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On estimation and detection of smooth high-dimensional function

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Abstract

We observe an unknown n-variables function $f(t), t \in [0,1]^n$ in the white Gaussian noise of a level $\varepsilon > 0$. We suppose that there exist 1-periodical (in each variable) σ-smooth extensions of functions $f(t)$ on $\mathbb{R}^n$ and f belongs to a Sobolev ball, i.e., $\|f\|_{\sigma,2} \le 1$, where $\|\cdot\|_{\sigma,2}$ is a Sobolev norm (we consider two variants of one). We consider two problems: to estimate f and to test of the null hypothesis $H_0 : f = 0$ against alternatives $\|f\|_2 \ge r_\varepsilon$.

We study the asymptotics (as $\varepsilon \to 0,\ n \to \infty$) of the minimax risk for square losses in the estimation problem, and of minimax error probabilities and of minimax separation rates in the detection problem. We show that if $n \to \infty$, then there exist "sharp separation rates" in the detection problem. The asymptotics of minimax risks of estimation and of separation rates of testing are of different type for $n \ll \log \varepsilon^{-1}$ and for $n \gg \log \varepsilon^{-1}$.

The problems under consideration are related with some version of "lattice problem" in the numerical theory.

1 Statement of the problem

In this paper we consider "observations" of the form

$$X = f + \varepsilon \dot{W},\ f = f(t),\ t = (t_1, \dots, t_n) \in [0,1]^n,\ f \in L_n^2 = L_2([0,1]^n), \quad (1)$$

where $\dot{W}$ is n-dimensional Gaussian white noise. According to the theory of generalized random fields, it means that, for any real-valued function $\phi \in L_n^2$, we can observe the random variable $\xi = X(\phi) \sim \mathcal{N}(a, \sigma^2)$, where

$$a = (f, \phi) = \int_{[0,1]^n} f(t)\phi(t)dt,\ \sigma^2 = \varepsilon^2 \|\phi\|_2^2 = \varepsilon^2 \int_{[0,1]^n} \phi^2(t)dt,$$

and for any $\xi_1 = X(\phi_1),\ \xi_2 = X(\phi_2)$ we have

$$\mathrm{Cov}(\xi_1, \xi_2) = \varepsilon^2(\phi_1, \phi_2) = \varepsilon^2 \int_{[0,1]^n} \phi_1(t)\phi_2(t)dt.$$

The observation (1) determines the Gaussian measure $P_{\varepsilon,f}$ on the Hilbert space L_n^2 with the mean function f and covariation operator $\varepsilon^2 I$ (see, for example, [11]).

We assume below that underlying function admits 1-periodical extension in each argument to $\mathbb{R}^n$ and this is σ-smooth, i.e.,

$$\|f\|_{\sigma,2} \le C.$$

Here $\|\cdot\|_{\sigma,2}$ is a Sobolev semi-norm (see below for definitions). To simplify, we assume $C = 1$ below. We denote $\mathcal{F}_n = \mathcal{F}_n(\sigma)$ the set of functions $f \in L^2_n$ under these constraints.

In estimation problem, we study the asymptotics, as $\varepsilon \to 0$, of minimax square risk of estimation and in structure of asymptotically minimax estimators.

Namely, for an estimator $\hat{f}_\varepsilon$ (it is a measurable function of observation X_ε taken values in L^2_n), its minimax square risk is

$$R^2_{\varepsilon,n}(\hat{f}_\varepsilon, \mathcal{F}_n) = \sup_{f \in \mathcal{F}_n} E_{\varepsilon,f}\|f - \hat{f}_\varepsilon\|^2_2,$$

where $E_{\varepsilon,f}$ stands for expectation over the measure $P_{\varepsilon,f}$, which corresponds to observation (1). The minimax square risk of estimation is defined by

$$R^2_{\varepsilon,n} = R^2_{\varepsilon,n}(\mathcal{F}_n) = \inf_{\hat{f}_\varepsilon} R^2_{\varepsilon,n}(\hat{f}_\varepsilon, \mathcal{F}_n), \tag{2}$$

where the infimum is taken over all possible estimators $\hat{f}_\varepsilon$. A family of estimators f^*_ε is called *asymptotically minimax*, if

$$R^2_{\varepsilon,n}(f^*_\varepsilon, \mathcal{F}_n) \sim R^2_{\varepsilon,n}.$$

Here and below limits are assumed as $\varepsilon \to 0$.

In detection problem, we test the null hypothesis $H_0 : f = 0$ against alternatives $H_1 : f \in \mathcal{F}_{\varepsilon,n}$ where the sets $\mathcal{F}_{\varepsilon,n}$ consist of functions $f \in \mathcal{F}_n$ that are bounded away from zero function. More precisely, taken a positive family $r_\varepsilon \to 0$, we set

$$\mathcal{F}_{\varepsilon,n} = \{f \in L^2_n : \|f\|_{\sigma,2} \le 1,\ \|f\|_2 \ge r_\varepsilon\}. \tag{3}$$

We are interesting in asymptotics of "minimax error probabilities" $\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n})$ and in the structure of "asymptotically minimax tests" in the problem.

Namely for a test ψ_ε (it is a measurable function of observation X_ε taken values in the interval $[0,1]$) we set

$$\alpha_\varepsilon(\psi_\varepsilon) = E_{\varepsilon,0}\psi_\varepsilon, \quad \beta_\varepsilon(\psi_\varepsilon, f) = E_{\varepsilon,f}(1 - \psi_\varepsilon), \quad \gamma_\varepsilon(\psi_\varepsilon, f) = \alpha_\varepsilon(\psi_\varepsilon) + \beta_\varepsilon(\psi_\varepsilon, f).$$

The minimax error probability defined by

$$\gamma_\varepsilon(\psi_\varepsilon, \mathcal{F}_{\varepsilon,n}) = \sup_{f \in \mathcal{F}_{\varepsilon,n}} \gamma_\varepsilon(\psi_\varepsilon, f), \quad \gamma_\varepsilon = \gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) = \inf_{\psi_\varepsilon} \gamma_\varepsilon(\psi_\varepsilon, \mathcal{F}_{\varepsilon,n}).$$

We call a family of tests ψ^*_ε *asymptotically minimax*, if

$$\gamma_\varepsilon(\psi^*_\varepsilon, \mathcal{F}_{\varepsilon,n}) = \gamma_\varepsilon + o(1), \quad \text{as} \quad \varepsilon \to 0.$$

Also we are interesting in "separation rates" in the problem.

Namely, we call *separation rates* for alternative (3) a family r_ε^* such that

$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) \to 0, \quad \text{if} \quad r_\varepsilon \gg r_\varepsilon^*,$$
$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) \to 1, \quad \text{if} \quad r_\varepsilon \ll r_\varepsilon^*.$$

It means that for small ε, one can detect all functions $f \in \mathcal{F}_{\varepsilon,n}$, if the ratio $r_\varepsilon/r_\varepsilon^*$ is large, whenever if this ratio is small, then it is impossible to distinguish between the null-hypothesis and alternative $\mathcal{F}_{\varepsilon,n}$ with small minimax error probabilities.

For the case $n = 1$ and $\sigma > 0$ is an integer, the Sobolev semi-norm if defined by

$$\|f\|_{\sigma,2} = \|f^{(\sigma)}\|_2,$$

where $f^{(\sigma)}$ is σ-derivative of f. For general case under the periodical constraints, this can be defined in terms of Fourier coefficients $\theta_l = (f, \phi_l)$ with respect to the standard Fourier basis in $L_1^2 = L_2([0,1])$. We present this basis in the form $\{\phi_l,\ l \in \mathbb{Z}\}$ where

$$\phi_0(t) = 1, \quad \phi_l(t) = \sqrt{2}\cos(2\pi lt), \quad \phi_{-l}(t) = \sqrt{2}\sin(2\pi lt) \quad \text{for } l > 0.$$

We have

$$\|f\|_{\sigma,2}^2 = \sum_{l \in \mathbb{Z}} (2\pi|l|)^{2\sigma}\theta_l^2.$$

Thus in the sequence space of Fourier coefficients $\theta_l = (f, \phi_l),\ l \in \mathbb{Z}$, the set $\mathcal{F}_1(\sigma)$ corresponds to the ellipsoid

$$\Theta(\sigma) = \{\theta : \sum_{l \in \mathbb{Z}} \theta_l^2 c_l^2 \le 1\}$$

of semi-axes $c_l^{-1} = (2\pi|l|)^{-\sigma}$.

In estimation problem for the Sobolev ball (just without periodical constraint), the rate asymptotics of the minimax square risk were obtained in [4],

$$R_{\varepsilon,1} = R_{\varepsilon,1}(\sigma) \asymp \varepsilon^{2\sigma/(2\sigma+1)}.$$

The relation $R_{\varepsilon,1}(\hat{f}_\varepsilon, \mathcal{F}_1(\sigma)) \asymp R_{\varepsilon,1}(\sigma)$ is provides by estimators $\hat{f}_\varepsilon$ of kernel or of projection types.

These results are extended to $n > 1$ (see [12] for regression model),

$$R_{\varepsilon,n} = R_{\varepsilon,n}(\sigma) \asymp \varepsilon^{2\sigma/(2\sigma+n)} \asymp R_{\varepsilon,1}(\tilde{\sigma}), \quad \tilde{\sigma} = \sigma/n, \tag{4}$$

i.e., n-dimensional case corresponds to 1-dimensional case with changed $\tilde{\sigma} = \sigma/n$.

The sharp asymptotics of the minimax square risk for $n = 1$ were obtained in [10] under periodical constraint. It was shown in [10] that

$$R_{\varepsilon,1} \sim c(\sigma)\varepsilon^{2\sigma/(2\sigma+1)}, \quad c^2(\sigma) = (\sigma/\pi(\sigma+1))^{2\sigma/(1+2\sigma)}(1+2\sigma)^{1/(1+2\sigma)}$$

The asymptotically minimax estimators are of linear type [1]

$$f_\varepsilon^* = \sum_{l\in\mathbb{Z}} \alpha_{\varepsilon,l}^* X(\phi_l), \quad \alpha_{\varepsilon,l}^* = (1-(2\pi|l|)^\sigma/T)_+\,,$$

where the quantities $T = T_\varepsilon$ are of the rate $T^2 \sim (2\sigma+1)/R_{\varepsilon,1}^2$.

In detection problem for the Sobolev ball (without periodical constraint), the separation rates $r_\varepsilon^* = r_{\varepsilon,1}^*(\sigma)$ were obtained in [5],

$$r_{\varepsilon,1}^*(\sigma) = \varepsilon^{4\sigma/(4\sigma+1)}. \tag{5}$$

Thus if $r_\varepsilon/r_\varepsilon^* \to 0$, then it is impossible to distinguish between the null-hypothesis, whenever if $r_\varepsilon/r_\varepsilon^* \to \infty$, then there exist test a family $\psi_\varepsilon = \mathbb{I}_{\{t_\varepsilon > T_\varepsilon\}}$ that provides distinguishability. These tests are based on statistics t_ε of χ^2-type.

