi(t)^2}{2r}. \tag{44}$$

We set $\phi(t) = 2/\psi(t)$, so $\phi$ is increasing and

$$\phi'(t) = -\frac{1}{2}\phi^2(t)\psi'(t) \geq \frac{2 - \phi(t)^2\mathcal{U}_\delta(r)^{-2}}{2r}. \tag{45}$$

Now take $r = r(t)$ such that $\phi(t) = \mathcal{U}_\delta(r(t))$. This gives $\phi'(t) = r'(t)\mathcal{U}_\delta'(r(t)) \geq (2r(t))^{-1}$ and thus

$$t \leq 2\int_0^t r(s)r'(s)\mathcal{U}_\delta'(r(s))ds = 2\int_0^{r(t)} u\mathcal{U}_\delta'(u)\,du \leq 2r(t)\mathcal{U}_\delta(r(t)). \tag{46}$$

Fix $R$ and choose $s$ such that $R = r(s)$. Then $s \leq 2r(s)\mathcal{U}_\delta(r(s)) = 2R\mathcal{U}_\delta(R)$, so $\phi(2R\mathcal{U}_\delta(R)) \geq \phi(s) = \mathcal{U}_\delta(R)$. Using now the definitions of $\psi$, $\phi$, and $f_t$ we get

$$p(0,0;4R\mathcal{U}_\delta(R)) \leq \frac{2}{\mathcal{U}_\delta(R)}. \tag{47}$$

We finish the proof using (38) and taking limit $\delta \to 0$ . □

We further prove the off-diagonal upper bounds. We will not estimate directly $p_\varepsilon(x,y;t)$ but only $\mathbb{P}_x[\sup_{s\leq t}|X_\varepsilon(s) - x| > D]$. As the notation used in the upper bounds in the quenched case is relatively complicated, we first present an argument leading to the upper bound and then formulate the obtained results as a lemma. In the proof we follow [Bas02] with modifications that are needed for random environment and sub-diffusive decay.

For the Brownian motion $W$ that is used in the time change (and that is not necessarily started at the origin) we define

$$T^r = \inf\{t : W(t) - W(0) \geq r\}. \tag{48}$$

Let

$$S_\varepsilon(r) = \int_{[W(0),W(0)+r)} \ell(T^r, x)\mu_\varepsilon(dx), \tag{49}$$

which means that $S_\varepsilon(r)$ is the time that $X_\varepsilon$ spends in $[X_\varepsilon(0), X_\varepsilon(0)+r)$ before hitting the right border of this interval. Note that due to scaling properties of Brownian motion $W$ and Lévy process $V$ (see (32)) the random variable $S_\varepsilon(r)$ has the same distribution as $r^{(1+\alpha)/\alpha}S_{\varepsilon/r}(1)$. Similarly, conditionally on $\rho = \rho_0$ and $W(0) = x$, the random variable $S_\varepsilon(r)$ has the same distribution as $r^{(1+\alpha)/\alpha}S_{\varepsilon/r}(1)$ conditionally on $W(0) = 0$ and $\rho = \mathsf{Sc}_{x,r}(\rho_0)$, where the scaling $\mathsf{Sc}_{x,r}$ of measure is defined by

$$\mathsf{Sc}_{x,r}(\rho)(dx) = \sum_i r^{-1/\alpha}v_i\delta_{(x_i-x)/r}(dx) \qquad \text{if} \qquad \rho(dx) = \sum_i v_i\delta_{x_i}(dx). \tag{50}$$

Note also that $\mathsf{Sc}_{x,r}(\rho)$ has the same distribution as $\rho$.

Take now $D$, $t$, $r$, and $n$ such that for some $a$ which will be fixed later these constants satisfy

$$n = \left[a\Big(\frac{D}{t^{\alpha\gamma}}\Big)^{1+\alpha}\right] \qquad \text{and} \qquad r = D/n. \tag{51}$$

Then, for fixed $\varepsilon$, using the equalities in distribution discussed in the previous paragraph, we get

$$\begin{aligned} &\mathbb{E}_x[e^{-nt^{-1}S_\varepsilon(r)}|\rho] = \mathbb{E}_0[e^{-a^{-1/\alpha}S_{\varepsilon/r}(1)}|\,\mathsf{Sc}_{x,r}(\rho)] =: \exp(-H^a_\varepsilon(x,r;\rho)),\\ &\mathbb{E}_x[e^{-nt^{-1}S_\varepsilon(r)}] = \mathbb{E}_0[e^{-a^{-1/\alpha}S_{\varepsilon/r}(1)}] =: \exp(-\mathcal{H}^a_{\varepsilon/r}). \end{aligned} \tag{52}$$

The constants $\mathcal{H}^a_\varepsilon$ converge to some constant $\mathcal{H}^a$ as $\varepsilon \to 0$ as follows from the vague convergence of measures $\mu_\varepsilon$ and (49). Similarly, the distribution of $H^a_\varepsilon(x,r;\rho)$ converges as $\varepsilon/r \to 0$. This condition corresponds to

$$r \gg \varepsilon \quad \text{or} \quad D \ll \varepsilon^{-1/\alpha} t a^{-1/\alpha}. \tag{53}$$

Finally, we fix $a$ and $\varepsilon_1$ small enough such that for all $\varepsilon < \varepsilon_1$

$$\mathbb{E}H^a_\varepsilon(x,r;\rho) \geq 2 \qquad \text{and} \qquad \mathcal{H}^a_\varepsilon \geq 2. \tag{54}$$

Let $T_0 = 0$ and define inductively $T_{i+1} = T^r \circ \theta_{T_i}$, where $\theta_t$ is the standard time shift. Then, by decomposition at $T_i$,

$$\begin{aligned} \mathbb{P}_x\big[\sup_{s\leq t} X_\varepsilon(s) - x \geq D\big|\rho\big] &\leq \mathbb{P}_x\Big[\sum_{i=0}^{n-1} S_\varepsilon(r)\circ\theta_{T_i} \leq t\Big|\rho\Big]\\ &\leq \mathbb{P}_x\Big[\prod_{i=0}^{n-1} \exp(-\lambda S_\varepsilon(r)\circ\theta_{T_i}) \geq e^{-\lambda t}\Big|\rho\Big] \end{aligned} \tag{55}$$

Take $\lambda = nt^{-1}$. Since $S_\varepsilon(r)\circ\theta_{T_i}$ are independent for different indices $i$, the last expression is bounded by

$$\leq e^n \prod_{i=0}^{n-1} \mathbb{E}_{x+ir}\big[\exp(-nt^{-1}S_\varepsilon(r))\big|\rho\big] = \exp\Big\{n - \sum_{i=0}^{n-1} H^a_\varepsilon(x+ir,r;\rho)\Big\}, \tag{56}$$

and similarly for the averaged case

$$\mathbb{P}_x\big[\sup_{s\leq t} X_\varepsilon(s) - X_\varepsilon(0) \geq D\big] \leq \exp\Big\{n - \sum_{i=0}^{n-1}\mathcal{H}^a_{\varepsilon/r}\Big\} \leq \exp(-n) \tag{57}$$

for all $\varepsilon/r < \varepsilon_1$. We have proved the claim (a) of the following lemma. The claim (b) can be proved using the same reasoning as in (55),(56) taking $\lambda = r = 1$, $a = 1$ instead of (51) and $\lambda = nt^{-1}$. Note that the estimate (b) is far to be optimal.

**Lemma 3.2.** *(a) Let* $D \le t(a\varepsilon_1\varepsilon)^{-1/\alpha}$ *(i.e.* $\varepsilon/r \le \varepsilon_1$*), and let* $n$ *be given by* (51)*. Then*

$$\mathbb{P}_x\big[\sup_{s\le t} X_\varepsilon(s) - x \ge D\big|\rho\big] \le \exp\Big\{n - \sum_{i=0}^{n-1} H_\varepsilon^a(x+ir,r;\rho)\Big\}, \quad (58)$$

$$\mathbb{P}_x\big[\sup_{s\le t} X_\varepsilon(s) - x \ge D\big] \le \exp\Big\{-a\Big(\frac{D}{t^{\alpha\gamma}}\Big)^{1+\alpha}\Big\}. \quad (59)$$

*(b) For all* $D \ge 0$*,*

$$\mathbb{P}_x\big[\sup_{s\le t} X_\varepsilon(s) - x \ge D\big|\rho\big] \le \exp\Big\{t - \sum_{i=0}^{D-1} H_\varepsilon^1(x+i,1;\rho)\Big\}, \quad (60)$$

$$\mathbb{P}_x\big[\sup_{s\le t} X_\varepsilon(s) - x \ge D\big] \le C\exp\{t - cD\}. \quad (61)$$

*(c) For all* $r > 0$ *the random variables* $H_\varepsilon^a(x+ir,r;\rho)$ *converge* $\rho$*-a.s. as* $\varepsilon \to 0$*.*

This lemma has a simple corollary, which follows from the weak convergence $X_\varepsilon(t) \to Z(t)$ as $\varepsilon \to 0$, namely the following sub-diffusive estimate holds.

**Corollary 3.3.** *For any* $t > 0$ *and* $x > 0$ *the singular diffusion* $Z$ *satisfies*

$$\mathbb{P}\big[|Z(t)| \ge x\big] \le C\exp\big(-c(x/t^{\alpha\gamma})^{1+\alpha}\big). \quad (62)$$

## 4. Short time behaviour of function Π

In this section we prove Theorem 1.1(a). We show that the second of the two strategies discussed in Introduction dominates. More precisely, we show that the traps with depth of order $f(t)$ give the largest contribution to $1 - \Pi(t, t+f(t))$. We will use expression (29) to compute $\Pi(t, t+f(t))$. Note that in the language of processes $X_\varepsilon$ the traps of original BTM with depth $f(t_\varepsilon)$ correspond to atoms of $\mu_\varepsilon$ with the weight

$$h(\varepsilon) := f(t_\varepsilon)/t_\varepsilon^\gamma = f(\varepsilon^{-1/\alpha\gamma})\varepsilon^{1/\alpha}. \quad (63)$$

From the assumptions of Theorem 1.1(a) follows that the function $h$ satisfies

$$\varepsilon^{(1-(1+\alpha)\mu)/\alpha} \le h(\varepsilon) \le \varepsilon^{(1-(1+\alpha)\kappa)/\alpha} \quad (64)$$

for all $\varepsilon$ small enough and therefore

$$\lim_{\varepsilon\to 0} h(\varepsilon) = 0. \quad (65)$$

To estimate the contributions of traps with different depth we define for $a, b \in [0,\infty]$ a random variable $\mathcal{Z}(\varepsilon; a, b)$,

$$\mathcal{Z}(\varepsilon; a,b) = h(\varepsilon)^{\alpha-1}\int_a^b \mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1)) \in du\big|\rho\big]\Big[1 - \exp\Big(-\frac{h(\varepsilon)}{u}\Big)\Big]. \quad (66)$$

By (29), the quantity we want to control satisfies

$$\left(\frac{f(t_\varepsilon)}{t_\varepsilon^\gamma}\right)^{\alpha-1}\big(1-\Pi(t_\varepsilon,t_\varepsilon+f(t_\varepsilon))\big)=\mathbb{E}\mathcal{Z}(\varepsilon;0,\infty)=:\mathbb{E}\mathcal{Z}(\varepsilon). \tag{67}$$

Theorems 1.1 (a) and 1.3 are therefore direct consequences of the following proposition

**Proposition 4.1.** *As $\varepsilon\to 0$ the family $\mathcal{Z}(\varepsilon)$ converges $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$ to a nontrivial random variable $\mathcal{Z}$ defined by*

$$\mathcal{Z}=\Gamma(\alpha+2)\frac{\alpha}{1-\alpha}\int_{-\infty}^{\infty}p_\rho(x)\,dx. \tag{68}$$

*The constant $K_1$ defined in Theorem 1.1 is equal to $\mathbb{E}\mathcal{Z}$.*

*Proof.* Choose two constants $\eta_1$, $\eta_2$ which satisfy

$$\frac{1}{\alpha}>\eta_1>\frac{1}{\alpha}-\frac{\mu(\alpha+1)}{\alpha}>1-\kappa(\alpha+1)>\eta_2>0. \tag{69}$$

With this choice $\varepsilon^{\eta_1}\le h(\varepsilon)\le\varepsilon^{\eta_2/\alpha}\le\varepsilon^{\eta_2}$ for $\varepsilon<1$.

