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A reduction approximation method for curved rods

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Abstract

We study the numerical approximation of a general linear model for three-dimensional clamped curved rods. We introduce a modified system and we show that the convergence of the numerical discretization is independent of the small parameters entering the coefficients of the differential equations.

1 Introduction

It is well known that standard finite element methods may become unstable when differential equations involving very small coefficients are solved. A recent thorough study of a simple parameter-dependent elliptic model problem is due to Havu and Pitkäranta [7]. Such difficulties commonly arise in numerical approaches to thin curved mechanical structures like arches, curved rods or shells (cf. Chenais and Paumier [4], Chenais and Zerner [5], Chapelle [3], Havu and Pitkäranta [8, 9]). They are know under the general term "locking phenomenon", and they are due to a parametric error amplification. Especially, if the discretization parameters are of the same order as the small parameter in the equation, then the obtained numerical results may be meaningless, deviating very much from the true solution of the problem. An example for this behavior may be found in Chenais and Paumier [4].

In this work, we aim to show that a careful modification of the bilinear functional governing the variational problem ensures the stability of the numerical scheme, even in the presence of very small parameters. This rather general idea has also been used by other authors (compare Chapelle [3], Havu and Pitkäranta [7]), but its successful realization strongly depends on the characteristics of the given problem. We also stress the fact that our approach can be adapted to many other applications (see Remark 3.3).

The problem under study is a linear model for three-dimensional clamped elastic curved rods that has been introduced in Ignat, Sprekels and Tiba [10]. It extends similar models of Reddy and Arunakirinathar [11] and Chapelle [3], in the sense that we also admit a deformation of the cross section of the rod. Our model involves nine unknown functions, while in the literature six unknowns are generally considered. Moreover, our smoothness assumptions on the parametrization of the geometry of the curved rod ($W^{2,\infty}$ instead of C^3) are lower than those in the available literature.

In Section 2, we introduce the "right" modification of the elliptic bilinear form in several steps. We also prove that both the original and the modified equations lead asymptotically to the same solution as the small parameter converges to zero.

In Section 3, it is proved that the convergence properties of the discretized solutions are independent of the small parameter appearing in the original problem. To obtain this, it will be essential to use the modified bilinear form. We note that a similar result was established by Chenais and Paumier [4] for the case of two-dimensional arches having constant curvatures. We underline that our approach makes it possible to use the simplest piecewise linear and continuous finite elements for the numerical solution, while in the literature higher order elements have to be used, in general.

The last part of the paper is devoted to some numerical experiments. We also provide a comparison with the standard technique which proves the importance of finding alternative methods for this type of stiff differential equations.

2 The model and its approximation

We start with an abstract scheme that puts into evidence our basic ideas in the approximation of the curved rods model (which is introduced later). We denote by $d > 0$ a "small" parameter, by V a Hilbert space, and by $A_d : V \times V \to \mathbb{R}$, $a_d : V \times V \to \mathbb{R}$, two bilinear bounded functionals depending on $d > 0$. In the subsequent analysis of the curved rods model, A_d will be the original bilinear functional according to Ignat, Sprekels and Tiba [10], while a_d is its first modification. A subsequent modification of a_d, denoted α_d, will be constructed as well.

We generally assume that

$$A_d(v,v) \geq C d^2 |v|_V^2, \quad \forall\, v \in V, \tag{2.1}$$

$$|A_d(v,w) - a_d(v,w)| \leq K\, d^3\, |v|_V\, |w|_V\,, \quad \forall\, v, w \in V, \tag{2.2}$$

where C, K are some positive constants, independent of $d > 0$. From (2.1), (2.2) it immediately follows that there is some $c > 0$ such that for any sufficiently small $d > 0$ it holds

$$a_d(v,v) \geq c\, d^2\, |v|_V^2\,, \quad \forall\, v \in V. \tag{2.3}$$

We compare the unique solutions $X_d \in V$, $x_d \in V$, of the variational equations

$$A_d(X_d, w) = (f_d, w)_{V \times V^*}\,, \quad \forall\, w \in V\,, \tag{2.4}$$

$$a_d(x_d, w) = (f_d, w)_{V \times V^*}\,, \quad \forall\, w \in V, \tag{2.5}$$

which exist thanks to the Lax–Milgram lemma. Here, $f_d \in V^*$ is given and $(\cdot,\cdot)_{V \times V^*}$ is the pairing in $V \times V^*$.