These results are extended to the case $n > 1$ in [6]. For any fixed $n > 1$ the separation rates are of the form

$$r_{\varepsilon,n}^*(\sigma) = \varepsilon^{4\sigma/(4\sigma+n)} = \varepsilon^{4\tilde\sigma/(4\tilde\sigma+1)} = r_{\varepsilon,1}^*(\tilde\sigma), \quad \tilde\sigma = \sigma/n. \tag{6}$$

Sharp asymptotics were obtained in [1] for $n = 1$ under the periodical constraint. It was shown in [1] that

$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,1}) = 2\Phi(-u_\varepsilon/2) + o(1), \tag{7}$$

where Φ is the distribution function of the standard Gaussian variable. The quantities u_ε are of asymptotics

$$u_\varepsilon \sim d(\sigma) r_\varepsilon^{2+1/2\sigma} \varepsilon^{-2}, \quad d^2(\sigma) = \pi(1+2\sigma)(1+4\sigma)^{-1-1/2\sigma} \quad . \tag{8}$$

The quantities u_ε characterize distinguishability in hypothesis testing problem for nonparametric alternative $\mathcal{F}_{\varepsilon,1}$. These are analogous to "signal-to-noise ratio" for known signal detection problem.

Asymptotically minimax tests are of the form $\psi_\varepsilon = \mathbb{I}_{t_\varepsilon > u_\varepsilon/2}$ and are based on statistics

$$t_\varepsilon = w_\varepsilon^{-1} \sum_{l\in\mathbb{Z}} w_{\varepsilon,l}(X_{\varepsilon,l}^2 - 1), \quad w_{\varepsilon,l} = ((1-|2\pi l/m|^{2\sigma})_+)^2, \quad w_\varepsilon^2 = \frac{1}{2}\sum_{l\in\mathbb{Z}} w_{\varepsilon,l}^2,$$

where $X_{\varepsilon,l} = \varepsilon^{-1}X(\phi_l)$ and $m = m_\varepsilon \asymp r_\varepsilon^{-1/\sigma}$.

The aim of the paper to study the asymptotics of minimax square risks and minimax error probabilities for the case $n > 1$ and as $n \to \infty$. A smoothness parameter $\sigma > 0$ is assumed to be fixed.

The paper is structured as follows.

[1] Here and below t_+ stands for the positive part of $t \in \mathbb{R}$, i.e., $t_+ = t$ for $t > 0$ and $t_+ = 0$ for $t \le 0$.

The main results are presented in Section 2. The results for estimation and detection problems are determined by similar extreme problems and these are formulated in parallel. In particular, we show that, as $n \to \infty$, there are different types of asymptotics for $n \ll \log \varepsilon^{-1}$ and for $n \gg \log \varepsilon^{-1}$.

The proofs are given in Sections 3–5. These are based on the study of the extreme problems noted above. For the case $n \ll \log \varepsilon^{-1}$, the proofs are based on analytical methods: we study an accuracy of evaluations of the sums by integrals. For the case $n \gg \log \varepsilon^{-1}$, we use probabilistic tools based on larger deviation machinery.

2 Main results

2.1 Sobolev semi-norms

Let $n > 1$. To specify a Sobolev semi-norm for an integer $\sigma > 0$, we consider two variants of one. The first one is

$$\|f\|_{\sigma,2}^2 = \sum_{k=1}^{n} \left\| \frac{\partial^\sigma f}{\partial t_k^\sigma} \right\|_2^2 . \tag{9}$$

The second one is

$$\|f\|_{\sigma,2}^2 = \sum \left\| \frac{\partial^\sigma f}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}} \right\|_2^2 , \tag{10}$$

where the sum is taken over all integer $k_1 \geq 0, \ldots, k_n \geq 0$ such that $k_1 + \ldots + k_n = \sigma$.

For general case $\sigma > 0$, we determine semi- norms in terms of Fourier coefficients

$$\theta_l = (f, \phi_l), \quad l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$$

with respect to the Fourier basis in L_n^2,

$$\phi_l(t) = \prod_{k=1}^{n} \phi_{l_k}(t_k).$$

Clearly,

$$\|f\|_2^2 = \sum_{l \in \mathbb{Z}^n} \theta_l^2,$$

and (9) corresponds to

$$\|f\|_{\sigma,2}^2 = \sum_{l \in \mathbb{Z}^n} \theta_l^2 c_l^2, \quad c_l^2 = \sum_{k=1}^{n} |2\pi l_k|^{2\sigma}, \tag{11}$$

whenever (10) corresponds to

$$\|f\|_{\sigma,2}^2 = \sum_{l \in \mathbb{Z}^n} \theta_l^2 c_l^2, \quad c_l^2 = \left(\sum_{k=1}^{n} (2\pi l_k)^2 \right)^\sigma . \tag{12}$$

In the sequence space of Fourier coefficients indexed by $l \in \mathbb{Z}^n$, the set $\mathcal{F}_n$ corresponds to ellipsoid of semi-axes c_l^{-1} determined by (11) or (12).

2.2 Extreme problems

There are similar extreme problems that determines the sharp asymptotics of minimax square risks in estimation problem and of minimax error probabilities in detection problem.

In estimation problem, this were formulated in [10]. Using the results [10] we obtain

Theorem 1 *Let $\mathcal{F}_n$ be the set determined by the norm $\|f\|_{\sigma,2}$ of the form (11), (12). Let $E^2_{\varepsilon,n}$ is the value of the extreme problem on the set of real-valued two-side sequences $\{v_l,\ l \in \mathbb{Z}^n\}$:*

$$E^2_{\varepsilon,n} = \varepsilon^2 \sup \sum_{l \in \mathbb{Z}^n} \frac{v_l^2}{v_l^2+1} \quad \textit{subject to} \quad \sum_{l \in \mathbb{Z}^n} c_l^2 v_l^2 \le \varepsilon^{-2}. \tag{13}$$

Then for any $n = n_\varepsilon \in \mathcal{N}$, one has

$$R^2_{\varepsilon,n} \sim E^2_{\varepsilon,n}.$$

The extreme sequence $\{\bar{v}_l^2\},\ l \in \mathbb{Z}^n$ in (13) is of the form

$$\bar{v}_l^2 = (T/c_l - 1)_+ \tag{14}$$

(we formally set $\bar{v}_l^2 = \infty$, if $c_l = 0$), where the quantities $T = T_\varepsilon > 0$ are taking in such way that

$$\sum_{c_l<T} c_l^2 \bar{v}_l^2 = T^2 \sum_{c_l<T} (c_l/T - (c_l/T))^2 = \varepsilon^{-2}, \quad T_\varepsilon \to \infty, \tag{15}$$

and the value of the problem is

$$E^2_{\varepsilon,n} = \varepsilon^2 \sum_{c_l<T} (1 - c_l/T). \tag{16}$$

The asymptotically minimax estimators are of linear type,

$$f^*_\varepsilon = \sum_{c_l<T} \alpha^*_{\varepsilon,l} X(\phi_l), \quad \alpha^*_{\varepsilon,l} = 1 - c_l/T.$$

Proof. Theorem 1 follows directly from [10], where minimax estimation problem for ellipsoids of general type were studied. Our case corresponds to $\sigma_l = \varepsilon,\ a_l = c_l^2,\quad P = 1$ in [10] (certainly there is nonessential difference $l \in \mathbb{Z}$ in [10] and $l \in \mathbb{Z}^n$ in our case). The assumption (19) in [10] corresponds to

$$N_n(t) = \#\{l \in \mathbb{Z}^n : c_l \le t\} < \infty, \quad \forall\ t > 0, \tag{17}$$

and fulfilled for the norms under consideration. Applying Lemma 1 in [10] and setting $T = \mu^{-1/2},\ v_l = \theta_l/\varepsilon$, we go to extreme problem (13). The extreme sequence is of the form (14), (15). It suffices to verify the assumption (25) in [10]. In our case, this is of the form

$$\varepsilon^{-2} E_{\varepsilon,n}^2 \to \infty,$$

and it follows from

$$I_n(T) = \sum_{c_l < T} (1 - c_l/T) \to \infty.$$

The last relation follows from

$$N_n(t) \to \infty, \quad \text{as} \quad t \to \infty \tag{18}$$

for our cases and, for any $\delta \in (0,1)$,

$$I_n(T) \geq (1-\delta) N_n(\delta T) \to \infty.$$

Thus the theorem follows from Theorem 1 in [10]. □

Let us go to detection problem. In the sequence space indexed by $l \in \mathbb{Z}^n$, the alternatives determined by (3) and (11) or (12) correspond to ellipsoids of semi-axes c_l^{-1} with a ball removed.

Let us start with general theorem based on methods and results [1], [8].

Theorem 2 *Let $\mathcal{F}_{\varepsilon,n}$ be the alternative determined by (3) and the norm $\|f\|_{\sigma,2}$ of the form (11), (12). Let $u_{\varepsilon,n}^2 = u_n^2(\varepsilon, r_\varepsilon)$ be the value of the problem*

$$u_n^2(\varepsilon, r_\varepsilon) = \inf \frac{1}{2} \sum_{l \in \mathbb{Z}^n} v_l^4 \quad \textit{subject to} \tag{19}$$

$$\sum_{l \in \mathbb{Z}^n} v_l^2 \geq (r_\varepsilon/\varepsilon)^2, \quad \sum_{l \in \mathbb{Z}^n} v_l^2 c_l^2 \leq \varepsilon^{-2},$$

For any $n = n_\varepsilon \in \mathcal{N}$,

$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) \to 1, \quad \textit{as} \quad u_{\varepsilon,n}^2 \to 0. \tag{20}$$

Moreover let $r_\varepsilon \to 0$. Then for any $n = n_\varepsilon \in \mathcal{N}$, one has

$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) = 2\Phi(-u_{\varepsilon,n}/2) + o(1). \tag{21}$$

The extreme sequence is of the form

$$\bar{v}_l^2 = u_0^2 (1 - (c_l/T)^2)_+, \tag{22}$$

where the quantities $u_0 > 0$, $T > 0$ are determined by the equation

$$\sum_{l \in \mathbb{Z}^n} \bar{v}_l^2 = u_0^2 \sum_{c_l < T} (1 - (c_l/T)^2) = (r_\varepsilon/\varepsilon)^2, \tag{23}$$

$$\sum_{l \in \mathbb{Z}^n} c_l^2 \bar{v}_l^2 = u_0^2 \sum_{c_l < T} c_l^2 (1 - (c_l/T)^2) = (1/\varepsilon)^2, \tag{24}$$

$$T \geq r_\varepsilon^{-1} \to \infty, \tag{25}$$

whenever the value of the problem is

$$u_{\varepsilon,n}^2 = \frac{1}{2}\sum_{l\in\mathbb{Z}^n} \bar{v}_l^4 = \frac{1}{2}u_0^4 \sum_{c_l<T}(1-(c_l/T)^2)^2. \tag{26}$$

Asymptotically minimax tests are of the form $\psi_{\varepsilon,n} = \mathbb{1}_{t_{\varepsilon,n}>u_{\varepsilon,n}/2}$ *and are based on statistics*

$$t_{\varepsilon,n} = w_\varepsilon^{-1}\sum_{l\in\mathbb{Z}^n} w_{\varepsilon,l}(X_{\varepsilon,l}^2-1), \quad w_{\varepsilon,l} = \Big((1-(c_l/T)^2)_+\Big)^2, \qquad w_\varepsilon^2 = \frac{1}{2}\sum_{l\in\mathbb{Z}^n} w_{\varepsilon,l}^2,$$

where $X_{\varepsilon,l} = \varepsilon^{-1}X(\phi_l)$.