To show that the second of two strategies dominates the behaviour of $\mathcal{Z}(\varepsilon)$ we first prove that the atoms with weights larger than $\varepsilon^{\eta_2}$ do not contribute, that is

$$\lim_{\varepsilon\to 0}\mathcal{Z}(\varepsilon;\varepsilon^{\eta_2},\infty)=0 \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}). \tag{70}$$

Indeed, since $1-e^{-x}\le x$, we have uniformly in $\rho$

$$\begin{aligned}h(\varepsilon)^{\alpha-1}\int_{\varepsilon^{\eta_2}}^{\infty}&\mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1))\in du\big|\rho\big]\Big(1-\exp\big(-\frac{h(\varepsilon)}{u}\big)\Big)\\&\le h(\varepsilon)^{\alpha-1}\big(1-\exp(h(\varepsilon)\varepsilon^{-\eta_2})\big)\le h(\varepsilon)^{\alpha}\varepsilon^{-\eta_2}.\end{aligned} \tag{71}$$

Using (64) we can bound the last expression by $C\varepsilon^{1-\kappa(1+\alpha)-\eta_2}$, by (69) $1-\kappa(1+\alpha)-\eta_2>0$. This proves (70).

We must now control the remaining part, that is $\mathcal{Z}(\varepsilon;0,\varepsilon^{\eta_2})$. Set

$$F_\varepsilon(u)=\mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1))\le u\big|\rho\big]. \tag{72}$$

Integrating by parts we get

$$\begin{aligned}\mathcal{Z}(\varepsilon;0,\varepsilon^{\eta_2})=h(\varepsilon)^{\alpha-1}\Big[&F_\varepsilon(\varepsilon^{\eta_2})\big(1-\exp(h(\varepsilon)\varepsilon^{-\eta_2})\big)\\&+\int_0^{\varepsilon^{\eta_2}}\frac{F_\varepsilon(u)h(\varepsilon)}{u^2}\exp\Big(-\frac{h(\varepsilon)}{u}\Big)\,du\Big].\end{aligned} \tag{73}$$

As $\varepsilon\to 0$ the first term becomes negligible as can be shown using a similar estimate as for (71). Dividing further the domain of integration in the second term into $[0,\varepsilon^{\eta_1})$ and $[\varepsilon^{\eta_1},\varepsilon^{\eta_2})$, and using the substitution

$v = h(\varepsilon)^{-1}u$, we get for the first part

$$h(\varepsilon)^\alpha \int_0^{\varepsilon^{\eta_1}} \frac{F_\varepsilon(u)}{u^2} \exp\Big( - \frac{h(\varepsilon)}{u}\Big)\, du$$
$$\leq h(\varepsilon)^{\alpha-1} \int_0^{\varepsilon^{\eta_1} h(\varepsilon)^{-1}} v^{-2} e^{-1/v}\, dv. \quad (74)$$

For small values of $\varepsilon$ the integrand is increasing on the whole domain of integration, so that we can bound the previous expression by $Q(\varepsilon^{-1})\exp(-\varepsilon^{-c})$ for some $c > 0$ and for some at most polynomially increasing $Q$. Therefore (74) tends to zero uniformly in $\rho$. The result of the previous paragraph and (70) imply that if the following limits exist, then $\lim_{\varepsilon\to 0} \mathcal{Z}(\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{Z}_1(\varepsilon)$, where

$$\mathcal{Z}_1(\varepsilon) = h(\varepsilon)^\alpha \int_{\varepsilon^{\eta_1}}^{\varepsilon^{\eta_2}} \frac{F_\varepsilon(u)}{u^2} \exp\Big( - \frac{h(\varepsilon)}{u}\Big)\, du. \quad (75)$$

We can see now that we need to study the behaviour of $F_\varepsilon$ for small values of $u$. In the next section we will prove

**Proposition 4.2.** *For $\mathbb{P}$-a.e. $\rho$, the function $F_\varepsilon$ defined in* (72) *can be written as*

$$F_\varepsilon(u) = \mathcal{C}_\varepsilon u^{1-\alpha} + f_\varepsilon(u) u^{1-\alpha}, \quad (76)$$

*where*

$$\mathcal{C}_\varepsilon = \frac{\alpha}{1-\alpha} \int_{-\infty}^{\infty} p_\varepsilon(x)\, dx. \quad (77)$$

*$\mathcal{C}_\varepsilon$ and $f_\varepsilon$ further satisfy*

$$\lim_{\varepsilon\to 0} \mathcal{C}_\varepsilon = \frac{\alpha}{1-\alpha} \int_{-\infty}^{\infty} p_\rho(x)\, dx =: \mathcal{C} \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}), \quad (78)$$

$$\lim_{\varepsilon\to 0} \sup \big\{ |f_\varepsilon(u)| : u \in (\varepsilon^{\eta_1}, \varepsilon^{\eta_2}) \big\} = 0 \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}). \quad (79)$$

*Remark.* It is necessary to exclude $u \in (0, \varepsilon^{\eta_1})$ from the supremum in (79), since the behaviour of $F_\varepsilon$ for such $u$ is influenced by the behaviour of the distribution of $\tau_0$ near the origin that is not specified.

First, we use Proposition 4.2 to finish the proof of Proposition 4.1. Applying again the substitution $v = h(\varepsilon)^{-1}u$ we obtain

$$\mathcal{Z}_1(\varepsilon) = \mathcal{C}_\varepsilon \int_{\varepsilon^{\eta_1} h(\varepsilon)^{-1}}^{\varepsilon^{\eta_2} h(\varepsilon)^{-1}} v^{-1-\alpha} \exp(-1/v)\, dv + R(\varepsilon). \quad (80)$$

Using (78) and the fact that, by (69), the integration domain converges to $(0,\infty)$, we obtain that the main term converges to

$$\mathcal{C} \int_0^\infty v^{-1-\alpha} \exp(-1/v)\, dv. \quad (81)$$

The last display is equal to $\mathcal{Z}$ (see (68)) as can be verified by a simple integration. The absolute value of the error term $R(\varepsilon)$ can be bounded for $\varepsilon$ small enough by

$$2\sup\big\{|f_\varepsilon(u)| : u \in (\varepsilon^{\eta_1}, \varepsilon^{\eta_2})\big\} \cdot \int_0^\infty v^{-1-\alpha} \exp(-1/v)\, dv, \tag{82}$$

which is negligible by (79). □

*Remark.* The claim (17) of Theorem 1.1(c) can be proved by methods that are very similar to the methods that we used in the just finished proof. They differ only in the way how the limits are taken. To prove Theorem 1.1(a) we needed to take the limit $\varepsilon \to 0$ and in the same time observe the atoms with the weight close to $h(\varepsilon) \to 0$. To show (17), the limit $\varepsilon \to 0$ must be taken first, and then the probability that $\rho(Z(1))$ is of order $\theta \to 0$ should be studied. It is easy to observe that this change does not produce any problems in the argumentation we presented. The equivalent of Proposition 4.2, that is $\mathbb{P}[\rho(Z(1)) \le u|\rho] = \mathcal{C}u^{1-\alpha}(1+o(1))$ as $u \to 0$ is a direct consequence of smoothness of $p_\rho$ and properties (24) of $\rho$. Similar remarks apply also to proofs of (18) and Theorem 1.2(b).

## 5. Proof of Proposition 4.2

To finish the proof of Theorem 1.1(a) we need to show Proposition 4.2. We first prove claim (78).

**Lemma 5.1.**

$$\int_{-\infty}^\infty p_\varepsilon(x)\, dx \to \int_{-\infty}^\infty p_\rho(x)\, dx \qquad \textit{as } \varepsilon \to 0 \quad \mathbb{P}\textit{-a.s. and in } L^1(\mathbb{P}). \tag{83}$$

*Proof.* First, we use Lemma 3.2 to estimate $\int_K^\infty p_\varepsilon(x)\,dx$ for large $K$. The integral over $(-\infty,-K]$ can be treated in the same way, while for the integral over $[-K,K]$ we will need different methods. The functions $p_\rho$ and also $p_\varepsilon$ with $\varepsilon$ small enough are decreasing on $[K,\infty)$ if $K$ is large enough. Therefore,

$$\int_K^\infty p_\varepsilon(x)\, dx \le \sum_{k=K}^\infty p_\varepsilon(k). \tag{84}$$

Further, for $k \ge K$,

$$\begin{aligned}\mathbb{P}[X_\varepsilon(1) \ge k|\rho] &\ge \mathbb{P}\big[X_\varepsilon(1) \in [k,k+1)\big|\rho\big]\\ &\ge p_\varepsilon(k+1)\mu_\varepsilon\big([k,k+1)\big)\end{aligned} \tag{85}$$

Using this and Lemma 3.2(b) we get

$$p_\varepsilon(k) \le \frac{C\prod_{j=0}^{k-2}\exp(-H^1_\varepsilon(j,1;\rho))}{\mu_\varepsilon\big([k-1,k)\big)} =: q_\varepsilon(k). \tag{86}$$

and

$$\int_K^\infty p_\varepsilon(x)\,dx \le \sum_{k=K}^\infty q_\varepsilon(k). \tag{87}$$

To control the denominator of (86) we need one technical lemma.