Proposition 2.1 *Assume that* $|f_d|_{V^*} \leq c\, d^2$. *Then there are some* $d_0 > 0$ *and some* $M > 0$, *independent of* $d \in]0, d_0[$, *such that*

$$|X_d - x_d|_V \leq M d\,. \tag{2.6}$$

Proof. We have:

$$0 = A_d(X_d, w) - a_d(x_d, w) = A_d(X_d - x_d, w) + A_d(x_d, w) - a_d(x_d, w)\,.$$

From (2.2), we infer that

$$|A_d(X_d - x_d, w)| = |A_d(x_d, w) - a_d(x_d, w)| \le K\, d^3\, |x_d|_V\, |w|_V\,, \quad \forall\, w \in V\,.$$

For $w = X_d - x_d$, we obtain from (2.1) that

$$|X_d - x_d|_V \le \frac{K}{C}\, |x_d|_V\, d\,.$$

The hypothesis on f_d, and (2.2), (2.5), show that $\{x_d\}$ is bounded in V. Together with the above inequality, we obtain (2.6), which finishes the proof of the assertion. □

Remark 2.1 The assumption on the order of f_d is justified by the linearity of (2.4), (2.5) and by the subsequent applications. Proposition 2.1 shows that it suffices to solve (2.5) instead of (2.4), provided that d is sufficiently small.

In the applications to clamped curved rods $d > 0$ is a measure for the area of the cross section of the rod, and it is well known that for "small" d the locking problem appears.

The bilinear form $A_d : V \times V \to \mathbb{R}$, $V = H_0^1(0, L)^9$ ($L > 0$ being the length of the rod) was introduced in Ignat, Sprekels and Tiba [10] as follows:

$$\begin{aligned}
A_d(\bar{u}, \bar{v}) = {} & \lambda \int_\Omega \sum_{i,j=1}^{3} \Big[N_i(x_3) h_{1i}(\bar{x}) + B_i(x_3) h_{2i}(\bar{x}) \\
& + \Big(\tau_i^{'}(x_3) + x_1 N_i^{'}(x_3) + x_2 B_i^{'}(x_3)\Big) h_{3i}(\bar{x})\Big] \\
& \times \Big[M_j(x_3) h_{1j}(\bar{x}) + D_j(x_3) h_{2j}(\bar{x}) + \Big(\mu_j^{'}(x_3) + x_1 M_j^{'}(x_3) + x_2 D_j^{'}(x_3)\Big) h_{3j}(\bar{x})\Big] \\
& \times \Big| \det J(\bar{x}) \Big| d\bar{x} + \mu \int_\Omega \sum_{i<j} \Big[N_i(x_3) h_{1j}(\bar{x}) + B_i(x_3) h_{2j}(\bar{x}) + \Big(\tau_i^{'}(x_3) + x_1 N_i^{'}(x_3) \\
& + x_2 B_i^{'}(x_3)\Big) h_{3j}(\bar{x}) + N_j(x_3) h_{1i}(\bar{x}) + B_j(x_3) h_{2i}(\bar{x}) + \Big(\tau_j^{'}(x_3) + x_1 N_j^{'}(x_3) \\
& + x_2 B_j^{'}(x_3)\Big) h_{3i}(\bar{x})\Big] \Big[M_i(x_3) h_{1j}(\bar{x}) + D_i(x_3) h_{2j}(\bar{x}) + \Big(\mu_i^{'}(x_3) + x_1 M_i^{'}(x_3) \\
& + x_2 D_i^{'}(x_3)\Big) h_{3j}(\bar{x}) + M_j(x_3) h_{1i}(\bar{x}) + D_j(x_3) h_{2i}(\bar{x}) + \Big(\mu_j^{'}(x_3) + x_1 M_j^{'}(x_3) \\
& + x_2 D_j^{'}(x_3)\Big) h_{3i}(\bar{x})\Big] \Big| \det J(\bar{x}) \Big| \, d\bar{x} \\
& + 2\mu \int_\Omega \sum_{i=1}^{3} \Big[N_i(x_3) h_{1i}(\bar{x}) + B_i(x_3) h_{2i}(\bar{x}) \\
& + \Big(\tau_i^{'}(x_3) + x_1 N_i^{'}(x_3) + x_2 B_i^{'}(x_3)\Big) h_{3i}(\bar{x})\Big] \Big[M_i(x_3) h_{1i}(\bar{x}) + D_i(x_3) h_{2i}(\bar{x}) \\
& + \Big(\mu_i^{'}(x_3) + x_1 M_i^{'}(x_3) + x_2 D_i^{'}(x_3)\Big) h_{3i}(\bar{x})\Big] \Big| \det J(\bar{x}) \Big| \, d\bar{x}\,.
\end{aligned} \tag{2.7}$$