Proof. Setting $v_l = \varepsilon^{-1}(f,\psi_l), \quad l\in\mathbb{Z}^n$, we pass to the sequence space of normalized Fourier coefficients. In this space the set $\mathcal{F}_{\varepsilon,n}$ corresponds to the ellipsoid with the semi-axes $(\varepsilon c_l)^{-1}$, minus its intersection with the ball of the radius $r_\varepsilon/\varepsilon$. The hypothesis testing problem for alternative of this type were studied in [7] and in [8]. The relation (20) follows from these results (compare with [8], Proposition 3.6). If $r_\varepsilon \to 0$, then analogously to [8], Chapter 4 we see that there exists the unique extreme sequence $\bar{v}_l^2,\ l\in\mathbb{Z}^n$ in the extreme problem (19). Using the Lagrange multipliers rule we see that the extreme sequence is of the form (22), (23), (24). Using (23), (24) we get the inequality

$$(1/\varepsilon)^2 = u_0^2\sum_{c_l<T} c_l^2(1-(c_l/T)^2) \leq T^2u_0^2\sum_{c_l<T}(1-(c_l/T)^2) = T^2(r_\varepsilon/\varepsilon)^2.$$

By $r_\varepsilon \to 0$ this yields (25).

In order to obtain (21) it suffices to verify that the extreme sequence satisfy

$$\omega_\varepsilon^2 = \sup_{l\in\mathbb{Z}^n}\bar{v}_l^4 \Big/ \sum_{l\in\mathbb{Z}^n}\bar{v}_l^4 = \sup_{l\in\mathbb{Z}^n}\bar{v}_l^4/2u_{\varepsilon,n}^2 = o(1), \tag{27}$$

compare with [8], Proposition 4.5. Set

$$I(T) = \sum_{l\in N_n(T)}(1-(c_l/T)^2)^2.$$

For any $\delta\in(0,1)$, we have

$$I(T) \geq (1-\delta^2)^2 N_n(\delta T) \to \infty,$$

where $N_n(t)$ is defined by (17) and (18) holds. By (22),

$$\omega_\varepsilon^2 \leq 1/I(T),$$

and we get (27). Theorem 2 follows. $\square$

2.3 Sharp asymptotics for fixed n

Using results of Section 2.2 we obtain the the sharp asymptotics of minimax square risks and minimax error probabilities for any fixed n.

Namely, for estimation problem we have

Theorem 3 *Assume $n \in \mathcal{N}$, $\sigma > 0$ be fixed. For the norms (11), (12), the quantities $E_{\varepsilon,n}$ are of asymptotics*

$$E_{\varepsilon,n} \sim c(\sigma,n)\varepsilon^{2\sigma/(2\sigma+n)}, \quad T^2 \sim E_{\varepsilon,n}^{-2}(2\sigma+n)/n. \tag{28}$$

For (11), the function $c(\sigma,n)$ is of the form

$$c^2(\sigma,n) = \left((1+2\sigma/n)^{n/2\sigma}\frac{\sigma\Gamma^n(1+1/2\sigma)}{\pi^n(\sigma+n)\Gamma(1+n/2\sigma)}\right)^{2\sigma/(n+2\sigma)}, \tag{29}$$

for (12) one has

$$c^2(\sigma,n) = \left((1+2\sigma/n)^{n/2\sigma}\frac{\sigma\Gamma^n(3/2)}{\pi^n(\sigma+n)\Gamma(1+n/2}\right)^{2\sigma/(n+2\sigma)}. \tag{30}$$

Proof of Theorem 3 is given in Section 3.

For any fixed n, the rates (4) follows from Theorem 3.

Let us go to detection problem. We obtain

Theorem 4 *Assume $n \in \mathcal{N}$, $\sigma > 0$ be fixed. For the norms (11), (12), the quantities $u_{\varepsilon,n}$ are of asymptotics*

$$u_{\varepsilon,n} \sim d(\sigma,n)r_\varepsilon^{2+n/2\sigma}\varepsilon^{-2}, \quad T^2 = r_\varepsilon^{-2}(4\sigma+n)/n. \tag{31}$$

For (11), the function $d(\sigma,n)$ is of the form

$$d^2(\sigma,n) = \pi^n(1+2\sigma/n)(1+4\sigma/n)^{-n/2\sigma-1}(\Gamma((2\sigma)^{-1}+1))^{-n}\Gamma((n/2\sigma)+1),$$

for (12) one has

$$d^2(\sigma,n) = \pi^n(1+2\sigma/n)(1+4\sigma/n)^{-n/2\sigma-1}(\Gamma(2^{-1}+1))^{-n}\Gamma(n/2+1).$$

Proof of Theorem 4 is given in Section 3.

For any fixed n, the rates (6) follows from Theorem 4.

2.4 Sharp asymptotics for $n \to \infty$

2.4.1 Estimation problem

For $n = n_\varepsilon \to \infty$ using Simpson's formula for $\Gamma(x),\ x \to \infty$, we can slightly modify the quantities $c_n(\sigma, n)$. For (11) we have,

$$c^2(\sigma, n) \sim \left(\frac{\Gamma(1/2\sigma)}{2\pi\sigma}\right)^{2\sigma} \frac{2\sigma e}{n}, \tag{32}$$

whenever for (12),

$$c^2(\sigma, n) \sim (e/2\pi n)^\sigma. \tag{33}$$

Theorem 5 *Let $n = n_\varepsilon \to \infty,\ n = o(\log \varepsilon^{-1})$. Then the relation (28) holds true with c_n defined by (32), (33).*

Proof of Theorem 5 is given in Section 4.

However if $n \gg \log \varepsilon^{-1}$, then we have asymptotics of different type.

Theorem 6

(1) Let $n = n_\varepsilon \to \infty,\ n/\log \varepsilon^{-1} \to \infty,\ \log n = o(\log \varepsilon^{-1})$. Then for the norm (11),

$$E^2_{\varepsilon,n} \sim \frac{\log n - \log\log \varepsilon^{-1}}{2(2\pi)^{2\sigma} \log \varepsilon^{-1}}, \tag{34}$$

whenever for the norm (12),

$$E^2_{\varepsilon,n} \sim \left(\frac{\log n - \log\log \varepsilon^{-1}}{8\pi^2 \log \varepsilon^{-1}}\right)^\sigma. \tag{35}$$

(2) Let $\liminf \log n/\log \varepsilon^{-1} > 0$. Then $\liminf R^2_{\varepsilon,n} > 0$. It means that there do not exist consistent estimators in this case.

Proof of Theorem 6 is given in Section 5.

2.4.2 Detection problem

We can modify the quantities $d_n(\sigma, n)$ for $n \to \infty$. For (11) we get

$$d^2(\sigma, n) \sim \mathrm{e}^{-2}\pi^n(\Gamma((2\sigma)^{-1}+1))^{-n}\Gamma((n/2\sigma)+1) \sim \frac{\pi^n(\pi n/\sigma)^{1/2}(n/2\sigma e)^{n/2\sigma}}{e^2\Gamma^n(1+1/2\sigma)}, \tag{36}$$

whenever for (12),

$$d^2(\sigma, n) \sim \mathrm{e}^{-2}\pi^n(\Gamma(3/2))^{-n}\Gamma(n/2+1) \sim (2\pi n/e)^{n/2} e^{-2}\sqrt{\pi n}. \tag{37}$$

Moreover, we can establish "sharp separation rates", as $n \to \infty$. Namely in hypothesis testing problem (3), we call *sharp separation rates* a family r_ε^* such that

$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) \to 0, \quad \text{as} \quad \liminf r_\varepsilon / r_\varepsilon^* > 1$$

and

$$\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) \to 1, \quad \text{as} \quad \limsup r_\varepsilon / r_\varepsilon^* < 1.$$

Note that sharp separation rates do not exist for the case of fixed or bounded n in hypothesis testing problems under consideration here. For hypothesis testing problems that were studied in [8], sharp separation rates exist for the problems of "degenerate type" and for "adaptive problems" (see [8]). Sharp separation rates were established in [9] for one-dimensional signal detection problems analogous to (3), where the removed L_2-ball is replaced be removed L_∞-ball.

Theorem 7 *Let* $n = n_\varepsilon \to \infty$, $r_\varepsilon = o(1)$.

(1) Let $\log u_\varepsilon = o(n)$. *For the norms (11), let*

$$\limsup \frac{n \log\log \varepsilon^{-1}}{\log \varepsilon^{-1}} < 4, \quad \textit{if} \quad \sigma \geq 1/2,$$

$$\limsup \frac{n \log\log \varepsilon^{-1}}{\log \varepsilon^{-1}} < 8\sigma, \quad \textit{if} \quad \sigma \in (0, 1/2).$$

For the norms (12), let

$$\limsup \frac{n \log\log \varepsilon^{-1}}{\log \varepsilon^{-1}} < 4.$$

Then the relation (31) holds true with the functions $d(\sigma, n)$ *defined by (36), (37).*

(2) Let $n = o(\log \varepsilon^{-1})$. *Then for the problem with the norm (11), the sharp separation rates are of the form*

$$r_\varepsilon^* = \varepsilon^{4\sigma/(4\sigma+n)} (2\sigma e/n)^{1/2} (\Gamma(1/2\sigma)/2\pi\sigma)^\sigma. \tag{38}$$

For the problem with the norm (12), the sharp separation rates are of the form

$$r_\varepsilon^* = \varepsilon^{4\sigma/(4\sigma+n)} (e/2\pi n)^{\sigma/2}. \tag{39}$$

Proof of Theorem 7 is given in Section 4.

The assumption $n = o(\log \varepsilon^{-1})$ is essential in Theorem 7 (2): if $n \gg \log \varepsilon^{-1}$, we have sharp separation rates of different type. Namely

Theorem 8

(1) Let $n = n_\varepsilon \to \infty$, $n/\log \varepsilon^{-1} \to \infty$, $\log n = o(\log \varepsilon^{-1})$. *Then for the norm (11), the sharp separation rates are of the form*

$$r_\varepsilon^* = \frac{1}{2(2\pi)^\sigma} \sqrt{\frac{\log n - \log\log \varepsilon^{-1}}{\log \varepsilon^{-1}}}, \tag{40}$$

whenever for the norm (12), the sharp separation rates are

$$r_\varepsilon^* = \left(\frac{\log n - \log\log \varepsilon^{-1}}{16\pi^2 \log \varepsilon^{-1}}\right)^{\sigma/2}. \tag{41}$$

(2) Let $\liminf \log n/\log \varepsilon^{-1} > b > 0$. *Then there exists* $r_0 = r_0(b) > 0$ *such that if* $\limsup r_\varepsilon < r_0$, *then* $\gamma_\varepsilon(\mathcal{F}_{\varepsilon,n}) \to 1$. *It means that there do not exist separation rates* $r_\varepsilon^* \to 0$.