**Lemma 5.2.** *The family $\tau_i^\varepsilon$ satisfy*

$$\sup_{\varepsilon\le 1}\mathbb{E}\big[\mu_\varepsilon\big([0,1)\big)^{-1}\big] = \sup_{\varepsilon\le 1}\mathbb{E}\Big[\Big\{\varepsilon^{1/\alpha}\sum_{i=1}^{\varepsilon^{-1}}\tau_i\Big\}^{-1}\Big] < \infty. \tag{88}$$

*Proof.* Let $F_\tau$ be the common distribution function of $\tau_i^\varepsilon$'s. Since $\tau_i^\varepsilon$'s satisfy (4) and (14), there exists $\kappa > 0$ small such that $F_\tau(x) \le 1-\kappa x^{-\alpha}$ for all $x \ge c$. Define

$$F_\sigma(x) = \begin{cases} 1-\kappa x^{-\alpha} & \text{for } x \ge c \\ 0 & \text{for } 0 \le x < c. \end{cases} \tag{89}$$

Since $F_\sigma(x) \ge F_\tau(x)$ for all $x \ge 0$ there exists a sequence of i.i.d. random variables $\sigma_i$ with the common distribution function $F_\sigma$ defined on the same probability space as $\tau_i$ satisfying $\sigma_i \le \tau_i$ for all $i$. Therefore

$$\mathbb{E}\Big[\Big\{\varepsilon^{1/\alpha}\sum_{i=0}^{\varepsilon^{-1}}\tau_i\Big\}^{-1}\Big] \le \mathbb{E}\Big[\Big\{\varepsilon^{1/\alpha}\max_{i\le\varepsilon^{-1}}\sigma_i\Big\}^{-1}\Big] := \mathbb{E}[Y_\varepsilon]. \tag{90}$$

The distribution function of $Y_\varepsilon$ satisfies

$$F_{Y_\varepsilon}(x) = \begin{cases} 1 & \text{for } x \ge \varepsilon^{-1/\alpha}c^{-1} \\ 1-(1-\kappa x^\alpha\varepsilon)^{\varepsilon^{-1}} & \text{for } x < \varepsilon^{-1/\alpha}c^{-1}. \end{cases} \tag{91}$$

The expectation of $Y_\varepsilon$ is therefore

$$\begin{aligned}\mathbb{E}[Y_\varepsilon] &= \varepsilon^{-1/\alpha}c^{-1}(1-\kappa c^{-\alpha})^{\varepsilon^{-1}} + \int_0^{\varepsilon^{-1/\alpha}c^{-1}} \kappa\alpha x^\alpha(1-\varepsilon\kappa x^\alpha)^{-1+\varepsilon^{-1}}\,dx \\ &\le C + c\int_0^\infty x^\alpha \exp[-\varepsilon\kappa x^\alpha(\varepsilon^{-1}-1)]\,dx.\end{aligned} \tag{92}$$

The last expression can be easily bounded uniformly for small $\varepsilon$. This finishes the proof. □

We can now bound (86) and (87). It follows from Lemma 3.2 that $H_\varepsilon^1(j,1;\rho)$ converge for and all $j \in \mathbb{N}$ as $\varepsilon \to 0$ to some nontrivial random variable. The random variables $H_\varepsilon^1(j,1;\rho)$ with different indices $j$ depend on $\rho$ on disjoint intervals, therefore they are mutually independent. The denominator of $q_\varepsilon(k)$ is also independent of any $H_\varepsilon^1(j,1;\rho)$ that appears in the numerator. From these claims and Lemma 5.2 it follows that there exist $q < 1$ and $C < \infty$ such that for all $\varepsilon$ small enough $\mathbb{E}[q_\varepsilon(k)] \le Cq^k$. Therefore, the expectation of (87) is finite.

For any $\delta > 0$ and for a.e. $\rho$ it is thus possible to choose $K = K(\rho)$ independent of $\varepsilon$ such that for all $\varepsilon$ small enough

$$\int_{|x|\geq K} p_\varepsilon(x)\,dx \leq \delta/2. \tag{93}$$

Similarly, one can choose $L$ such that

$$\mathbb{E}\int_{|x|\geq L} p_\varepsilon(x)\,dx \leq \delta/2. \tag{94}$$

Inside of the interval $[-K, K]$ it is not difficult to conclude from the regularity properties of $p_\varepsilon$ and from the point process convergence of fixed time distributions of $X_\varepsilon$ (Proposition 2.4) that

$$p_\varepsilon(x) \to p_\rho(x) \qquad \text{as } \varepsilon \to 0 \text{ uniformly for } x \in [-K, K],\ \mathbb{P}\text{-a.s.} \tag{95}$$

Indeed, fix $\delta > 0$ and choose $\delta'$ small enough such that the set $I(\delta')$ of atoms of $\rho$

$$I(\delta') = \{x : \rho[x] \geq \delta'\} \tag{96}$$

satisfies

$$\sup_{y\in[-K,K]} \min_{x\in I(\delta')} |x - y| \leq \delta, \tag{97}$$

and further, writing $I(\delta') =: \{x_1, \dots x_r\}$, with $x_i < x_{i+1}$ for all $i \in \{1, \dots, r-1\}$,

$$\left|p_\rho(x_i) - p_\rho(x_{i+1})\right| \leq \delta \qquad i \in \{1, \dots, r-1\}. \tag{98}$$

From the properties of $p_\rho$ (Lemma 2.5) it is not difficult to show that $I(\delta')$ is finite a.s. Further, by the point process convergence of fixed time distributions and of the sequence $\mu_\varepsilon$ (see Proposition 2.4), there exists a set $I(\delta', \varepsilon) = \{y_1^\varepsilon, \dots, y_r^\varepsilon\}$ of atoms of $\mu_\varepsilon$ such that $y_i^\varepsilon \to x_i$, $\mu^\varepsilon[y_i^\varepsilon] \to \rho[x_i]$, and $\mathbb{P}[X_\varepsilon(1) = y_i^\varepsilon|\rho] \to \mathbb{P}[Z(1) = x_i|\rho]$ as $\varepsilon \to 0$ (see [BČ04], Proposition 2.5). Since $p_\varepsilon(y) = \mathbb{P}[X_\varepsilon(1) = y|\rho]/\mu_\varepsilon(y)$, for all $\varepsilon$ small enough and for all $i \in \{1, \dots, r\}$

$$|y_i^\varepsilon - x_i| \leq \delta^2 \qquad \text{and} \qquad |p_\varepsilon(y_i^\varepsilon) - p_\rho(x_i)| \leq \delta^2. \tag{99}$$

Using the regularity properties of $p_\varepsilon$ and the last expression it is possible for $\delta$ small enough to identify three (not necessary disjoint) subintervals of $[-K, K]$ that cover $[-K, K]$ so that for all $\varepsilon$ small enough $p_\varepsilon$ should be increasing on the first, concave on the second, and decreasing on the third interval. Using (99) it is then easy to get uniform convergence of $p_\varepsilon$ to $p_\rho$ on any of these three intervals and thus on $[-K, K]$. This proves (95).

The uniform convergence (95) implies

$$\int_{-K}^{K} p_\varepsilon(x)\,dx \to \int_{-K}^{K} p_\rho(x)\,dx \qquad \mathbb{P}\text{-a.s.} \tag{100}$$

and this together with (93) gives the a.s. convergence of Lemma 5.1.

It remains to verify the $L^1$ convergence of the integral in (94). Note that the function $p_\varepsilon$ can become very large here. It is thus a priori not clear if $\mathbb{E}\big[\int_{-L}^{L} p_\rho \, dx\big]$ is finite. To control this integral we estimate $\sup_{y\in[-L,L]} p_\varepsilon(y)$.

From the well known bound

$$p_\varepsilon(y) = p_\varepsilon(0, y; 1) \le \big(p_\varepsilon(0, 0; 1)p_\varepsilon(y, y; 1)\big)^{1/2} \tag{101}$$

it follows that it is sufficient to bound $\sup_{y\in[-C,C]} p_\varepsilon(y, y; 1)$. Using Lemma 3.1 we get for all $a \ge 16L$

$$\mathbb{P}[p_\varepsilon(y, y; 1) \ge a] \le \mathbb{P}\Big[\frac{2}{\mathcal{V}_\varepsilon(y, r(\varepsilon)-)} \ge a\Big], \tag{102}$$

where $r(\varepsilon)$ is defined by

$$r(\varepsilon) = \inf\{r : 4r\mathcal{V}_\varepsilon(y, r) \ge 1\}. \tag{103}$$

We recall that $\mathcal{V}_\varepsilon(x, r)$ denotes the $\mu_\varepsilon$-measure of the closed ball $\mathcal{B}(x, r)$ with radius $r$ and centre $x$. Therefore,

$$\begin{aligned}\mathbb{P}[p_\varepsilon(y, y; 1) \ge a] &\le \mathbb{P}\big[r(\varepsilon) \ge a/8\big] = \mathbb{P}\big[\mathcal{V}_\varepsilon(y, a/8) \le 2/a\big] \\ &\le \mathbb{P}\big[\mathcal{V}_\varepsilon(0, a/8 - 2L) \le 2/a\big].\end{aligned} \tag{104}$$

For the last inequality we used the fact that

$$\mathcal{B}(0, a/8 - 2L) \subset \mathcal{B}(y, a/8) \qquad \text{for all } y \in [-L, L]. \tag{105}$$

Expression (104) can be bounded using the same strategy as we used in the proof of Lemma 5.2. Using the same notation as there, setting $b = a/8 - 2L$,

$$\begin{aligned}\mathbb{P}[\mathcal{V}_\varepsilon(b) \le 2/a] &\le \mathbb{P}\Big[\varepsilon^{1/\alpha}\sum_{i=0}^{b/\varepsilon}\tau_i \le 2/a\Big] \le \mathbb{P}\big[\max_{i\le b/\varepsilon}\sigma_i \le 2\varepsilon^{-1/\alpha}a^{-1}\big] \\ &\le \big[F_\sigma(2\varepsilon^{-1/\alpha}a^{-1})\big]^{b/\varepsilon} \le (1 - \kappa 2^{-\alpha}\varepsilon a^\alpha)^{b/\varepsilon} \le e^{-ca^\alpha b}.\end{aligned} \tag{106}$$

Therefore

$$\mathbb{P}[p_\varepsilon(y, y; 1) \ge a] \le c \exp(-c' a^{1+\alpha}) \tag{107}$$

for all large $a$, uniformly for all $y \in [-L, L]$ and for $\varepsilon$ small enough. It follows from (101) and (107) that the family of random variables $\sup_{y\in[-L,L]} p_\varepsilon(y)$ is uniformly integrable and therefore $\int_{-L}^{L} p_\varepsilon \, dx$ is uniformly integrable. In view of (94) and the already proved a.s. convergence this implies the $L^1$ convergence of Lemma 5.1. □

*Proof of* (76) *and* (79). We want to show that the function $F_\varepsilon(u) := \mathbb{P}[\mu_\varepsilon(X_\varepsilon(1)) \le u|\rho]$ can be written as $\mathcal{C}_\varepsilon u^{1-\alpha} + f_\varepsilon(u)u^{1-\alpha}$, where $f_\varepsilon$ is small in the sense of (79).