Here, $\Omega = \omega \times]0, L[$ with $\omega \subset \mathbb{R}^2$, $\text{meas}(\omega) = d$, being the area of the cross section of the rod, $\lambda \ge 0$ and $\mu > 0$ are the Lamé constants of the material,

$(\tau_i, N_i, B_i)_{i=\overline{1,3}} \in H_0^1(0,L)^9$ are the unknowns, and $(\mu_i, M_i, D_i)_{i=\overline{1,3}} \in H_0^1(0,L)^9$ are arbitrary test functions.

The coefficients $(h_{ij})_{i,j=\overline{1,3}}$ depend on $d > 0$ and are obtained from the geometry of the curved rod as explained below.

We denote by $\bar\theta \in W^{2,\infty}(0,L)^3$ the parametrization of the line of centroids of the curved rod (which is assumed to be a unit speed curve) and by $\bar t, \bar n, \bar b \in W^{1,\infty}(0,L)^3$ the corresponding local frame. It differs, in general, from the classical Frenet or Darboux frames (cf. Cartan [2]), since our (regularity) assumptions are very weak. A new specific construction under such conditions is reported in [10]. The curved rod $\tilde\Omega$ is given as the image of the cylinder Ω under a transformation $F : \Omega \to \tilde\Omega$,

$$\tilde\Omega = F(\Omega)\,, \tag{2.8}$$

$$\begin{aligned}(x_1, x_2, x_3) &= \bar x \in \Omega \mapsto F\bar x = \tilde x = (\tilde x_1, \tilde x_2, \tilde x_3)\\ &= \bar\theta(x_3) + x_1 \bar n(x_3) + x_2 \bar b(x_3), \quad \forall\, \bar x \in \Omega\,.\end{aligned} \tag{2.9}$$

Then $J(\bar x) = DF(\bar x)$ is the Jacobian of F, and the relation $(h_{ij}(\bar x))_{i,j=\overline{1,3}} = J(\bar x)^{-1}$ yields the coefficients in (2.7). More precisely, it holds

$$J(\bar x)^{-1} = \begin{bmatrix} n_i - \dfrac{c\, t_i\, x_2}{1 - \beta\, x_1 - a\, x_2} \\ b_i + \dfrac{c\, t_i\, x_1}{1 - \beta\, x_1 - a\, x_2} \\ \dfrac{t_i}{1 - \beta\, x_1 - a\, x_2} \end{bmatrix}_{i=\overline{1,3}}. \tag{2.10}$$

A thorough construction of A_d and the proof of (2.1), starting from the linear elasticity system, was performed in Ignat, Sprekels and Tiba [10]. Moreover,

$$\det J(\bar x) = 1 - \beta(x_3) x_1 - a(x_3) x_2, \quad \forall\, \bar x \in \Omega\,. \tag{2.11}$$

The presence of $(x_1, x_2) \in \omega$ in (2.10), (2.11) shows the dependence of (2.9) on $d > 0$.