Proof of Theorem 8 is given in Section 5.

Let us compare the rates of $E_{\varepsilon,n}$ in estimation problem and separation rates r_ε^* in detection problem. It follows from Theorems 5–8 that if $n = o(\log \varepsilon^{-1})$, then

$$r_\varepsilon^* = o(E_{\varepsilon,n}).$$

This relation is well-known for fixed n.

However if $n/\log \varepsilon^{-1} \to \infty$, then for the norm (11) we have

$$r_\varepsilon^* \sim 2^{-1/2} E_{\varepsilon,n},$$

and for the norm (12),

$$r_\varepsilon^* \sim 2^{-\sigma/2} E_{\varepsilon,n}.$$

Remark 2.1 Note that for the norm (12), the asymptotics of the minimax square risk sharp separation rates are closely related with the following version of the lattice problem: *what are the asymptotics of numbers of lattice points in n-dimensional Euclidean ball of radius m, as $n \to \infty$, $m \to \infty$?*

If n is fixed, then, roughly, this number is close to the volume of the ball (see the paper [3] and references in this paper for details). However if $n \to \infty$, $m \to \infty$, then this holds for $n = o(m^2)$, whenever if $n \gg m^2$, then the asymptotics are of different type.

Analogously, the case of the norms (11) is related with the number of lattice points in n-dimensional ball in l_η-norm, $\eta = 2\sigma$, of radius $m \to \infty$ and we have analogous effects in this problem.

See Remarks 3.1–5.1 in the proofs of Theorems.

3 Proof of Theorems 3, 4

3.1 Proof of Theorems 4

In order to prove Theorem 4, we need to study the equations (23), (24), (26) assuming $T \to \infty$ by (25). Set

$$I_1 = \sum_{c_l < T} (1 - (c_l/T)^2), \tag{42}$$

$$I_2 = \sum_{c_l<T} (c_l/T)^2(1-(c_l/T)^2), \tag{43}$$

$$I_0 = \sum_{c_l<T} (1-(c_l/T)^2)^2 = I_1 - I_2. \tag{44}$$

It follows from (23), (24), (26) that

$$T^2 = r_\varepsilon^{-2} I_1/I_2, \tag{45}$$

$$u_\varepsilon^2 = \frac{1}{2}(r_\varepsilon/\varepsilon)^4 I_0/I_1^2. \tag{46}$$

Let us study the asymptotics of the quantities $I_k,\ k=0,1,2$ as $T\to\infty$ assuming n be fixed.

3.1.1 Norms (11)

Set

$$\eta = 2\sigma, \quad T=(2\pi m)^\sigma, \quad x_{li}=l_i/m, \quad \delta x_{li}=1/m, \quad \delta x_l = \delta x_{l1}\dots\delta x_{ln} = m^{-n}, \tag{47}$$

and denote

$$|x|_\eta = (\sum_{i=1}^n |x_i|^\eta)^{1/\eta}, \quad x\in \mathbb{R}^n.$$

If $\eta\ge 1$, then $|x|_\eta$ is l_η-norm in $\mathbb{R}^n$, if $\eta\in(0,1)$, then it is quasi-norm. We get

$$(c_l/T)^2 = \sum_{i=1}^n |x_{li}|^\eta = |x_l|_\eta^\eta,$$

and by replacing the integral sums by the integrals (this is possible for any fixed n since $m\to\infty$), we have

$$I_1 = m^n \sum_{c_l<T} (1-|x_l|_\eta^\eta)\delta x_l \sim m^n J_1, \tag{48}$$

$$I_2 = m^n \sum_{c_l<T} |x_l|_\eta^\eta (1-|x_l|_\eta^\eta)\delta x_l \sim m^n J_2, \tag{49}$$

$$I_0 = m^n \sum_{c_l<T} (1-|x_l|_\eta^\eta)^2\delta x_l \sim m^n J_0, \tag{50}$$

where

$$J_1 = \int_{D_\eta^n(1)} (1-|x|_\eta^\eta)dx_1\dots dx_n, \tag{51}$$

$$J_2 = \int_{D_\eta^n(1)} |x|_\eta^\eta(1-|x|_\eta^\eta)dx_1\dots dx_n, \tag{52}$$

$$J_0 = \int_{D_\eta^n(1)} (1-|x|_\eta^\eta)^2 dx_1\dots dx_n = J_1 - J_2, \tag{53}$$

and $D^n_\eta(1) = \{x \in \mathbb{R}^n : |x|_\eta \le 1\}$ is the unit l_η-ball. We can rewrite (45), (46) in the form

$$\begin{aligned} (2\pi m)^{2\sigma} &\sim r_\varepsilon^{-2} J_1/J_2, & (54)\\ u_\varepsilon^2 &\sim \frac{1}{2}(r_\varepsilon/\varepsilon)^4 m^{-n} J_0/J_1^2. & (55) \end{aligned}$$

The integrals $J_0 - J_2$ are reduced to integrals of the form

$$\begin{aligned} J_n(\tau,\eta) &= \int_{D^n_\eta(1)} (\sum_{i=1}^n |x_i|^\eta)^\tau dx_1 \dots dx_n \\ &= \sigma^{-n} \int_{\Delta_n} (\sum_{i=1}^n y_i)^\tau y_1^{1/\eta-1} dy_1 \dots y_n^{1/\eta-1} dy_n, \end{aligned}$$

where

$$\Delta_n = \{y \in \mathbb{R}^n : y_i \ge 0,\ \sum_{i=1}^n y_i \le 1\}$$

is the standard simplex in $\mathbb{R}^n$, $\eta > 0,\ \tau \ge 0$. Using the Liouville formula (see [2], Section XVIII, n.14)

$$\int_{\Delta_n} \phi(\sum_{i=1}^n y_i) y_1^{p-1} dy_1 \dots y_n^{p-1} dy_n = \frac{\Gamma^n(p)}{\Gamma(np)} \int_0^1 \phi(u) u^{np-1} du, \quad p > 0,$$

we get

$$J_n(\tau,\eta) = \frac{n 2^n \Gamma^n(1+1/\eta)}{(\eta\tau + n)\Gamma(1+n/\eta)}. \tag{56}$$

This yields

$$\begin{aligned} J_1 &= J_n(0,2\sigma) - J_n(1,2\sigma) = \frac{\sigma 2^{n+1}\Gamma^n(1+1/2\sigma)}{(2\sigma+n)\Gamma(1+n/2\sigma)}, & (57)\\ J_2 &= J_n(1,2\sigma) - J_n(2,2\sigma) = \frac{n\sigma 2^{n+1}\Gamma^n(1+1/2\sigma)}{(2\sigma+n)(4\sigma+n)\Gamma(1+n/2\sigma)}, & (58)\\ J_0 &= J_n(0,2\sigma) - 2J_n(1,2\sigma) + J_n(2,2\sigma) \\ &= \frac{\sigma^2 2^{n+3}\Gamma^n(1+1/2\sigma)}{(2\sigma+n)(4\sigma+n)\Gamma(1+n/2\sigma)}. & (59) \end{aligned}$$

Using this relations jointed with (54), (55) we obtain the statements of Theorem 4 for the norm (11).

3.1.2 Norms (12)

Using (47) we have

$$(c_l/T)^2 = (\sum_{i=1}^n x_{li}^2)^\sigma = |x_l|^{2\sigma}, \quad x_l \in \mathbb{R}^n,$$

where $|x| = |x|_2$ is the Euclidean norm. Analogously,

$$I_1 = m^n \sum_{c_l<T} (1 - |x_l|^{2\sigma})\delta x_l \sim m^n J_1, \tag{60}$$

$$I_2 = m^n \sum_{c_l<T} |x_l|^{2\sigma}(1 - |x_l|^{2\sigma})\delta x_l \sim m^n J_2, \tag{61}$$

$$I_0 = m^n \sum_{c_l<T} (1 - |x_l|^{2\sigma})^2\delta x_l \sim m^n J_0, \tag{62}$$

where

$$J_1 = \int_{D_2^n(1)} (1 - |x|^{2\sigma}) dx_1 \dots dx_n = J_n(0,2) - J_n(\sigma,2),$$

$$J_2 = \int_{D_2^n(1)} |x|^{2\sigma}(1 - |x|^{2\sigma}) dx_1 \dots dx_n = J_n(\sigma,2) - J_n(2\sigma,2),$$

$$J_0 = \int_{D_2^n(1)} (1 - |x|^{2\sigma})^2 dx_1 \dots dx_n = J_n(0,2) - 2J_n(\sigma,2) + J_n(2\sigma,2),$$

the region $D_2^n(1)$ is the unit Euclidean ball in $\mathbb{R}^n$. Using (56) ones again we get

$$J_1 = \frac{\sigma 2^{n+1}\Gamma^n(3/2)}{(2\sigma+n)\Gamma(1+n/2)}, \tag{63}$$

$$J_2 = \frac{n\sigma 2^{n+1}\Gamma^n(3/2)}{(2\sigma+n)(4\sigma+n)\Gamma(1+n/2)}, \tag{64}$$

$$J_0 = \frac{\sigma^2 2^{n+3}\Gamma^n(3/2)}{(2\sigma+n)(4\sigma+n)\Gamma(1+n/2)}. \tag{65}$$

Using this relations jointed with (45), (46) we obtain the statements of Theorem 4 for the norm (12). □

3.2 Proof of Theorems 3

In order to prove Theorem 4, we need to study the equations (15), (16) assuming $T \to \infty$. Set, analogously to (42),

$$I_1 = \sum_{c_l<T} (1 - c_l/T), \tag{66}$$

$$I_2 = \sum_{c_l<T} (c_l/T)(1 - (c_l/T)). \tag{67}$$

We have

$$T^2 = (2\pi m)^{2\sigma} = 1/(\varepsilon^2 I_2), \quad E_{\varepsilon,n}^2 = \varepsilon^2 I_1. \tag{68}$$