Recall that $\mu_\varepsilon(i\varepsilon) = \varepsilon^{1/\alpha}\tau_i^\varepsilon$. We define $\tilde{\tau}_i^\varepsilon(u) = \tau_i^\varepsilon \mathbb{1}\{\varepsilon^{1/\alpha}\tau_i^\varepsilon \le u\}$ and $\bar{\tau}_i^\varepsilon(u) = \tilde{\tau}_i^\varepsilon(u) - \mathbb{E}\tilde{\tau}_i^\varepsilon(u)$. It follows from (4) and [Fel71] Theorem VIII.9.2 that for all $m \in \mathbb{N}$

$$\mathbb{E}[(\tilde{\tau}_i^\varepsilon(u))^m] = \frac{\alpha}{m-\alpha}(u\varepsilon^{-1/\alpha})^{m-\alpha}(1+o(1)) \quad \text{as} \quad u\varepsilon^{-1/\alpha} \to \infty. \tag{108}$$

This implies that

$$\mathbb{E}[(\bar{\tau}_i^\varepsilon(u))^m] = C(u\varepsilon^{-1/\alpha})^{m-\alpha} \qquad \text{as} \qquad u\varepsilon^{-1/\alpha} \to \infty. \tag{109}$$

Using this notation we can write

$$u^{\alpha-1}F_\varepsilon(u) = \varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i=-\infty}^{\infty} p_\varepsilon(i\varepsilon)\mathbb{E}\tilde{\tau}_i^\varepsilon(u) + \varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i=-\infty}^{\infty} p_\varepsilon(i\varepsilon)\bar{\tau}_i^\varepsilon(u). \tag{110}$$

Applying (108) we get for the difference of the first term and $\mathcal{C}_\varepsilon$ (see (77))

$$\left|\varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i=-\infty}^{\infty} p_\varepsilon(i\varepsilon)\mathbb{E}\tilde{\tau}_i^\varepsilon(u) - \frac{\alpha}{1-\alpha}\int p_\varepsilon(x)\,dx\right| \to 0 \tag{111}$$

as $\varepsilon \to 0$, uniformly for $u \in (\varepsilon^{\eta_1}, \varepsilon^{\eta_2})$, $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. Therefore, the error that we make by replacing the first term of (110) by $\mathcal{C}_\varepsilon$ can be included into $f_\varepsilon(u)$.

To bound the second term of (110) we use the following lemma that is proved later in this section.

**Lemma 5.3.** *For any finite interval $I \subset \mathbb{R}$ let*

$$\begin{aligned} f_\varepsilon^u(I) &= \varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i:i\varepsilon^{-1}\in I}\bar{\tau}_i^\varepsilon(u), \\ f_\varepsilon(I) &= \sup\left\{\left|f_\varepsilon^u(I)\right| : u \in (\varepsilon^{\eta_1}, \varepsilon^{\eta_2})\right\}. \end{aligned} \tag{112}$$

*Then*

$$\lim_{\varepsilon\to 0} f_\varepsilon(I) = 0 \qquad \mathbb{P}\text{-a.s, in } L^1(\mathbb{P}), \text{ and in } L^2(\mathbb{P}). \tag{113}$$

We can now control the second term of (110). We start with the contribution of terms with $i\varepsilon \ge K$ for some large $K$, the terms with $i\varepsilon \le -K$ can be controlled in the same way. Fix $\delta > 0$. For $u \in (\varepsilon^{\eta_1}, \varepsilon^{\eta_2})$ we get using estimate (86)

$$\varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i:i\varepsilon\ge K} p_\varepsilon(i\varepsilon)\bar{\tau}_i^\varepsilon \le \sum_{k=K}^{\infty} q_\varepsilon(k) f_\varepsilon([k,k+1)). \tag{114}$$

As we have already discussed, $q_\varepsilon(k)$ depends only on $\rho$ in interval $[0,k)$ and satisfies $\mathbb{E}(q_\varepsilon(k)) \le Cq^k$ for some $q < 1$. Therefore

$$\mathbb{E}\Big[\sum_{k=K}^{\infty} q_\varepsilon(k) f_\varepsilon([k,k+1))\Big] \le C\mathbb{E}[f_\varepsilon([0,1))]\sum_{k=K}^{\infty} q^k < \infty. \tag{115}$$

Therefore, for all $\delta$ small and for a.e. $\rho$ it is possible to choose constants $K(\rho)$ and $L$ such that for all $\varepsilon$ small

$$\varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i:i\varepsilon\geq K(\rho)}p_\varepsilon(i\varepsilon)\bar{\tau}_i^\varepsilon\leq\delta\quad\text{and}\quad\mathbb{E}\Big[\varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i:i\varepsilon\geq L}p_\varepsilon(i\varepsilon)\bar{\tau}_i^\varepsilon\Big]\leq\delta. \tag{116}$$

It remains to estimate the sum over $|i\varepsilon|<K$.

$$\sup_{u\in(\varepsilon^{\eta_1},\varepsilon^{\eta_2})}\varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i:|i\varepsilon|<K}p_\varepsilon(i\varepsilon)\bar{\tau}_i^\varepsilon\leq f_\varepsilon((-K,K))\sup_{x\in(-K,K)}p_\varepsilon(x). \tag{117}$$

From uniform convergence of $p_\varepsilon$ to $p_\rho$ (see (95)) it follows that for all $\varepsilon$ small $\sup_{x\in(-K,K)}p_\varepsilon(x)\leq\sup_{x\in(-K,K)}p_\rho(x)+\delta$. Since $f_\varepsilon((-K,K))\to 0$ a.s., the left-hand side of (117) converges to 0 a.s. Taking expectation in (117) and using the Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathbb{E}\Big[&\sup_{u\in(\varepsilon^{\eta_1},\varepsilon^{\eta_2})}\varepsilon^{1/\alpha}u^{\alpha-1}\sum_{i:|i\varepsilon|<L}p_\varepsilon(i\varepsilon)\bar{\tau}_i^\varepsilon\Big]\\&\leq\mathbb{E}\big[f_\varepsilon((-L,L))^2\big]^{1/2}\mathbb{E}\big[(\sup_{x\in(-L,L)}p_\rho(x)+\delta)^2\big]^{1/2}.\end{aligned} \tag{118}$$

The $L^1$ convergence then follows from $L^2$ convergence of $f_\varepsilon(I)$ that we proved in Lemma 5.3 and the fact that $\mathbb{E}\big[(\sup_{x\in(-L,L)}p_\rho(x)+\delta)^2\big]$ is finite by (107) and (101). □

*Proof of Lemma 5.3.* Without loss of generality we can take $I=[0,1]$. Fix $a>0$. Using a standard $2k$-th moment method we get

$$\begin{aligned}&\mathbb{P}\big[f_\varepsilon^u(I)>a\big]\\&\quad\leq a^{-2k}u^{2k(\alpha-1)}\varepsilon^{2k/\alpha}\mathbb{E}\Big[\Big(\sum_{i=0}^{\varepsilon^{-1}}\bar{\tau}_i^\varepsilon(u)\Big)^{2k}\Big]\\&\quad=a^{-2k}u^{2k(\alpha-1)}\varepsilon^{2k/\alpha}\sum_{\substack{l_1,\dots,l_r\\\sum l_i=2k}}c(l_1,\dots,l_r)\prod_{j=1}^r\sum_{i_j=0}^{\varepsilon^{-1}}\mathbb{E}\big[(\bar{\tau}_{i_j}^\varepsilon(u))^{l_j}\big].\end{aligned} \tag{119}$$

Since $\mathbb{E}\bar{\tau}_i^\varepsilon=0$, the first sum runs over all collections of integers $l_1,\dots,l_r$ satisfying $l_i\geq 2$ for all $i=1,\dots,r$. Using (109) we get

$$\begin{aligned}\mathbb{P}\big[f_\varepsilon^u(I)>a\big]&\leq a^{-2k}u^{2k(\alpha-1)}\varepsilon^{2k/\alpha}\sum_{l_1,\dots,l_r}C(l_1,\dots,l_r)\prod_{i=1}^r\varepsilon^{-1}(u\varepsilon^{-1/\alpha})^{\ell_j-\alpha}\\&\leq Ca^{-2k}u^{2k\alpha}\sum_{r=1}^k u^{-r\alpha}.\end{aligned} \tag{120}$$

Since $u\to 0$ as $\varepsilon\to 0$ the largest contribution comes from the term $r=k$. Therefore, for all $\varepsilon$ small,

$$\mathbb{P}\big[f_\varepsilon^u(I)>a\big]\leq Ca^{-2k}u^{k\alpha}. \tag{121}$$

Let $\delta > 0$ small. Define $i_1(\varepsilon)$ and $i_2(\varepsilon)$ by $i_j(\varepsilon) := \max\{i \in \mathbb{Z} : \varepsilon^{\eta_j} \geq (1+\delta)^i\}$, and $u_i := (1+\delta)^i$. Then

$$\mathbb{P}\Big[\bigcup_{i=i_1}^{i_2+1} \Big(f_\varepsilon^{u_i}(I) > a\Big)\Big] \leq \sum_{i=i_1}^{i_2+1} Ca^{-2k} u_i^{k\alpha} \leq Ca^{-2k}\delta^{-1}\varepsilon^{k\alpha\eta_2}\log(\varepsilon^{-1}). \tag{122}$$

Using the definition (108) of $\bar{\tau}_j^\varepsilon$, and the observation that $\tilde{\tau}_j^\varepsilon(u_i) \leq \tilde{\tau}_j^\varepsilon(u) < \tilde{\tau}_j^\varepsilon(u_{i+1})$ for all for $u \in [u_i, u_{i+1})$ we obtain

$$\begin{aligned} f_\varepsilon^u(I) &= \varepsilon^{1/\alpha} u^{\alpha-1} \sum_{j=0}^{\varepsilon^{-1}} \Big(\tilde{\tau}_j^\varepsilon(u) - \mathbb{E}[\tilde{\tau}_0^\varepsilon(u)]\Big) \\ &\leq f_\varepsilon^{u_{i+1}}(I) + \varepsilon^{1/\alpha} u^{\alpha-1}\varepsilon^{-1}\big(\mathbb{E}[\tilde{\tau}_0^\varepsilon(u_{i+1})] - \mathbb{E}[\tilde{\tau}_0^\varepsilon(u)]\big) \\ &\leq f_\varepsilon^{u_{i+1}}(I) + cu^{\alpha-1}\big[u_{i+1}^{1-\alpha} - u^{1-\alpha}\big] \leq f_\varepsilon^{u_{i+1}}(I) + c\delta. \end{aligned} \tag{123}$$