Above, β, c and $a \in L^\infty(0,L)$ are "curvatures" of the line of centroids (recall that we are not using the classical Frenet frame) that may be obtained via the "equations of motion" of the local frame:

$$\begin{aligned}\bar t^{\,\prime}(x_3) &= a(x_3)\, \bar b(x_3) \;+\; \beta(x_3)\, \bar n(x_3)\,,\\ \bar b^{\,\prime}(x_3) &= -a(x_3)\, \bar t(x_3) \;+\; c(x_3)\, \bar n(x_3)\,,\\ \bar n^{\,\prime}(x_3) &= -\beta(x_3)\, \bar t(x_3) \;-\; c(x_3)\, \bar b(x_3)\,.\end{aligned} \tag{2.12}$$

Relations (2.12) are a simple consequence of

$$|\bar t(x_3)|^2_{\mathbb{R}^3} = |\bar n(x_3)|^2_{\mathbb{R}^3} = |\bar b(x_3)|^2_{\mathbb{R}^3} = 1\,.$$

We continue now with the construction of the modified bilinear form $a_d(\cdot,\cdot)$, and we also prove the condition (2.2). We notice that A_d in (2.7) consists of three different sums (or terms) which we denote by S_1, S_2, S_3 (in the order they appear in (2.7)). The functional a_d is obtained by modifying each of the terms S_1, S_2, S_3, and can be put in the form

$$a_d(u,v) = d a_0(u,v) + d^2 a_1(u,v) + d^3 a_2(u,v), \quad \forall\, u,v \in V. \tag{2.13}$$

By inspecting (2.10) and (2.7), we see that it is possible to approximate just $[\det J(\bar{x})]^{-1}$ by

$$\frac{1}{1-\beta x_1 - a x_2} \cong 1 + \beta x_1 + a x_2 + (\beta x_1 + a x_2)^2\,, \tag{2.14}$$

in order to get only polynomial coefficients in x_1, x_2. This will be used in Section 4, in the applications, to perform all integrations over the cross section ω exactly. Moreover, the terms in S_1, S_2, S_3 which already have a "simple" structure will not be modified according to (2.14) and will be preserved as they are. The approximation properties for a_d are improved in this way. It readily follows from (2.14) that (2.2) is fulfilled.

A lengthy, but elementary, calculation based on the above ideas leads to the following approximations (we write just the part corresponding to S_3 which is the shortest one):

$$\begin{aligned}
[S_3]_a \;=\; & 2\mu \int_0^L \sum_{i=1}^{3} \Big\{ N_i M_i (n_i^2\, C_{00} \,+\, c^2\, t_i^2\, C_{02} \,+\, c^2\, t_i^2\, \beta^2\, C_{22} \,+\, c^2\, t_i^2\, a^2\, C_{04}) \\
& + (N_i\, D_i \,+\, B_i\, M_i)(n_i\, b_i\, C_{00} \,-\, 2c^2\, t_i^2 a\, \beta\, C_{22}) \,+\, B_i\, D_i (b_i^2\, C_{00} \,+\, c^2\, t_i^2\, C_{20} \\
& + c^2\, t_i^2\, \beta^2\, C_{40} \,+\, c^2\, t_i^2\, a^2\, C_{22}) \,+\, (M_i\, \tau_i' \,+\, N_i\, \mu_i')(n_i\, t_i\, C_{00} \,-\, c\, t_i^2\, a\, C_{02}) \\
& + (B_i\, \mu_i' \,+\, D_i\, \tau_i')(b_i\, t_i\, C_{00} \,+\, c\, t_i^2\, \beta\, C_{20}) \,-\, 2(N_i\, M_i' \,+\, N_i'\, M_i) c\, t_i^2\, a\, \beta\, C_{22} \\
& + (B_i\, M_i' \,+\, N_i'\, D_i) c\, t_i^2 (C_{20} \,+\, \beta^2\, C_{40} \,+\, a^2\, C_{22}) \,-\, (N_i\, D_i' \,+\, M_i\, B_i') c\, t_i^2 \\
& \times (C_{02} \,+\, \beta^2\, C_{22} \,+\, a^2\, C_{04}) \,+\, 2\,(B_i\, D_i' \,+\, B_i'\, D_i) c\, t_i^2\, a\, \beta\, C_{22} \,+\, \tau_i'\, \mu_i'\, t_i^2 (C_{00} \\
& + \beta^2\, C_{20} \,+\, a^2\, C_{02}) \,+\, \tau_i'\, \mu_i'\, t_i^2 (C_{00} \,+\, \beta^2\, C_{20} \,+\, a^2\, C_{02}) \,+\, (\tau_i'\, M_i' \,+\, \mu_i'\, N_i') \\
& \times t_i^2\, \beta\, C_{20} \,+\, (\tau_i'\, D_i' \,+\, \mu_i'\, B_i')\, t_i^2\, a\, C_{02} \,+\, 2\,(N_i'\, D_i' \,+\, B_i'\, M_i')\, t_i^2\, a\, \beta\, C_{22} \\
& + N_i'\, M_i'\, t_i^2 (C_{20} \,+\, \beta^2\, C_{40} \,+\, a^2\, C_{22}) \,+\, B_i'\, D_i'\, t_i^2 (C_{02} \,+\, \beta^2\, C_{22} \\
& + a^2\, C_{04}) \Big\}\, dx_3\,.
\end{aligned} \tag{2.15}$$