The study of the sums (66), (67) is the same as in Section 3.1. We have

$$I_1 \sim m^n J_1, \quad I_2 \sim m^n J_2, \tag{69}$$

where, for the norm (11),

$$J_1 = J_n(0, 2\sigma) - J_n(1/2, 2\sigma), \quad J_2 = J_n(1/2, 2\sigma) - J_n(1, 2\sigma),$$

whenever for the norm (12),

$$J_1 = J_n(0, 2) - J_n(\sigma/2, 2), \quad J_2 = J_n(\sigma/2, 2) - J_n(\sigma, 2),$$

and $J_n(\tau, \eta)$ are defined by (56). Thus, for the norm (11),

$$J_1 = \frac{\sigma 2^n \Gamma^n(1 + 1/2\sigma)}{(\sigma + n)\Gamma(1 + n/2\sigma)}, \quad J_2 = \frac{n\sigma 2^n \Gamma^n(1 + 1/2\sigma)}{(\sigma + n)(2\sigma + n)\Gamma(1 + n/2\sigma)}, \tag{70}$$

whenever for the norm (12),

$$J_1 = \frac{\sigma 2^n \Gamma^n(3/2)}{(\sigma + n)\Gamma(1 + n/2)}, \quad J_2 = \frac{n\sigma 2^n \Gamma^n(3/2)}{(\sigma + n)(2\sigma + n)\Gamma(1 + n/2)}; \tag{71}$$

$$\Gamma^n(3/2) = \pi^{n/2} 2^{-n}.$$

It follows from (68), (69) that

$$\begin{aligned} m &\sim \left(\varepsilon^2 J_2 (2\pi)^{2\sigma}\right)^{-1/(2\sigma+n)}, \\ E^2_{\varepsilon,n} &\sim \varepsilon^2 m^n J_1 \sim \varepsilon^{4\sigma/(2\sigma+n)} (J_1/J_2) J_2^{2\sigma/(2\sigma+n)} (2\pi)^{-2n\sigma/(2\sigma+n)}. \end{aligned} \tag{72}$$

Combine (70), (71), (72) we obtain (28), (29), (30). □

Remark 3.1 The sums I_k, $k = 0, 1, 2$ are related with the numbers of lattice points $N_n(T)$ in the η-ball in $\mathbb{R}^n$ (see (17)). If n is fixed, then for the norm (11) analogously to (57)–(59),

$$N_n(T) \sim m^n V_n(\eta),$$

where $V_n(\eta) = J_n(0, \eta)$ is the Euclidean volume of the unit ball $D^n_\eta(1)$,

$$V_n(\eta) = \frac{2^n \Gamma^n(1 + 1/\eta)}{\Gamma(1 + n/\eta)},$$

here we used the equality $\Gamma(1 + x) = x\Gamma(x)$.

Note that the relations (57)–(59) could be rewritten in the form

$$J_1 = \frac{2\sigma V_n(2\sigma)}{n + 2\sigma}, \quad J_2 = \frac{2n\sigma V_n(2\sigma)}{(n + 2\sigma)(n + 4\sigma)}, \quad J_0 = \frac{8\sigma^2 V_n(2\sigma)}{(n + 2\sigma)(n + 4\sigma)}.$$

Analogously for the norm (12),

$$N_n(T) \sim m^n V_n(2), \quad V_n(2) = \pi^{n/2}/\Gamma(1 + n/2),$$

and the relations (63)–(65) are the same form with the change $V_n(2\sigma)$ by $V_n(2)$.

4 Proof of Theorems 7, 5

4.1 Proof of Theorems 7

To study the case $n \to \infty$ we need to evaluate the accuracy of of replacing the integral sums by integrals in the evaluations in Section 3.

4.1.1 Norms (11)

Because of $|x|_\eta$ is not a norm for $\eta \in (0,1)$, we need the following statement.

Lemma 4.1 *Let*

$$x \in \mathbb{R}^n, \ \delta \in \mathbb{R}^n, \ \max_{1\le i\le n} |\delta_i| \le 1/2.$$

Then

$$|x|_\eta - n^{1/\eta}/2 \le |x+\delta|_\eta \le |x|_\eta + n^{1/\eta}/2, \quad \textit{for} \quad \eta \ge 1, \tag{73}$$

$$|x|_\eta^\eta - n2^{-\eta} \le |x+\delta|_\eta^\eta \le |x|_\eta^\eta + n2^{-\eta}, \quad \textit{for} \quad \eta \in (0,1). \tag{74}$$

Proof of Lemma 4.1. The inequality (73) follows directly from the triangle inequality since $|\delta|_\eta \le n^{1/\eta}/2$. The inequality (74) follows the inequality

$$|x+y|_\eta^\eta \le |x|_\eta^\eta + |y|_\eta^\eta, \quad \eta \in (0,1), \tag{75}$$

since $|\delta|_\eta^\eta \le n2^{-\eta}$. Let us prove (75). It is of the form

$$\sum_{i=1}^n |x_i + y_i|^\eta \le \sum_{i=1}^n (|x_i|^\eta + |y_i|^\eta),$$

and it suffices to consider the case $n = 1$. Let $|x| \ge |y|, \ x, y \in \mathbb{R}$. Then using the inequality

$$(1+z)^\eta \le 1 + \eta z, \quad z > -1$$

(this follows from concavity of the function $(1+z)^\eta$ in $z > -1$ for $\eta \in (0,1)$), we have

$$|x+y|^\eta = |x|^\eta(1+|y|/|x|)^\eta \le |x|^\eta(1+\eta|y|/|x|) \le |x|^\eta + |y|/|x|^{1-\eta} \le |x|^\eta + |y|^\eta.$$

□

Let

$$\Delta_l = \{x \in \mathbb{R}^n : x_i - 1/2 \le l_i < x_i + 1/2\}$$

be the cube with the centre $l \in \mathbb{Z}^n$ of the size 1. For $m > 0$ let

$$D_\eta^n(m) = \{x \in \mathbb{R}^n : |x|_\eta \le m\}$$

be the l_η-ball of radius m in $\mathbb{R}^n$ and

$$L^n_\eta(m) = \{l \in \mathbb{Z}^n : |l|_\eta \leq m\} = D^n_\eta(m) \cap \mathbb{Z}^n.$$

Set also

$$\tilde{D}^n_\eta(m) = \bigcup_{l \in L^n_\eta} \Delta_l.$$

Set, for any $b > 0$,

$$\begin{aligned} \delta_n &= \begin{cases} n^{1/\eta}/2m, & \eta \geq 1, \\ n/(2m)^\eta, & \eta \in (0,1), \end{cases} \\ C^\pm_{n,b} &= \begin{cases} 1 \pm n^{1/\eta}/2bm, & \eta \geq 1, \\ (1 \pm n/(2bm)^\eta)^{1/\eta}, & \eta \in (0,1), \end{cases} \end{aligned}$$

and put

$$C_{n,b} = C^+_{n,b}, \quad c_{n,b} = C^-_{n,b}.$$

Using Lemma 4.1 we have the embedding,

$$\tilde{D}^n_\eta(bm) \subset D^n_\eta(C_{n,b}bm) \tag{76}$$

and, for $c_{n,b} > 0$,

$$D^n_\eta(c_{n,b}bm) \subset \tilde{D}^n_\eta(bm). \tag{77}$$

In fact, let

$$x \in \Delta_l, \quad |l|_\eta \leq bm, \quad \zeta = l - x, \quad |\zeta|_\eta \leq n^{1/\eta}/2.$$

If $\eta \geq 1$, then

$$|x|_\eta \leq |l|_\eta + |\zeta|_\eta \leq bm + n^{1/\eta}/2 = bmC_{n,b}.$$

If $\eta \in (0,1)$, then

$$|x|^\eta_\eta \leq |l|^\eta_\eta + |\zeta|^\eta_\eta \leq b^\eta m^\eta + n/2^\eta = b^\eta m^\eta C^\eta_{n,b}.$$

These yield (76). Let us verify (77). For $|x|_\eta \leq c_{n,b}bm$, let us take $l = (l_1, ..., l_n) \in \mathbb{Z}^n$ such that $|x_i - l_i| \leq 1/2,\ i = 1, ..., n$, i.e., $x \in \Delta_l$. Setting

$$\zeta = l - x, \quad |\zeta|_\eta \leq n^{1/\eta}/2,$$

analogously to above we have, for $\eta \geq 1$,

$$|l|_\eta \leq |x|_\eta + |\zeta|_\eta \leq bm,$$

and for $\eta \in (0,1)$,

$$|l|^\eta_\eta \leq |x|^\eta_\eta + |\zeta|^\eta_\eta \leq b^\eta m^\eta.$$

These yield (77).

Analogously, if $c_{n,b} > 0$, then the evaluations above yield the inequalities

$$c_{n,b}|l|_\eta \leq |x|_\eta \leq C_{n,b}|l|_\eta, \quad \forall\, l \in \mathbb{Z}^n,\ |l|_\eta \geq bm, \quad x \in \Delta_l. \tag{78}$$

Our aim is to evaluate the sums in (48)–(50)

$$I_k = I_{k,Z} = \sum_{l \in L^n_\eta(m)} f_k(l), \quad k = 0, 1, 2,$$

where

$$\begin{aligned} f_1(x) &= 1 - (|x|_\eta/m)^\eta, \\ f_2(x) &= (|x|_\eta/m)^\eta f_1(x), \\ f_0(x) &= f_1^2(x) = f_1(x) - f_2(x). \end{aligned}$$

Set

$$\tilde f_k(x) = f_k(l), \quad \text{for} \quad x \in \Delta_l, \quad l \in L^n_\eta(m); \quad \tilde f_k(x) = 0, \quad \text{for} \quad x \in \mathbb{R}^n \setminus \tilde D^n_\eta(m).$$

Note that

$$I_{k,Z} = \int_{\mathbb{R}^n} \tilde f_k(x) dx, \quad k = 0, 1, 2.$$

Fix any $b \in (0, 1)$. Introduce the functions

$$\begin{aligned} f_1^+(x) &= (1 - (|x|_\eta/C_{n,b}m)^\eta)_+, \quad f_1^-(x) = (1 - (|x|_\eta/c_{n,b}m)^\eta)_+, \\ f_2^+(x) &= (|x|_\eta/c_{n,b}m)^\eta f_1^+(x), \quad f_2^-(x) = (|x|_\eta/C_{n,b}m)^\eta f_1^-(x), \\ f_0^\pm(x) &= (f_1^\pm(x))^2. \end{aligned}$$

In view of inequalities (73) we have

$$f_k^-(x) \le \tilde f_k(x) \le f_k^+(x), \quad k = 0, 1, 2, \quad \forall x \in \mathbb{R}^n \setminus \tilde D^n_\eta(bm). \tag{79}$$

Consider the integrals

$$I^\pm_{k,R} = \int_{\mathbb{R}^n} f_k^\pm(x) dx, \quad k = 0, 1, 2.$$

By making change of variables $y = x/C_{n,b}$ or $y = x/c_{n,b}$ we get

$$I^+_{k,R} = (C_{n,b}m)^n J_k, \quad I^-_{k,R} = (c_{n,b}m)^n J_k, \quad k = 0, 1 \tag{80}$$

$$I^+_{2,R} = (C_{n,b}/c_{n,b})^\eta (C_{n,b}m)^n J_2, \quad I^-_{2,R} = (c_{n,b}m/C_{n,b})^\eta (c_{n,b}m)^n J_2, \tag{81}$$

where the integrals $J_k,\ k = 0, 1, 2$ are of the form (51)–(53) and these were calculated in (57)–(59). Recall that by Remark 3.1 we have

$$J_k \asymp V_n(\eta)/n, \quad k = 1, 2, \quad J_0 \asymp V_n(\eta)/n^2. \tag{82}$$

Introduce the assumptions:

A1. $\delta_n \to 0$ as $n \to \infty$.

A2. $n\delta_n \to 0$ as $n \to \infty$.