Therefore,

$$\mathbb{P}\big[\sup\{f_\varepsilon^u(I) : u \in [\varepsilon^{\eta_1}, \varepsilon^{\eta_2}]\} \geq a + c\delta\big] \leq Ca^{-2k}\delta^{-1}\varepsilon^{k\alpha\eta_2}\log(\varepsilon^{-1}). \tag{124}$$

The last bound is valid for all $a > 0$ and $\delta$ small. Therefore, the $L^1$ and $L^2$ convergences follows easily. Taking $k$ large enough, the a.s. convergence can be proved using the Borel-Cantelli argument. □

## 6. Long time behaviour of function Π

In this section we prove Theorem 1.1(b). We want to show that

$$\lim_{t\to\infty} \Big(\frac{t^\gamma}{g(t)}\Big)^{-\alpha} \Pi(t, t+g(t)) = K_2, \tag{125}$$

for $g(t)$ much larger than $t^\gamma$. In the language of processes $X_\varepsilon$ it is equivalent (see (29)) to show

$$\lim_{\varepsilon\to 0} h(\varepsilon)^\alpha \int_0^\infty \mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1)) \in du\big] \exp\Big(-\frac{h(\varepsilon)}{u}\Big) = K_2, \tag{126}$$

where

$$h(\varepsilon) := g(t_\varepsilon)/t_\varepsilon^\gamma = g(\varepsilon^{-1/\alpha\gamma})\varepsilon^{1/\alpha} \to \infty \qquad \text{as } \varepsilon \to 0. \tag{127}$$

The event that $X_\varepsilon$ stays in one trap for an unusually long time is determined by the event: "At time $t = 1$ the process $X_\varepsilon$ is in an unusually deep trap." Indeed, for any constant $K$

$$\begin{aligned} h(\varepsilon)^\alpha \int_0^K \mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1)) \in du\big] \exp\Big(-\frac{h(\varepsilon)}{u}\Big) \\ \leq h(\varepsilon)^\alpha \exp\Big(-\frac{h(\varepsilon)}{K}\Big) \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{128}$$

To control the remaining part of the integral in (126) we need the following lemma.

**Lemma 6.1.** *Let* $U_\varepsilon(u) = \mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1)) \geq u\big]$. *Then*

$$U_\varepsilon(u) = \mathcal{K}(\varepsilon)u^{-\alpha}(1 + f_\varepsilon(u)) \tag{129}$$

*where for some* $\varepsilon_0$

$$\lim_{u\to\infty} \sup_{\varepsilon<\varepsilon_0} \big|f_\varepsilon(u)\big| = 0. \tag{130}$$

*The function* $\mathcal{K}(\varepsilon)$ *converges as* $\varepsilon \to 0$ *to* $\mathbb{E}[R_Z]$, *where* $R_Z$ *is the range of* $Z$ *up to time* $t = 1$ *defined by*

$$R_Z = \sup\{Z(t) : t \leq 1\} - \inf\{Z(t) : t \leq 1\}. \tag{131}$$

We can now finish the proof of Theorem 1.1(b). Integrating by parts we get

$$\begin{aligned} h(\varepsilon)^\alpha &\int_K^\infty \mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1)) \in du\big] \exp\Big(-\frac{h(\varepsilon)}{u}\Big) \\ &= h(\varepsilon)^\alpha \Big\{U_\varepsilon(K)e^{-h(\varepsilon)/K} + \int_K^\infty U_\varepsilon(u)e^{-h(\varepsilon)/u}\frac{h(\varepsilon)}{u^2}\,du\Big\}. \end{aligned} \tag{132}$$

The contribution of the first term in the braces becomes negligible as $h(\varepsilon) \to \infty$. We give the upper bound for the second term. Fix $\delta > 0$ and choose $K$ such that $U_\varepsilon(u) \leq (1 + \delta)u^{-\alpha}\mathbb{E}[R_Z]$ for all $u \geq K$ and $\varepsilon \leq \varepsilon_0$. This is possible by Lemma 6.1. The contribution of the second term in (132) is then bounded from above by

$$(1 + \delta)h(\varepsilon)^\alpha \int_K^\infty \mathbb{E}[R_Z]u^{-\alpha}e^{-h(\varepsilon)/u}\frac{h(\varepsilon)}{u^2}\,du. \tag{133}$$

Combining (128),(132) with the last expression and using the substitution $v = u/h(\varepsilon)$ we get

$$\begin{aligned} \limsup_{\varepsilon\to 0} h(\varepsilon)^\alpha &\int_0^\infty \mathbb{P}\big[\mu_\varepsilon(X_\varepsilon(1)) \in du\big]e^{-h(\varepsilon)/u} \\ &\leq (1 + \delta)\mathbb{E}[R_Z] \int_0^\infty v^{-\alpha-2}e^{-1/v}\,dv =: (1 + \delta)K_2 \in (0,\infty). \end{aligned} \tag{134}$$

In the same way we get a corresponding lower bound. Since $\delta$ was arbitrary the proof of Theorem 1.1(b) is finished. It remains to show Lemma 6.1.

*Proof of Lemma 6.1.* We show that the probability that $X_\varepsilon$ hits a trap with the depth larger than $u$ during the time interval $[0, 1]$ decreases as $u^{-\alpha}$. If $X_\varepsilon$ hits such trap at a time $T < 1$, it has a very large probability to be there also at time $t = 1$. Formally, let $I_\varepsilon(u)$ be the set of atoms of $\mu_\varepsilon$ with the weight larger than $u$, $I_\varepsilon(u) = \{x : \mu_\varepsilon(x) \geq u\}$. Let $L$ be a large constant. We define

$$
\begin{aligned}
T = T_\varepsilon(u) &= \inf\{t \geq 0 : X_\varepsilon(t) \in I_\varepsilon(u)\},\\
A = A_\varepsilon(u) &= \{T_\varepsilon(u) \leq 1\},\\
B = B_\varepsilon(u) &= \{|X_\varepsilon(s)| \leq L \text{ for all } s \leq 1 \wedge T_\varepsilon(u)\},\\
C = C_\varepsilon(u) &= \{X_\varepsilon(1) = X_\varepsilon(T_\varepsilon(u))\}.
\end{aligned} \tag{135}
$$

Using these definitions we get

$$\mathbb{P}[A] \geq U_\varepsilon(u) \geq P[A \cap B \cap C]. \tag{136}$$

We will show that it is possible to choose $L = L(u)$ such that

$$\mathbb{P}[A] = \mathcal{K}(\varepsilon)u^{-\alpha}(1 + \kappa_\varepsilon(u)), \tag{137}$$

$$\mathbb{P}[A \cap B^c] \leq u^{-\alpha}\lambda_\varepsilon(u), \tag{138}$$

$$\mathbb{P}[A \cap B \cap C^c] \leq u^{-\alpha}\eta_\varepsilon(u), \tag{139}$$

for some $\kappa_\varepsilon$, $\lambda_\varepsilon$, $\eta_\varepsilon$ satisfying the same relation (130) as $f_\varepsilon$. The lemma follows then from (136)–(139).

We first introduce some additional notation. Let for all $u > 0$ $\hat{U}_i$ and $\bar{U}_i$, $i \in \mathbb{Z}$, be two independent sequences of i.i.d. random variables independent of $V$ and $W$, having the same distribution as $\mu_\varepsilon(0)$ conditioned on being larger, resp. smaller than $u$, that is

$$
\begin{aligned}
\mathbb{P}[\hat{U}_i \geq a] &= \mathbb{P}[\mu_\varepsilon(0) \geq a | \mu_\varepsilon(0) \geq u]\\
\mathbb{P}[\bar{U}_i \geq a] &= \mathbb{P}[\mu_\varepsilon(0) \geq a | \mu_\varepsilon(0) < u]
\end{aligned} \tag{140}
$$

Let $Y_i$, $i \in \mathbb{Z}$, be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$ and let $\mathcal{I}_\varepsilon(u) = \{i\varepsilon : Y_i \leq \mathbb{P}[\mu_\varepsilon(0) \geq u]\}$. We define two random measures

$$
\begin{aligned}
\bar{\nu}_\varepsilon^u &= \sum_{i : i\varepsilon \notin I_\varepsilon(u)} \mu_\varepsilon(i\varepsilon)\delta_{i\varepsilon} + \sum_{i : i\varepsilon \in I_\varepsilon(u)} \bar{U}_i \delta_{i\varepsilon}\\
\hat{\nu}_\varepsilon^u &= \sum_{i : i\varepsilon \notin \mathcal{I}_\varepsilon(u)} \nu_\varepsilon^u(i\varepsilon)\delta_{i\varepsilon} + \sum_{i : i\varepsilon \in \mathcal{I}_\varepsilon(u)} \hat{U}_i \delta_{i\varepsilon}.
\end{aligned} \tag{141}
$$

The measure $\bar{\nu}_\varepsilon^u$ is therefore almost equal to the measure $\mu_\varepsilon$, only the weights of large atoms are changed to be smaller than $u$. We then reinsert large atoms to $\hat{\nu}_\varepsilon^u$ in the way that is independent of $\rho$. Let $\bar{X}_\varepsilon^u$ and $\hat{X}_\varepsilon^u$ be processes defined as the time change of Brownian motion with the speed measure $\bar{\nu}_\varepsilon^u$, resp. $\hat{\nu}_\varepsilon^u$. It is not difficult to verify that $(\hat{\nu}_\varepsilon^u, \hat{X}_\varepsilon^u)$ has the same distribution as $(\mu_\varepsilon, X_\varepsilon)$ for all $u > 0$. Further, for any bounded interval $J$ there is $u_0$ such that if $u > u_0$, then $I_\varepsilon(u) \cap J = \emptyset$. Therefore $\bar{\nu}_\varepsilon^u$ converges to $\mu_\varepsilon$ as $u \to \infty$ vaguely and in the point process sense for a.e. realisation of $\rho$. Therefore, as follows from Theorem 2.1 of [FIN02], $\bar{X}_\varepsilon^u(t)$ converges to $X_\varepsilon(t)$ as $u \to \infty$ weakly and in the point process sense.