Assume A1. For any $b > 0$, we have, for some $a_n \asymp A_n \asymp 1$,

$$C_{n,b} = 1 + A_n\delta_n \sim 1, \quad c_{n,b} = 1 - a_n\delta_n \sim 1.$$

Taking $b \in (0,1)$, denote

$$\begin{aligned}
I_{k,1,Z} &= \sum_{l\in L^n_\eta(bm)} f_k(l) = \int_{\tilde{D}^n_\eta(bm)} \tilde{f}_k(x)dx, \\
I_{k,2,Z} &= \sum_{l\in L^n_\eta(m)\setminus L^n_\eta(bm)} f_k(l) = \int_{\mathbb{R}^n\setminus\tilde{D}^n_\eta(bm)} \tilde{f}_k(x)dx = I_{k,Z} - I_{k,1,Z}, \\
I^-_{k,1,R} &= \int_{D^n_\eta(C_{n,b}bm)} f^-_k(x)dx, \quad I^-_{k,2,R} = \int_{\mathbb{R}^n\setminus D^n_\eta(C_{n,b}bm)} f^-_k(x)dx, \\
I^+_{k,1,R} &= \int_{D^n_\eta(c_{n,b}bm)} f^+_k(x)dx, \quad I^+_{k,2,R} = \int_{\mathbb{R}^n\setminus D^n_\eta(c_{n,b}bm)} f^+_k(x)dx, \\
I^\pm_{k,2,R} &= I^\pm_{k,R} - I^\pm_{k,1,R}, \quad k = 0,1,2.
\end{aligned}$$

By the embeddings (76), (77) and inequalities (79), we have

$$I^-_{k,2,R} \le I_{k,2,Z} \le I^+_{k,2,R}, \quad k = 0,1,2. \tag{83}$$

For a set $D \subset \mathbb{R}^n$, let $V_n(D)$ be its volume in $\mathbb{R}^n$. Note that, for any $b > 0$,

$$V_n(D^n_\eta(bm)) = (bm)^n V_n(D^n_\eta(1)) = (bm)^n V_n(\eta).$$

It follows from (76),

$$\begin{aligned}
I^\pm_{k,1,R} &\le (C_{n,b}bm)^n V_n(\eta), \quad k = 0,1, \quad I^\pm_{2,1,R} \le b^\eta (C_{n,b}bm)^n V_n(\eta), && (84) \\
I_{k,1,Z} &\le (C_{n,b}bm)^n V_n(\eta), \; k = 0,1,2. && (85)
\end{aligned}$$

By (82)–(85) we have, for some $a_n \asymp A_n \asymp 1$,

$$m^n J_k (1 - a_n\delta_n)^n \le I_{k,Z} \le m^n J_k (1 + A_n\delta_n)^n, \quad k = 0,1,2. \tag{86}$$

We can rewrite (86) in the form

$$I_{k,Z} = m^n J_k \exp(n\zeta_n)), \quad \zeta_n = O(\delta_n). \quad k = 0,1,2. \tag{87}$$

Let us evaluate the ratio

$$R_n = I_{1,Z}/I_{2,Z}.$$

Clearly, $R_n \ge 1$ because of $f_2(l) \le f_1(l)$ for $l \in L^n_\eta(m)$. On the other hand for any $b \in (0,1)$ in view of (85), (87) we have $I_{1,Z} \sim I_{1,2,Z}$. However if $l \in L^n_\eta(m) \setminus L^n_\eta(bm)$, then $f_2(l) \ge b^\eta f_1(l)$, which yields

$$R_n \le \frac{\sum_{l\in L^n_\eta(m)\setminus L^n_\eta(bm)} f_1(l)}{\sum_{l\in L^n_\eta(m)\setminus L^n_\eta(bm)} f_2(l)}(1 + o(1)) \le b^{-\eta}(1 + o(1)).$$

Since we can take $b \in (0,1)$ arbitrarily close to 1, this yields $R_n \le 1 + o(1)$. Thus

$$I_{1,Z} \sim I_{2,Z}. \tag{88}$$

In view of (87), (88), (47) under A1, we can rewrite (45), (46),

$$\begin{aligned} 2\pi m &\sim r_\varepsilon^{-1/\sigma}, \\ u_\varepsilon^2 &\sim \frac{1}{2}(r_\varepsilon/\varepsilon)^4 m^{-n} J_0/J_1^2 \exp(n\tau_n), \quad \tau_n = o(1). \end{aligned} \tag{89}$$

By Remark 3.1 this yields

$$u_\varepsilon^2 = r_\varepsilon^{4+n/\sigma}\varepsilon^{-4}(2\pi)^n (V_n(2\sigma))^{-1} e^{n\tau_n}, \quad \tau_n = o(1). \tag{90}$$

If $\log u_\varepsilon = o(n)$, then, as $n \to \infty$,

$$\begin{aligned} r_\varepsilon &= (u_\varepsilon \varepsilon^2)^{2\sigma/(4\sigma+n)}((2\pi)^{-n} V_n(2\sigma))^{\sigma/(4\sigma+n)} e^{\tau_n n\sigma/(4\sigma+n)} \\ &\sim \varepsilon^{4\sigma/(4\sigma+n)}(\Gamma(1/2\sigma)/2\pi\sigma)^\sigma (2\sigma e/n)^{1/2} = r_\varepsilon^*, \end{aligned}$$

where the rates r_ε^* are defined by (38). For any $C > 1$, one can see that if $r_\varepsilon > C r_\varepsilon^*$, then $u_\varepsilon \to \infty$, and if $r_\varepsilon < r_\varepsilon^*/C$, then $u_\varepsilon \to 0$.

Let us verify the assumption A1. This is of the form

$$n \ll m^{2\sigma} \asymp r_\varepsilon^{-2},$$

and for $r_\varepsilon \asymp r_\varepsilon^*$ this is equivalent to

$$n^{1/2} r_\varepsilon^* \asymp \varepsilon^{4\sigma/(4\sigma+n)} \to 0, \quad \frac{4\sigma \log \varepsilon^{-1}}{4\sigma + n} \to \infty,$$

or

$$n = o(\log \varepsilon^{-1}).$$

This yields the statement of Theorem 7 (2) for the norm (11).

Assume A2. Then (87) yields

$$I_{k,Z} \sim m^n J_k, \quad k = 0, 1, 2. \tag{91}$$

This yields the relations (54), (55), (36), and the asymptotics of u_ε in the theorem. In fact, using (51), we get

$$\begin{aligned} \frac{I_{1,Z}}{I_{2,Z}} &= 1 + \frac{I_{0,Z}}{I_{2,Z}} = 1 + \frac{J_0}{J_2}(1 + o(1)) = 1 + \frac{4\sigma + o(1)}{n}, \\ \frac{I_{0,Z}}{I_{1,Z}^2} &\sim \frac{J_0}{m^n J_1^2} \sim \frac{4\sigma^{n+1}\Gamma(2 + n/2\sigma)}{m^n (4\sigma + n)\Gamma^n(1/2\sigma)}. \end{aligned}$$

In view of (45), (46), (47) this yields

$$m = (2\pi r_\varepsilon^{1/\sigma})^{-1}\left(1+\frac{4\sigma+o(1)}{n}\right)^{1/2\sigma}, \quad m^n \sim (2\pi)^{-n} r_\varepsilon^{-n/\sigma} e^2,$$
$$u_\varepsilon^2 \sim r_\varepsilon^{4+n/\sigma}\varepsilon^{-4}\frac{\pi^n\Gamma(1+n/2\sigma)}{e^2\Gamma^n(1+1/2\sigma)}.$$

Let us verify the assumption A2. It suffices to consider the case $r_\varepsilon \asymp r_\varepsilon^*$. In view of (89) for $\eta \geq 1$, the assumption A2 is of the form

$$n^{-\sigma-1/2} \gg r_\varepsilon^*$$

and it is equivalent to $n = o(\varepsilon^{-4/(4\sigma+n)})$ or

$$(4\log\varepsilon^{-1})/(4\sigma+n) - \log n \to \infty.$$

It suffices

$$\limsup n\log n/\log\varepsilon^{-1} < 4.$$

The last relation holds for

$$n \leq C\log\varepsilon^{-1}/\log\log\varepsilon^{-1}, \quad C < 4. \tag{92}$$

For $\eta \in (0,1)$, the assumption A2 is of the form

$$n^{-1} \gg r_\varepsilon^*$$

and it is equivalent to

$$n = o(\varepsilon^{-8\sigma/(4\sigma+n)})$$

or

$$(8\sigma\log\varepsilon^{-1})/(4\sigma+n) - \log n \to \infty. \tag{93}$$

The relation (93) follows from

$$\limsup n\log n/\log\varepsilon^{-1} < 8\sigma.$$

The last relation holds for

$$n \leq C\log\varepsilon^{-1}/\log\log\varepsilon^{-1}, \quad C < 8\sigma.$$

This yields the statement of Theorem 7 (1) for the norm (11).

4.1.2 Norms (12)

The proof follows to the scheme of Section 4.1.1 and we note the differences only. We set

$$\delta_n = n^{1/2}/2m, \quad C_{n,b} = 1 + \delta_n/b, \quad c_{n,b} = 1 - \delta_n/b, \quad \forall\, \eta > 0.$$

We change the balls $D^n_\eta(m)$ by the Euclidean balls $D^n_2(m)$, and the sets $L^n_\eta(m)$ by $L^n_2(m)$. The embedding (76)–(77) and relations (78) hold true as well. We consider the functions

$$\begin{aligned} f_1(x) &= 1 - |x/m|_2^\eta, \\ f_2(x) &= |x/m|_2^\eta f_1(x), \\ f_0(x) &= f_1^2(x) = f_1(x) - f_2(x), \end{aligned}$$

and functions $\tilde{f}_k,\ f_k^\pm$ of analogous structure such that (79) hold true. Next evaluations are the same. These yield the relation (87) with J_k determined by (63)–(65). The relation (88) holds true as well. These lead to sharp rates (39) under A1, and the sharp asymptotics (31), (37), under A2.

Let us verify the assumptions A1, A2. It suffices assume $r_\varepsilon \asymp r_\varepsilon^*$. In view of (89) and (39), the assumption A2 is of the form

$$n^{3/2} \ll m \asymp r_\varepsilon^{-1/\sigma} \asymp n^{1/2}\varepsilon^{-4/(4\sigma+n)},$$

This is equivalent to $n = o(\varepsilon^{-4/(4\sigma+n)})$ and is fulfilled under (92).