To show (137) we write

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}\big[\{X_\varepsilon(t) : t \le 1\} \cap I_\varepsilon(u) \neq \emptyset\big] \\ &= \mathbb{P}\big[\{\hat{X}^u_\varepsilon(t) : t \le 1\} \cap \mathcal{I}_\varepsilon(u) \neq \emptyset\big] \\ &= \mathbb{P}\big[\{\bar{X}^u_\varepsilon(t) : t \le 1\} \cap \mathcal{I}_\varepsilon(u) \neq \emptyset\big].\end{aligned} \tag{142}$$

For the second equality we used the equality of the distributions of $X_\varepsilon$ and $\hat{X}^u_\varepsilon$, and for the third equality we used the fact that before the first hit of $\mathcal{I}_\varepsilon(u)$ the processes $\hat{X}^u_\varepsilon$ and $\bar{X}^u_\varepsilon$ behave in the same way. This is true since for any measurable $G \subset \mathbb{R} \setminus \mathcal{I}_\varepsilon(u)$ the corresponding speed measures satisfy $\hat{\nu}^u_\varepsilon(G) = \bar{\nu}^u_\varepsilon(G)$. Let $R(\bar{X}^u_\varepsilon) = \max_{t\le 1} \bar{X}^u_\varepsilon(t) - \min_{t\le 1} \bar{X}^u_\varepsilon(t)$. Since $\mathcal{I}_\varepsilon(u)$ is independent of $\bar{X}^u_\varepsilon$,

$$\mathbb{P}[A] = \mathbb{E}\Big[1 - \big(1 - \mathbb{P}[\mu_\varepsilon(0) \ge u]\big)^{1+R(\bar{X}^u_\varepsilon)/\varepsilon}\Big]. \tag{143}$$

By definition (27) of $\mu_\varepsilon$ and by (4), $\mathbb{P}[\mu_\varepsilon(0) \ge u] = (u\varepsilon^{-1/\alpha})^{-\alpha}(1 + \kappa'_\varepsilon(u))$, where $\kappa'$ satisfies (130). Therefore,

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{E}\big[R(\bar{X}^u_\varepsilon)\varepsilon^{-1}(u\varepsilon^{-1/\alpha})^{-\alpha}(1 + \kappa'_\varepsilon(u)) + O(\varepsilon^{-1}(u\varepsilon^{-1/\alpha})^{-2\alpha})\big] \\ &= \mathbb{E}[R(\bar{X}^u_\varepsilon)]u^{-\alpha}(1 + \kappa_\varepsilon(u)).\end{aligned} \tag{144}$$

In the last computation we used the fact that $\mathbb{E}[\exp(\lambda R(\bar{X}^u_\varepsilon))]$ exists for some $\lambda > 0$ independent of $\varepsilon$ and $u$ if $\varepsilon$ is small and $u$ large enough as can be proved as in Lemma 3.2. Since $\bar{X}^u_\varepsilon$ converges to $X_\varepsilon$ a.s. as $u \to \infty$ and $X_\varepsilon$ converges to $Z$ a.s. as $\varepsilon \to 0$, it is not difficult to show

$$\lim_{u\to\infty} \mathbb{E}[R(\bar{X}^u_\varepsilon)] \to \mathbb{E}[R(X_\varepsilon)] =: \mathcal{K}(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathcal{K}(\varepsilon) = \mathbb{E}[R_Z]. \tag{145}$$

This proves (137).

To prove (138) we write (recall that $\mathcal{B}(a) = [-a, a]$)

$$\begin{aligned}\mathbb{P}[A \cap B^c] &= \sum_{i=0}^{\infty} \mathbb{P}\Big[\sup_{t\le 1\wedge T} \big|X_\varepsilon(t)\big| \in (2^i L, 2^{i+1}L] \cap A\Big] \\ &\le \sum_{i=0}^{\infty} \mathbb{P}\Big[\sup_{t\le 1\wedge T} \big|X_\varepsilon(t)\big| \in (2^i L, 2^{i+1}L] \cap \big\{\mathcal{B}(2^{i+1}L) \cap I_\varepsilon(u) \neq \emptyset\big\}\Big] \\ &= \sum_{i=0}^{\infty} \mathbb{P}\Big[\sup_{t\le 1\wedge T} \big|\hat{X}^u_\varepsilon(t)\big| \in (2^i L, 2^{i+1}L] \cap \big\{\mathcal{B}(2^{i+1}L) \cap \mathcal{I}_\varepsilon(u) \neq \emptyset\big\}\Big].\end{aligned} \tag{146}$$

Since $\bar{\nu}^u_\varepsilon(x) \le \hat{\nu}^u_\varepsilon(x)$ for all $x \in \mathbb{R}$,

$$\mathbb{P}\big[\sup_{t\le 1\wedge T} |\hat{X}^u_\varepsilon(t)| \ge a\big] \le \mathbb{P}\big[\sup_{t\le 1} |\bar{X}^u_\varepsilon(t)| \ge a\big]. \tag{147}$$

Therefore, using also independence of $\mathcal{I}_\varepsilon(u)$ and $\bar{X}^u_\varepsilon$,

$$\begin{aligned}\mathbb{P}[A\cap B^c] &\le \sum_{i=0}^{\infty}\mathbb{P}\big[\sup_{t\le 1}\big|\bar{X}^u_\varepsilon(t)\big|\ge 2^iL\big]\mathbb{P}\big[\mathcal{B}(2^{i+1}L)\cap\mathcal{I}_\varepsilon(u)\neq\emptyset\big]\\ &\le Cu^{-\alpha}\sum_{i=0}^{\infty}e^{-c2^iL}2^iL.\end{aligned}\tag{148}$$

Here we used the fact that uniformly for all large $u$ and small $\varepsilon$, $\mathbb{P}[\sup_{t\le 1}|\bar{X}^u_\varepsilon(t)|\ge a]$ decreases at least exponentially with $a$, and further the fact that $\mathbb{P}[\mathcal{B}(K)\cap\mathcal{I}_\varepsilon(u)\neq\emptyset]$ is smaller than $CKu^{-\alpha}$ for all $\varepsilon$ small enough. If we choose $L=L(u)$ such that $\lim_{u\to\infty}L(u)=\infty$, then the sum in the last display tends to 0 as $u\to\infty$. This proves (138).

To show (139) we first estimate $\mathbb{P}_x\big[X_\varepsilon(s)\neq x|\rho,\mu_\varepsilon[x]\ge u\big]$ for $s\le 1$ and $x\in[-L,L]$. If $\mu_\varepsilon[x]\ge u$, then it follows from the definition of $p_\varepsilon$ (33) and Lemma 2.5 that $p_\varepsilon(x,y,t)\le u^{-1}$ for all $y\in\mathbb{R}$ and $t\ge 0$. Let $K=b\log u$ with $b$ large. Then

$$\begin{aligned}&\mathbb{P}_x\big[X_\varepsilon(s)\neq x\big|\rho,\mu_\varepsilon[x]\ge u\big]\\ &=\mathbb{P}_x\big[|X_\varepsilon(s)|>K\big|\rho,\mu_\varepsilon[x]\ge u\big]+\int_{[-K,K]\setminus\{x\}}p_\varepsilon(x,y;s)\mu_\varepsilon(dy)\\ &\le\mathbb{P}_x\big[|X_\varepsilon(s)|>K\big|\rho,\mu_\varepsilon[x]\ge u\big]+\big\{u^{-1}\mu_\varepsilon\big(\mathcal{B}(K)\setminus\{x\}\big)\wedge 1\big\},\end{aligned}\tag{149}$$

It can be shown using the same methods as in Lemma 3.2 that, uniformly for all $x\in\mathbb{R}$,

$$\mathbb{P}_x\big[|X_\varepsilon(s)|\ge K\big|\mu_\varepsilon[x]\ge u\big]\le Ce^{-cK}\le Cu^{-cb}.\tag{150}$$

Using the two previous displays we obtain

$$\begin{aligned}\mathbb{P}[A\cap B\cap C^c]&\le\mathbb{E}_\rho\Big[\sum_{x\in\varepsilon\mathbb{Z}\cap\mathcal{B}(L)}\mathbb{1}\{x\in I_u(\varepsilon)\}\mathbb{P}[X_\varepsilon(T_\varepsilon)=x|\rho]\\ &\qquad\times\big\{u^{-1}\mu_\varepsilon\big(\mathcal{B}(K)\setminus\{x\}\big)\wedge 1\big\}\Big]+Cu^{-cb}\\ &\le\mathbb{E}_\rho\Big[\sum_{x\in\varepsilon\mathbb{Z}\cap\mathcal{B}(L)}\mathbb{1}\{x\in I_u(\varepsilon)\}\big\{u^{-1}\mu_\varepsilon\big(\mathcal{B}(K)\setminus\{x\}\big)\wedge 1\big\}\Big]+Cu^{-cb}\end{aligned}\tag{151}$$

Since $\{x\in I_u(\varepsilon)\}$ is independent of $\mu_\varepsilon(\mathcal{B}(K)\setminus\{x\})$, the last display is bounded by

$$\begin{aligned}&\sum_{x\in\varepsilon\mathbb{Z}\cap\mathcal{B}(L)}\mathbb{P}[x\in I_\varepsilon(u)]\mathbb{E}_\rho\big[u^{-1}\mu_\varepsilon(\mathcal{B}(K))\wedge 1\big]\\ &\qquad\le u^{-\alpha}\big[Cu^{-\alpha}K(u)L+Cu^{-cb+\alpha}\big].\end{aligned}\tag{152}$$

Here we used the fact that $\mathbb{E}[u^{-1}\mu_\varepsilon(\mathcal{B}(K))\wedge 1]$ behaves like $Ku^{-\alpha}$ as can be verified easily. It is now possible to take $b$ large enough and $L(u)\to\infty$ such that the expressions in the brackets in the last display tends to zero as $u \to \infty$. This finishes the proof of (137)–(139) and thus of Lemma 6.1. $\square$

## 7. Long time behaviour of function $R$

We prove Theorem 1.2 in this section. We first rewrite the object of our interest in the language of processes $X_\varepsilon$. Using (28) we get

$$\Big(\frac{f(t)}{t}\Big)^{\alpha\gamma}\mathbb{P}\big[X(t+f(t))=X(t)\big]=h(\varepsilon)^{\alpha\gamma}\mathbb{P}\big[X_\varepsilon(1+h(\varepsilon))=X_\varepsilon(1)\big], \tag{153}$$

where in this section $h=h(\varepsilon):=f(t_\varepsilon)/t_\varepsilon$. From the assumptions of the theorem it follows that $\lim_{\varepsilon\to0}h(\varepsilon)=\infty$.