Analogously, the assumption A1 is of the form

$$n^{1/2} \ll m \asymp n^{1/2}\varepsilon^{-4/(4\sigma+n)},$$

and it is equivalent to $n = o(\log \varepsilon^{-1})$. □

4.2 Proof of Theorems 5

The proof is based on the considerations of Section 4.1 under assumption A1. We study the sums $I_k = I_{k,Z}$ defined by (66), (67) and show that

$$I_{1,Z} \sim I_{2,Z}, \quad I_{k,Z} = m^n J_k e^{n\tau_n}, \quad \tau_n = o(1), \quad k = 1, 2,$$

where J_k were calculated in (70), (71). By (68) we have,

$$E^2_{\varepsilon,n} = \varepsilon^2 I_1 = T^{-2} I_1/I_2 \sim T^{-2} = (2\pi m)^{-2\sigma}. \tag{94}$$

On the other hand,

$$\begin{aligned} m^{2\sigma} &= \left(\varepsilon^2 J_2 (2\pi)^{2\sigma} e^{n\tau_n}\right)^{-2\sigma/(2\sigma+n)} = \left(\varepsilon^2 J_2 (2\pi)^{2\sigma}\right)^{-2\sigma/(2\sigma+n)} e^{2\sigma n\tau_n/(2\sigma+n)} \\ &\sim \left(\varepsilon^2 J_2 (2\pi)^{2\sigma}\right)^{-2\sigma/(2\sigma+n)}. \end{aligned} \tag{95}$$

Combining (94), (95), (70), (71), we obtain the asymptotics of Theorem. As above, the assumption A1 corresponds to $n = o(\log \varepsilon^{-1})$. □

Remark 4.1 The quantities $N_n(T)$ (see (17)) correspond to the sums $I_{k,Z}$ for function $f_k(x) = 1$. Therefore as $n \to \infty$, $T^2 = (2\pi m)^\eta \to \infty$, $\eta = 2\sigma$, under assumption A2, the considerations above yield the relation

$$N_n(T) \sim m^n V_n(\eta),$$

which are analogous to (91). Under assumption A1, we obtain somewhat rough relation,

$$\log N_n(T) = \log(m^n V_n(\eta)) + o(n),$$

which are analogous to (87). These extend the asymptotics in Remark 3.1.

5 Proof of Theorems 8, 6

We use a different way for estimations of quantities I_0–I_2 in (42)–(44) and in (66)–(67). This is based on probabilistic machinery.

5.1 Proof of Theorems 8

5.1.1 Norm (11)

We start with the case $\log n = o(\log \varepsilon^{-1})$. Set[2]

$$H = m^{2\sigma} = T^2/(2\pi)^{2\sigma},\ L = [H^{1/2\sigma}]$$

and supposes

$$H \to \infty. \tag{96}$$

Let $X = X(l)$ be integer-valued random variable defined on the set $\Omega = \{l = 0, \pm 1, \dots, \pm L\}$,

$$X(l) = l, \quad P_0(X = l) = 1/(2L+1), \quad l \in \Omega, \tag{97}$$

and

$$Y = Y(X) = |X|^{2\sigma}.$$

Note that

$$E_0(Y) = L^{2\sigma} \frac{2}{2L+1} \sum_{l=1}^{L} (l/L)^{2\sigma} \sim L^{2\sigma} \int_0^1 x^{2\sigma} dx = \frac{L^{2\sigma}}{2\sigma+1} \sim \frac{H}{2\sigma+1}.$$

Let $X, X_1, \dots, X_n$ be i.i.d., $Y_i = Y(X_i)$ and let

$$S_n = \sum_{i=1}^{n} Y_i,$$

[2]Here and below $[t]$ stands for the integer part of $t \in \mathbb{R}$, i.e., $t = [t] + \delta$, $\delta \in [0, 1)$.

be defined on the set Ω^n. For $l \in \Omega^n$ in the relations (42)–(44), we have

$$(c_l/T)^2 = S_n(l)/H,$$

i.e., we consider quantities c_l^2 as realisations of random variable $S_n T^2/H$. The constraint $c_l < T$ corresponds to $S_n < H$. The quantities $I_k,\ k = 0,1,2$ can be presented in the form

$$\begin{aligned} I_1 &= (2L+1)^n E_1, \quad E_1 = E_{0,n}(1 - S_n/H)\mathbb{1}_{S_n<H}, \\ I_2 &= (2L+1)^n E_2, \quad E_2 = E_{0,n}(S_n/H)(1 - S_n/H)\mathbb{1}_{S_n<H}, \\ I_0 &= (2L+1)^n E_0, \quad E_0 = E_{0,n}(1 - S_n/H)^2\mathbb{1}_{S_n<H}, \end{aligned}$$

where $E_{0,n}$ is the expectation with respect to P_0^n-probability.

For the study of expectations $E_k,\ k = 0,1,2$ we use large deviation methods. Let us pass to measure P_h such that

$$dP_h/dP_0 = e^{-hY}/\Psi(h), \quad \Psi(h) = E_0 e^{-hY},$$

i.e.,

$$P_h(Y = |l|^{2\sigma}) = Z_h^{-1}\exp(-h|l|^{2\sigma}), \quad Z_h = (2L+1)\Psi(h).$$

The quantity $h = h_\varepsilon$ is taking such that

$$E_h Y = H/n, \quad \text{i.e.} \quad E_{h,n} S_n = H.$$

This choice is possible because of the function

$$E_h Y = -d\log\Psi(h)/dh$$

decreases in h and takes all values in $(0, E_0(Y))$ for $h \in (0,\infty)$.

We consider the case $H/n \to 0$ because of the case $H/n \to \infty$ corresponds to Theorem 7 (2). Note that if $h \asymp 1$, then

$$\begin{aligned} Z_h &= \sum_{|l|\le L}\exp(-h|l|^{2\sigma}) \asymp 2\int_0^\infty e^{-hx^{2\sigma}}dx = \\ &= \frac{1}{h^{1/2\sigma}\sigma}\int_0^\infty y^{1/2\sigma-1}\exp(-y)dy = \frac{\Gamma(1/2\sigma)}{\sigma h^{1/2\sigma}}, \\ E_h Y &= \frac{1}{Z_h}\sum_{|l|\le L}|l|^{2\sigma}\exp(-h|l|^{2\sigma}) \asymp \frac{2}{Z_h}\int_0^\infty x^{2\sigma}\exp(-x^{2\sigma})dx \\ &= \frac{1}{\sigma h^{1+1/2\sigma}Z_h}\int_0^\infty y^{1/2\sigma}\exp(-y)dy = \frac{\Gamma(1+1/2\sigma)}{h^{1+1/2\sigma}\sigma Z_h} \asymp \frac{1}{2\sigma h} \asymp 1. \end{aligned}$$

Therefore the case $E_h(Y) = H/n \to 0$ corresponds to $h \to \infty$.

Set

$$\sigma_h^2 = \mathrm{Var}_h Y, \quad \sigma_{h,n}^2 = n\sigma_h^2 = \mathrm{Var}_{h,n} S_n, \quad \xi_n = (S_n - H)/\sigma_{h,n},$$

i.e., $E_{h,n}\xi_n = 0$, $\mathrm{Var}_{h,n}\xi_n = 1$. Then the expectations are of the form

$$E_1 = \Psi^n(h)e^{hH}K_1, \quad K_1 = E_{h,n}f_1(\xi_n)e^{h\sigma_{h,n}\xi_n}\mathbb{I}_{\xi_n<0}, \tag{98}$$
$$E_2 = \Psi^n(h)e^{hH}K_2, \quad K_2 = E_{h,n}f_2(\xi_n)e^{h\sigma_{h,n}\xi_n}\mathbb{I}_{\xi_n<0}, \tag{99}$$
$$E_0 = \Psi^n(h)e^{hH}K_0, \quad K_0 = E_{h,n}f_0(\xi_n)e^{h\sigma_{h,n}\xi_n}\mathbb{I}_{\xi_n<0}, \tag{100}$$

where

$$f_1(\xi_n) = -\sigma_{h,n}\xi_n/H, \tag{101}$$
$$f_2(\xi_n) = (1+\sigma_{h,n}\xi_n/H)f_1(\xi_n), \tag{102}$$
$$f_0(\xi_n) = f_1^2(\xi_n) = f_1(\xi_n) - f_2(\xi_n). \tag{103}$$

We rewrite (45), (46) in the form

$$H \sim (2\pi)^{-2\sigma}r_\varepsilon^{-2}K_1/K_2, \tag{104}$$
$$u_\varepsilon^2 = \frac{1}{2}(r_\varepsilon/\varepsilon)^4 Z_h^{-n}e^{-hH}K_0/K_2^2. \tag{105}$$

Thus, we need to evaluate quantities h, Z_h and K_k, $k = 0,1,2$, as $h \to \infty$. Setting

$$p = 2e^{-h} \to 0, \quad \eta = 2\sigma,$$

we have

$$Z_h = \sum_{|l|\le L} \exp(-h|l|^\eta) = 1 + 2e^{-h} + \exp(-2^\eta h)f(h),$$

where

$$f(h) = \sum_{2\le|l|\le L} \exp(-h(|l|^\eta - 2^\eta)).$$

Clearly, $f(h)$ decreases in h and for $h > 1$,

$$f(h) \le f(1) \le \exp(-2^\eta)Z(1) \asymp 1.$$

This yields

$$Z_h = 1 + 2e^{-h} + O(\exp(-2^\eta h)) \sim 1 + p. \tag{106}$$

Analogously,

$$E_h(Y) = \frac{1}{Z_h}\sum_{|l|\le L} |l|^\eta \exp(-h|l|^\eta) = \frac{2e^{-h} + O(\exp(-2^\eta h))}{1 + 2e^{-h} + O(\exp(-2^\eta h))} \sim p. \tag{107}$$

The relations (106), (107) yield

$$p \sim H/n, \quad h \sim \log p^{-1}.$$

Analogously to (106), (107), we have

$$E_hY^2 = \frac{1}{Z_h}\sum_{|l|\le L} |l|^{2\eta} \exp(-h|l|^\eta) = \frac{2e^{-h} + O(\exp(-2^\eta h))}{1 + 2e^{-h} + O(\exp(-2^\eta h))} \sim p,$$

which yields $\sigma_h^2 \sim p$,

$$\sigma_{h,n}^2 \sim np \sim H \to \infty, \quad \sigma_{h,n}/H \sim H^{-1/2} \to 0, \quad h\sigma_{h,n} \sim H^{1/2} \log p^{-1}, \tag{108}$$

$$e^{hH} = (2n/H)^H e^{o(H)}, \quad Z_h^n = \exp(H(1+o(1))). \tag{109}$$

Analogous calculations yield

$$E_h Y^3 \sim p.$$

Therefore

$$E_h|Y - E_h Y|^3 \leq Bp,$$

and the Lyapunov ratio is controlled by

$$\frac{E_h|Y - E_h Y|^3}{n^{1/2}\sigma_h^3} \leq BH^{-1/2} \to 0.$$

This yields the Central Limit Theorem for S_n under $P_{h,n}$-probability,

$$\sup_{t\in\mathbb{R}} |P_{h,n}(\xi_n < t) - \Phi(t)| \to 0, \quad \text{as } n \to \infty.$$

Let us evaluate the quantities $K_k,\ k = 0, 1, 2$. Clearly,

$$K_k \leq 1.$$

On the other hand, for any $\delta \in (0, 1]$, let $\tilde{K}_k$ be analogous quantities with the change $\mathbb{I}_{\xi_n<0}$ by $\mathbb{I}_{\xi_n\in[-2\delta,-\delta]}$ and note that by the Central Limit Theorem,

$$P_h(\xi_n \in [-2\delta, -\delta]) \geq B\delta.$$

We have

$$\begin{aligned} K_k &\geq \tilde{K}_k \geq B\delta^2 H^{-1} \exp(-2h\sigma_{n,h}\delta) P_h(\xi_n \in [-2\delta, -\delta]) \\ &\geq BH^{-1}\delta^3 \exp(-\sqrt{H} \log p^{-1}(2\delta + o(1))). \end{aligned}$$

This yields

$$K_k = (H/n)^{\tau_{k,n}\sqrt{H}}, \quad \tau_{k,n} \to 0, \quad k = 0, 1, 2. \tag{110}$$

Let us evaluate the ratio K_1/K_2. Clearly,

$$K_1/K_2 \geq 1.$$

On the other hand, for any $C > 0$ let $K_k^{(1)}(C)$ and $K_k^{(2)}(C)$ be the quantities analogous to K_k with the change $\mathbb{I}_{\xi_n<0}$ by $\mathbb{I}_{\xi_n<-C}$ and by $\mathbb{I}_{\xi_n\in[-C,0)}$ respectively,

$$K_k^{(1)}(C) + K_k^{(2)}(C) = K_k.$$

We have

$$K_2^{(1)}(C) \leq \exp(-Ch\sigma_{n,h}) < \exp(-C\sqrt{H}\log p^{-1}(1 + o(1))) < (H/n)^{C\sqrt{H}/2}.$$

Taking into account (110) and since

$$1 \geq 1 + \sigma_{h,n}\xi_n/H \geq 1 + o(1), \quad \text{for} \quad \xi_n \in (-C, 0],$$

we get

$$K_2^{(1)}(C) = o(K_2), \quad K_1/K_2 \leq K_1^{(2)}(C)/K_2^{(2)}(C)(1 + o(1)) \leq 1 + o(1),$$

which yields

$$K_1/K_2 \sim 1.$$

The evaluations above yield

$$I_1 \sim I_2, \quad I_k = Z_h^n e^{hH} K_k = (2en/H)^{H+o(H)}. \tag{111}$$

Using (111) we can rewrite (104), (105) in the form

$$r_\varepsilon \sim (2\pi)^{-\sigma} H^{-1/2}, \tag{112}$$
$$u_\varepsilon^2 = \varepsilon^{-4} H^{-2} (H/2en)^{H(1+o(1))} = \varepsilon^{-4} (H/n)^{H(1+o(1))}. \tag{113}$$

Let

$$\log u_\varepsilon = o(H). \tag{114}$$

The relation (113), (114) implies

$$4 \log \varepsilon^{-1} \sim H(\log n - \log H). \tag{115}$$

Setting

$$D_\varepsilon = n/4 \log \varepsilon^{-1} \to \infty, \quad Y_\varepsilon = n/H, \tag{116}$$

we rewrite (115)

$$Y_\varepsilon / \log Y_\varepsilon \sim D_\varepsilon \tag{117}$$

The equation (117) yields the asymptotics

$$Y_\varepsilon \sim D_\varepsilon \log D_\varepsilon,$$

which yields

$$H \sim \frac{4 \log \varepsilon^{-1}}{\log D_\varepsilon} \sim \frac{4 \log \varepsilon^{-1}}{\log n - \log \log \varepsilon^{-1}}. \tag{118}$$

Note that $Y_\varepsilon \to \infty$, which yields $h \sim \log Y_\varepsilon \to \infty$, which corresponds to the original assumption.

Taking into account the relation (112) and (118), we obtain the rates (40)

$$r_\varepsilon \sim \frac{1}{2(2\pi)^\sigma} \sqrt{\frac{\log n - \log \log \varepsilon^{-1}}{\log \varepsilon^{-1}}} = r_\varepsilon^*.$$

The assumption $\log n = o(\log \varepsilon^{-1})$ corresponds to $r_\varepsilon = o(1)$. In view of (25) this yields (96).

Using (112), (113) one can easily verify for any $C > 1$, that, if $r_\varepsilon > Cr_\varepsilon^*$, then $u_\varepsilon \to \infty$, and if $r_\varepsilon < r_\varepsilon^*/C$, then $u_\varepsilon \to 0$, i.e., these are the sharp separation rates. The statement (1) follows. □

Let us go to the statement (2). Let

$$\liminf \log n / \log \varepsilon^{-1} > b > 0. \tag{119}$$

By Theorem 2 (see (20)) it suffices to show that the value $u_{\varepsilon,n}^2$ of the extreme problem (19) tends to 0. First note the inequality:

$$u_{\varepsilon,n}^2 \le (2T^4\varepsilon^4 N_n(T))^{-1}, \quad T = r_\varepsilon^{-1}. \tag{120}$$

the quantities $N_n(T)$ are defined by (17). In fact, consider the sequence

$$v_l = u\mathbb{I}_{\{c_l \le T\}}, \quad l \in \mathbb{Z}^n, \quad u^2 = r_\varepsilon^2/\varepsilon^2 N_n(T).$$

The constraints of the extreme problem (19) are fulfilled and

$$u_{\varepsilon,n}^2 \le \frac{1}{2}\sum_{l\in\mathbb{Z}^n} v_l^4 = (2T^4\varepsilon^4 N_n(T))^{-1}.$$

Let $T^2 = (2\pi)^{2\sigma}H,\ H \in \mathbb{N},\ H < n$. Then

$$N_n(T) \ge 2^H C_n^H. \tag{121}$$

In fact, it suffice to consider the the set $\mathcal{C}_{n,H}$ that consists of collections

$$l = (l_1, \dots, l_n),\ l_i \in \{0, \pm 1\}, \quad \sum_{i=1}^n |l_i| = H.$$

If $l \in \mathcal{C}_{n,H}$, then $c_l^2 = \sum_{i=1}^n (2\pi|l_i|)^{2\sigma} = T^2$ and $\#(\mathcal{C}_{n,H}) = 2^H C_n^H$.
Under assumption (119) take $H \in \mathbb{N},\ H > 4/b$. Then

$$n^H\varepsilon^4 \to \infty, \quad 2^H C_n^H \sim (2n)^H/H! \tag{122}$$

It follows from (120), (121), (122) that if $r_\varepsilon \le r_0 = T^{-1} = (2\pi)^{-\sigma}H^{-1/2}$, then $u_{\varepsilon,n}^2 \to 0$. □

5.1.2 Norm (12)

First, let $\log n = o(\log \varepsilon^{-1})$. Set

$$H = m^2 = T^{2/\sigma}/(2\pi)^2 \to \infty, \quad L = [H^{1/2}].$$

Assuming $H \to \infty$, we consider the random variable $Y = X^2$ and the sum $S_n = \sum_{i=1}^n Y_i$. We have, in the relations (42)–(44),

$$(c_l/T)^2 = (S_n(l)/H)^\sigma.$$

Next considerations repeat ones for the norm (11) with the change of the functions (101)–(103) by

$$\begin{aligned} f_1(\xi_n) &= 1-(1+\sigma_{h,n}\xi_n/H)^\sigma, && (123)\\ f_2(\xi_n) &= (1+\sigma_{h,n}\xi_n/H)^\sigma f_1(\xi_n), && (124)\\ f_0(\xi_n) &= f_1^2(\xi_n). && (125)\end{aligned}$$

This change is not essential for the results. In view of (45), the relation (104) is changed by

$$(2\pi)^{2\sigma}H^\sigma \sim r_\varepsilon^{-2}K_1/K_2.$$

Next, we obtain the relation (113), whenever (112) is replaced by

$$r_\varepsilon \sim ((2\pi)^2H)^{-\sigma/2}.$$

This leads to the sharp separation rates

$$r_\varepsilon^* = \left(\frac{\log n - \log\log\varepsilon^{-1}}{16\pi^2\log\varepsilon^{-1}}\right)^{\sigma/2}.$$

The statement (1) follows. □

Next, assume (119). Analogously to (121) we show that if

$$T^2 = (2\pi)^{2\sigma}H^\sigma, \quad H\in\mathbb{N}, \quad H<n,$$

then

$$N_n(T) \ge 2^H C_n^H.$$

Taking $H\in\mathbb{N}$, $H>4/b$ and using (120) we see that if $r_\varepsilon < r_0 = T^{-1}$, then $u_{\varepsilon,n}\to 0$. By (20) the statement (2) follows. □

5.2 Proof of Theorem 6

The proof of Theorem 6 (1) is analogous to the proof of Theorem 8 (1). The study of the sums (66), (67) follows to Section 5.1. As above, for the norm (11) we set

$$T = H^{1/2}(2\pi)^\sigma, \tag{126}$$

whenever the norm (12),

$$T = H^{\sigma/2}(2\pi)^\sigma. \tag{127}$$

We repeat the considerations above with the change σ by 1/2 for the norm (11), and by $\sigma/2$ for the norm (12), in (123)–(125). These yield the relations (111). Applying these to (68) we get

$$H(1+o(1))\log(2en/H) + 2\log T = 2\log\varepsilon^{-1}, \quad E_{\varepsilon,n}^2 \sim T^{-2}, \tag{128}$$

By (116), (126), (127), the first equation (128) yields (compare with (118))

$$H \sim \frac{2\log\varepsilon^{-1}}{\log n - \log\log\varepsilon^{-1}}.$$

By the second equation (128) for the norm (11), we have (34), and for the norm (12), we have (35).

Theorem 6 (2) follows from Theorem 8 (2). In fact, if there exist a family of estimators $\hat{f}_\varepsilon$ such that $d_\varepsilon^2 = R^2_{\varepsilon,n}(\hat{f}_\varepsilon, \mathcal{F}_n) \to 0$, then by taken plug-in tests

$$\psi_\varepsilon = \mathbb{1}_{\{\|f-\hat{f}_\varepsilon\|_2 > r_\varepsilon/2\}},$$

we have $\gamma_\varepsilon(\psi_\varepsilon, \mathcal{F}_{\varepsilon,n}) \to 0$, for any family $r_\varepsilon/d_\varepsilon \to \infty$ (see [8], Proposition 2.17). In particular, we can take $r_\varepsilon = d_\varepsilon^{1/2} \to 0$. This contradicts to Theorem 8 (2). $\square$

Remark 5.1 The quantities $N_n(T)$ (see (17)) are of the form

$$N_n(T) = (2L+1)^n P_0^n(S_n < T)$$

and can be studied by the same way. Analogously to (98)–(100) we have

$$P_0^n(S_n < T) = \Phi^n(h)e^{hH}K,$$

where the expectations K is analogous to K_k for function $f(\xi_n) = 1$. Let $n \to \infty$, $T^2 = (2\pi m)^\eta \to \infty$, and for the norm (11), $T^2 = H(2\pi)^\eta = o(n)$, whenever for the norm (12), $T^{2/\sigma} = (2\pi)^2 H = o(n)$. The considerations above yield the relation

$$\log N_n(T) \sim H\log(2en/H),$$

which are analogous to the second relation (111). This yields the asymptotics of differ type than the asymptotics in Remark 4.1.

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