7.1. **Lower bound.** To get the lower bound we consider the event "$X_\varepsilon$ hits a trap deeper than $h(\varepsilon)^\gamma$ before time one". Lemma 6.1 implies that this event has probability of order $h(\varepsilon)^{-\alpha\gamma}$. We will show that hitting such deep trap, the process has a non-negligible probability to be there also at times 1 and $1+h(\varepsilon)$. To prove such behaviour we consider (similarly as in the proof of Lemma 6.1) following events. Let $u=u(\varepsilon)=h(\varepsilon)^\gamma$. For some large constants $K$ and $L$ we define

$$\begin{aligned}
&T=T_\varepsilon=\min\big\{t\ge0;\mu_\varepsilon(X_\varepsilon(t))\ge u(\varepsilon)\big\}\\
&A=A_\varepsilon=\{T_\varepsilon\le1\}\\
&B=B_\varepsilon=\{|X_\varepsilon(t)|\le L\forall t\le1\}\\
&C=C_\varepsilon=\{X_\varepsilon(1)=X_\varepsilon(T_\varepsilon)\}\\
&D=D_\varepsilon=\{X_\varepsilon(1+h(\varepsilon))=X_\varepsilon(T_\varepsilon)\}\\
&E=E_\varepsilon=\{\mu_\varepsilon[-L,L]\le Kh(\varepsilon)^\gamma\}
\end{aligned} \tag{154}$$

Using these definitions we get

$$h(\varepsilon)^{\alpha\gamma}\mathbb{P}\big[X_\varepsilon\big(1+h(\varepsilon)\big)=X_\varepsilon(1)\big]\ge h(\varepsilon)^{\alpha\gamma}\mathbb{P}[A,B,C,D,E] \tag{155}$$

Using the same methods as in the proof of (137)–(139) it can be shown that it is possible to fix $L(\varepsilon)$ large enough such that

$$\lim_{\varepsilon\to0}h(\varepsilon)^{\alpha\gamma}\mathbb{P}[A,B,C]>\mathbb{E}[R_Z]/2. \tag{156}$$

In particular (see (148)) $L(\varepsilon)$ can be chosen to increase slower than any positive power of $\varepsilon^{-1}$ as $\varepsilon\to0$. Further, it is easy to see that $\mathbb{P}[E^c]\le cLK^{-\alpha}h(\varepsilon)^{-\alpha\gamma}$. Therefore we can choose $K$ large such that $\liminf_{\varepsilon\to0}h(\varepsilon)^{\alpha\gamma}\mathbb{P}[A,B,C,E]>0$. To finish the proof of the lower bound we should therefore show that

$$\liminf_{\varepsilon\to0}\mathbb{P}[D|A,B,C,E]>0. \tag{157}$$

Since

$$\mathbb{P}[D|A,B,C,E]=\mathbb{E}\big[\mu_\varepsilon(X_\varepsilon(T))p_\varepsilon\big(X_\varepsilon(T),X_\varepsilon(T),h(\varepsilon)\big)\big|A,B,C,E,\rho\big] \tag{158}$$

we should find first estimate $p_\varepsilon(x,x,h(\varepsilon))$ from bellow. Let $R \geq L$. The Markov property and the Cauchy inequality imply that

$$\begin{aligned} p_\varepsilon(x,x;h) &\geq \int_{-R}^{R} p_\varepsilon^2(x,y;h/2)\mu_\varepsilon(dy) \\ &\geq \frac{1}{\mu_\varepsilon(\mathcal{B}(R))}\big\{\mathbb{P}_x\big[|X_\varepsilon(h/2)| \leq R\big|\rho\big]\big\}^2 \geq \frac{1}{\mu_\varepsilon(\mathcal{B}(R))}\mathbb{P}_x[G_\varepsilon(R,L)|\rho]^2, \end{aligned} \tag{159}$$

where $G_\varepsilon(R,L)$ is the event: "Before exit from $[-R,R]$ the process $X_\varepsilon$ spends in the set $H := \mathcal{B}(R)\setminus\mathcal{B}(L)$ time larger than $h(\varepsilon)/2$ ."

Fix now $R = R(\varepsilon) = h(\varepsilon)^{\alpha\gamma}$. We use the following scaling argument to bound (159). Let $\bar V(x)$, $x \in [-1,1]$ be a Lévy process independent of $V$ that has the same distribution as $V$ restricted to $[-1,1]$. Define $\bar V_\varepsilon(x)$ by

$$\bar V_\varepsilon(x) = \begin{cases} (R-L)^{1/\alpha}\bar V((x-L)/(R-L)) & \text{for } x \in [L,R] \\ (R-L)^{1/\alpha}\bar V((x+L)/(R-L)) & \text{for } x \in [-R,-L] \\ 0 & \text{for } x \in [-L,L] \\ \bar V(R), \text{ resp. } \bar V(-R) & \text{for } x > R,\ x < -R. \end{cases} \tag{160}$$

Let $\bar\rho_\varepsilon$, $\bar\rho$ be the Lebesgue-Stieltjes measures associated to $\bar V_\varepsilon$ and $\bar V$ and let $\bar\mu_\varepsilon$ be constructed from $\bar\rho_\varepsilon$ in the same way (26), (27) as $\mu_\varepsilon$ was constructed from $\rho$. Then $\bar\mu_\varepsilon$ has the same distribution as the restriction of $\mu_\varepsilon$ to the set $H$. Let further $\bar X_\varepsilon$ be defined as the time change of Brownian motion using $\mu_\varepsilon \mathbb{1}\{H^c\} + \bar\mu_\varepsilon \mathbb{1}\{H\}$ as the speed measure, and let $\bar G_\varepsilon$ be defined as $G_\varepsilon$ using process $\bar X_\varepsilon$. Using the scaling properties of our model (32) it is not difficult to show that for all $x \in [-L,L]$ the probability $\mathbb{P}_x[\bar G_\varepsilon(R,L)|\rho,\bar\rho]$ equals to the probability of the event: "$X_{\varepsilon/R}$ started at $x/R$ spends in $\mathcal{B}(1)\setminus\mathcal{B}(L/R)$ a time larger than $1/2$ before hitting $-1$ or 1", conditioned on the random environment being equal to $\mathsf{Sc}_{0,R}(\bar\mu_\varepsilon)$ (see (50)). Since $L$ and $R$ can be chosen such that $L/R \to 0$, it can be proved using methods of [FIN02] or [BČ04] that $\mathsf{Sc}_{0,R(\varepsilon)}\,\mu_\varepsilon \to \bar\rho$ as $\varepsilon \to 0$ vaguely and in the point-process sense, $\bar\rho$-a.s. Using Theorem 2.1 of [FIN02] we get

$$\lim_{\varepsilon\to 0}\mathbb{P}_x[\bar G_\varepsilon(R,L)|\rho,\bar\rho] = \mathbb{P}_0[\sup_{s\leq 1/2}|Z(s)| \leq 1|\bar\rho] \tag{161}$$

and therefore, $\rho$ and $\bar\rho$-a.s. and in $L^1(\mathbb{P})$,

$$\lim_{\varepsilon\to 0}\frac{h^\gamma\mathbb{P}_x[\bar G_\varepsilon(R,L)|\rho,\bar\rho]^2}{\bar\mu_\varepsilon(H)+Kh^\gamma} = \frac{\mathbb{P}_0[\sup_{s\leq 1/2}|Z(s)|\leq 1|\bar\rho]^2}{\bar\rho(\mathcal{B}(1))+K}. \tag{162}$$

We can now insert (159)–(162) into (158). Using the fact that $A$, $B$, $C$, and $E$ are independent of $\bar\rho$ we get

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\mathbb{E}\big[\mu_\varepsilon(X_\varepsilon(T))p_\varepsilon\big(X_\varepsilon(T),X_\varepsilon(T),h\big)\big|A,B,C,E,\rho\big]\\
&\qquad\ge \liminf_{\varepsilon\to 0}\mathbb{E}\left[\frac{\mu_\varepsilon(X_\varepsilon(T))}{\mu_\varepsilon(\mathcal{B}(R))}\mathbb{P}_{X_\varepsilon(T)}[G_\varepsilon(R,L)]^2\middle|A,B,C,E,\rho\right]\\
&\qquad\ge \liminf_{\varepsilon\to 0}\mathbb{E}\left[\frac{h^\gamma}{\bar\mu_\varepsilon(\mathcal{B}(H))+Kh^\gamma}\mathbb{P}_{X_\varepsilon(T)}[\bar G_\varepsilon(R,L)]^2\middle|A,B,C,E,\rho,\bar\rho\right]\\
&\qquad\ge \liminf_{\varepsilon\to 0}\mathbb{E}\left[\frac{\mathbb{P}[\sup_{s\le 1/2}|Z(s)|\le 1|\rho]^2}{\bar\rho[-1,1]+K}\right]>0.
\end{aligned}\tag{163}$$

This proves (156) and thus the lower bound of Theorem 1.2.

7.2. **Upper bound.** To prove the corresponding upper bound we use Lemma 3.1. To be able to apply it we introduce some notation that is mainly technical. Its purpose will be clarified later. Set

$$\begin{aligned}
L&=L(\varepsilon)=\kappa(\alpha)\log h(\varepsilon)\\
B&=B(\varepsilon)=\{\sup_{t\le 1}|X_\varepsilon(1)|\le L(\varepsilon)\}\\
D&=D(\varepsilon)=\{\mu_\varepsilon[-3L,3L]\le h(\varepsilon)(8L)^{-1}\}.
\end{aligned}\tag{164}$$

Lemma 3.2 implies that it is possible to fix $\kappa(\alpha)$ large enough such that

$$h(\varepsilon)^{\alpha\gamma}\mathbb{P}[B(\varepsilon)^c]\to 0\qquad\text{as }\varepsilon\to 0.\tag{165}$$

Using the scaling properties (32) of the Lévy process $V$ we get

$$\begin{aligned}
h(\varepsilon)^{\alpha\gamma}\mathbb{P}\big[D(\varepsilon)^c\big]&\le ch(\varepsilon)^{\alpha\gamma}\mathbb{P}\big[V(1)\ge CL(\varepsilon)^{-1-1/\alpha}h(\varepsilon)\big]\\
&\le ch(\varepsilon)^{\alpha(\gamma-1)}L(\varepsilon)^{1+\alpha}\to 0\qquad\text{as }\varepsilon\to 0.
\end{aligned}\tag{166}$$

Expressions (165), (166) imply that to finish the proof we should find an upper bound for

$$\begin{aligned}
&h(\varepsilon)^{\alpha\gamma}\mathbb{P}\big[\{X_\varepsilon(1+h(\varepsilon))=X_\varepsilon(1)\},B,D\big]\\
&\qquad=h(\varepsilon)^{\alpha\gamma}\mathbb{E}\Big[\mathbb{1}_D(\rho)\sum_{\substack{x\in\varepsilon\mathbb{Z}\\|x|\le L}}\mathbb{P}[X_\varepsilon(1)=x,B|\rho]\mathbb{P}_x[X_\varepsilon(h)=x|\rho]\Big].
\end{aligned}\tag{167}$$

We first estimate the contribution of traps with $\mu_\varepsilon(x)\ge h(\varepsilon)^\gamma$ to the sum in (167). Using Lemma 6.1 with $u=h(\varepsilon)^\gamma$ we get

$$\begin{aligned}
&h(\varepsilon)^{\alpha\gamma}\mathbb{E}\Big[\mathbb{1}_D(\rho)\sum_{x:\mu_\varepsilon(x)\ge h(\varepsilon)^\gamma}\mathbb{P}[X_\varepsilon(1)=x,B|\rho]\mathbb{P}_x[X_\varepsilon(h)=x|\rho]\Big]\\
&\qquad\le h(\varepsilon)^{\alpha\gamma}\mathbb{P}[\mu_\varepsilon(X_\varepsilon(1))\ge h(\varepsilon)^\gamma]\le C<\infty.
\end{aligned}\tag{168}$$

We will to apply Lemma 3.1 to control the contribution of the remaining traps in (167). To avoid the problems with jumps of $\mathcal{V}_\varepsilon$ that is used in this lemma, we define $\mathcal{U}_\varepsilon(x,r):=\mathcal{V}_\varepsilon(x,r)=\mu_\varepsilon(\mathcal{B}(x,r))$ for all $r \in \varepsilon\mathbb{N}_0$, and by the linear interpolation for all other $r \geq 0$. Note that $\mathcal{U}_\varepsilon$ satisfies (38) with $\delta = \varepsilon$. The function $p_\varepsilon$ thus satisfies upper bound (47). Define now $r(x,\varepsilon)$ by

$$4r(x,\varepsilon)\mathcal{U}_\varepsilon(x, r(x,\varepsilon)) = h(\varepsilon). \tag{169}$$

Then

$$\mathbb{P}_x[X_\varepsilon(h) = x|\rho] \leq \frac{2\mu_\varepsilon(x)}{\mathcal{U}_\varepsilon(x, r(x,\varepsilon))} = \frac{8\mu_\varepsilon(x) r(x,\varepsilon)}{h(\varepsilon)}. \tag{170}$$

If $D(\varepsilon)$ holds, then $\mathcal{B}(L) \subset \mathcal{B}(x, r(x,\varepsilon))$ for all $x \in \mathcal{B}(L)$. Indeed, the last event holds if $r(x,\varepsilon) \geq 2L$, that means $8L\mathcal{U}_\varepsilon(x, 2L) \leq h(\varepsilon)$, and this is satisfied if $D(\varepsilon)$ is true.

To estimate $r(x,\varepsilon)$ we define $\mathcal{W}_\varepsilon(x,R)$ by

$$\mathcal{W}_\varepsilon(x,r) = \begin{cases} \mathcal{U}_\varepsilon(x,r) - \mathcal{U}_\varepsilon(x,L) + \mu_\varepsilon[x] & \text{for } r \geq L \\ \mu_\varepsilon[x] & \text{for } r < L. \end{cases} \tag{171}$$

Let $R(x,\varepsilon)$ be given by

$$R(x,\varepsilon) = \sup\{s : 4s\mathcal{W}_\varepsilon(x,s) \leq h(\varepsilon)\} \tag{172}$$

Note that the distribution of $R(x,\varepsilon)$ depends on $\rho$ in $\mathcal{B}(L)$ only through the value $\mu_\varepsilon(x)$. From (171) it follows that $\mathcal{W}_\varepsilon(x,r) \leq \mathcal{U}_\varepsilon(x,r)$ for all $x$ and $r$, and therefore $R(x,\varepsilon) \geq r(x,\varepsilon)$.

For any $a > 0$ and for any $K \in [0,\infty)$ such that $4aK \leq 1$ the random variable $R$ satisfies

$$\begin{aligned} &\mathbb{P}\big[R(x,\varepsilon) \geq ah^{\alpha\gamma} \big| \mu_\varepsilon[x] = Kh^\gamma\big] \\ &\qquad = \mathbb{P}\big[4ah^{\alpha\gamma}\big(\mathcal{U}_\varepsilon(x, ah^{\alpha\gamma}) - \mathcal{U}_\varepsilon(L) + Kh^\gamma\big) \leq h\big], \\ &\qquad = \mathbb{P}\big[\mathcal{U}_\varepsilon(ah^{\alpha\gamma} - L) \leq h^\gamma((4a)^{-1} - K)\big]. \end{aligned} \tag{173}$$

The above probability is 0 for all $a \geq (4K)^{-1}$. After a calculation very similar to that one used to prove (106) we get that (173) can be bounded for all $\varepsilon$ small enough and for $4aK \leq 1$ by

$$\mathbb{P}\big[R(x,\varepsilon) \geq ah^{\alpha\gamma} \big| \mu_\varepsilon[x] = Kh^\gamma\big] \leq C\exp\big\{-ca\big((4a)^{-1} - K\big)^{-\alpha}\big\}. \tag{174}$$

Therefore, there exists a function $G(K)$ such that

$$\limsup_{\varepsilon\to 0} \mathbb{E}[R(x,\varepsilon)h^{-\alpha\gamma}|\mu_\varepsilon[x] = Kh^\gamma] \leq G(K) \leq G(0) < \infty. \tag{175}$$

Taking the expectation of (170) over the random environment in the exterior of $[-L, L]$, using the fact that $R(x,\varepsilon)$ depends on the restriction $\rho_{\text{in}}$ of $\rho$ to $[-L, L]$ only through $\mu_\varepsilon(x)$, we get

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \mathbb{P}_x\big[X_\varepsilon(h) = x \big| \rho_{\text{in}}, \mu_\varepsilon[x] = Kh^\gamma\big] \\ &\qquad \leq \frac{8Kh^\gamma h^{\alpha\gamma}\mathbb{E}[R(x,\varepsilon)h^{-\alpha\gamma}|\mu_\varepsilon[x] = Kh^\gamma]}{h} \leq CKG(K). \end{aligned} \tag{176}$$

We can now estimate the contribution of traps shallower than $h^\gamma$. Inserting (176) into (167) this contribution can be bounded by

$$Ch^{\alpha\gamma}\mathbb{E}_{\rho_{\text{in}}}\Big[\mathbb{1}_D(\rho)\int_0^{h^\gamma}\mathbb{P}[\mu_\varepsilon(x)\in du\cap B|\rho_{\text{in}}]uh^{-\gamma}G(0)\Big]. \tag{177}$$

Taking first $u \le h^{1-2\alpha\gamma}$ we have

$$\begin{aligned}h^{\alpha\gamma}\mathbb{E}_{\rho_{\text{in}}}\Big[\mathbb{1}_D(\rho)\int_0^{h^{1-2\alpha\gamma}}\mathbb{P}[\mu_\varepsilon(x)\in du\cap B|\rho_{\text{in}}]uh^{-\gamma}G(0)\Big]&\\ \le Ch^{\alpha\gamma}h^{1-2\alpha\gamma}h^{-\gamma}G(0)\le C<\infty.& \quad (178)\end{aligned}$$

Finally, for $u\in[h^{1-2\alpha\gamma},h^\gamma]$ we set $i_0(\varepsilon)=\lfloor\log_2(h^{1-2\alpha\gamma}/h^\gamma)\rfloor$. Then, using Lemma 6.1,

$$\begin{aligned}h^{\alpha\gamma}\mathbb{E}\Big[\mathbb{1}_D(\rho)&\int_{h^{1-2\alpha\gamma}}^{h^\gamma}\mathbb{P}[\mu_\varepsilon(x)\in du\cap B|\rho_{\text{in}}]uh^{-\gamma}G(uh^{-\gamma})\Big]\\ &\le h^{\alpha\gamma}\sum_{i=i_0}^{-1}\mathbb{P}[\mu_\varepsilon[X_\varepsilon(1)]\ge 2^ih^\gamma]2^{i+1}G(2^i)\\ &\le C\sum_{i=-\infty}^{-1}2^{-i\alpha}2^iG(0)<\infty.\end{aligned} \tag{179}$$

This finishes the proof of (167) and thus of the upper bound of Theorem 1.2.

## References

[Bas02] Richard F. Bass, *On Aronson's upper bounds for heat kernels*, Bull. London Math. Soc. **34** (2002), no. 4, 415–419.

[BB03] E. Bertin and J.-P. Bouchaud, *Sub-diffusion and localisation in the one dimensional trap model*, Physical Review E **67** (2003), 026128.

[BBG03a] Gérard Ben Arous, Anton Bovier, and Véronique Gayrard, *Glauber dynamics of the random energy model. I. Metastable motion on the extreme states*, Comm. Math. Phys. **235** (2003), no. 3, 379–425.

[BBG03b] ———, *Glauber dynamics of the random energy model. II. Aging below the critical temperature*, Comm. Math. Phys. **236** (2003), no. 1, 1–54.

[BČ04] Gérard Ben Arous and Jiří Černý, *Bouchaud's model exhibits two aging regimes in dimension one*, to appear in Annals of Applied Probability (2004).

[BCK04] M. T. Barlow, T. Coulhon, and T. Kumagai, *Characterization of sub-gaussian heat kernel estimates on strongly recurrent graphs*, preprint, 2004.

[BCKM98] J.-P. Bouchaud, L. Cugliandolo, J. Kurchan, and M. Mézard, *Out of equilibrium dynamics in spin-glasses and other glassy systems*, Spin glasses and random fields (A. P. Young, ed.), World Scientific, Singapore, 1998.

[BČM03] Gérard Ben Arous, Jiří Černý, and Thomas Mountford, *Aging for Bouchaud's model in dimension two*, preprint, 2003.

[BM97] J.-P. Bouchaud and M. Mézard, *Universality classes for extreme-value statistics*, J. Phys. A: Math. Gen. **30** (1997), 7997–8015.

[Bou92] J.-P. Bouchaud, *Weak ergodicity breaking and aging in disordered systems*, J. Phys. I (France) **2** (1992), 1705–1713.

[Čer03] Jiří Černý, *On two properties of strongly disordered systems, aging and critical path analysis*, Ph.D. thesis, EPF Lausanne, 2003.

[DM76] H. Dym and H. P. McKean, *Gaussian processes, function theory, and the inverse spectral problem*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976, Probability and Mathematical Statistics, Vol. 31.

[Fel71] William Feller, *An introduction to probability theory and its applications. Vol. II.*, Second edition, John Wiley & Sons Inc., New York, 1971.

[FIN02] L. R. G. Fontes, M. Isopi, and C. M. Newman, *Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension*, Ann. Probab. **30** (2002), no. 2, 579–604.

[KW82] S. Kotani and S. Watanabe, *Kreĭn's spectral theory of strings and generalized diffusion processes*, Functional analysis in Markov processes, Lecture Notes in Math., vol. 923, Springer, Berlin, 1982, pp. 235–259.

[MB96] C. Monthus and J.-P. Bouchaud, *Models of traps and glass phenomenology*, J. Phys. A **29** (1996), 3847–3869.

[RMB00] B. Rinn, P. Maass, and J.-P. Bouchaud, *Multiple scaling regimes in simple aging models*, Phys. Rev. Lett **84** (2000), 5403–5406.

[RW00] L. C. G. Rogers and David Williams, *Diffusions, Markov processes, and martingales. Vol. 2*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000.

[Sto63] Charles Stone, *Limit theorems for random walks, birth and death processes, and diffusion processes*, Illinois J. Math. **7** (1963), 638–660.