Here, $C_{ij} = \int_\omega x_1^i\, x_2^j\, dx_1\, dx_2$ and $C_{00} = \int_\omega dx_1\, dx_2 = d$, in particular. That is, the integration over ω is already performed in (2.15). The forms of $[S_1]_a$ and of $[S_2]_a$ are much more complicated than (2.15).

Notice that C_{00} is of order d, C_{02} and C_{20} are of order d^2, and C_{22}, C_{04}, C_{40} are of order d^3, due to their definition. Then, we may introduce the bilinear forms $a_0(\cdot,\cdot), a_1(\cdot,\cdot), a_2(\cdot,\cdot)$ by "collecting" from $[S_1]_a, [S_2]_a$, and $[S_3]_a$, respectively, the terms containing C_{00} or C_{02}, C_{20} or C_{22}, C_{04}, C_{40}. In relation (2.16) below, we show just the form of $a_0(u,u)$ which is the shortest one. Recalling that $u = (\tau_1, \tau_2, \tau_3, N_1, N_2, N_3, B_1, B_2, B_3) \in H_0^1(0,L)^9$, we obtain that

$$\begin{aligned} a_0(u,u) &= \lambda \int_0^L \left[\sum_{i=1}^3 (N_i\, n_i \,+\, B_i\, b_i \,+\, \tau_i^{'}\, t_i) \right]^2 dx_3 \\ &\quad + \mu \int_0^L \sum_{i<j} \left(N_i\, n_j + N_j\, n_i + B_i\, b_j + B_j\, b_i + \tau_i^{'}\, t_j + \tau_j^{'}\, t_i \right)^2 dx_3 \qquad (2.16) \\ &\quad + 2\,\mu \int_0^L \sum_{i=1}^3 \left(N_i\, n_i \,+\, B_i\, b_i \,+\, \tau_i^{'}\, t_i \right)^2 dx_3\,. \end{aligned}$$

In view of Proposition 2.1, this provides a stable (with respect to d) approximation to the equation (2.4), as well. Using (2.13), and invoking the hypothesis of Proposition 2.1, we can rewrite (2.5) in the form

$$a^d(x_d, v) = (\ell, v)_{V \times V^*}\,, \qquad (2.17)$$

where $\ell = d^{-2} f_d \in V^*$ and $a^d(\cdot,\cdot) = d^{-2} a_d(\cdot,\cdot)$. By (2.3), we have $a^d(v,v) \geq c\,|v|_V^2$, for every $v \in V$. Moreover, (2.16) shows that a_0 is positive. It is not strictly positive. Indeed, introducing the set

$$G = \{ w \in V\,;\, a_0(w,v) \,=\, 0\,, \quad \forall\, v \in V \}\,, \qquad (2.18)$$

we have:

Lemma 2.2 $G \neq \{0\}$.

Proof. Clearly, (2.18) may be rewritten as $G = \{w \in V; a_0(w,w) = 0\}$. Then $u = (\tau_1, \tau_2, \tau_3, N_1, N_2, N_3, B_1, B_2, B_3) \in G$ if and only if

$$t_i \tau_i^{'} + b_i B_i + n_i N_i = 0\,, \qquad (2.19)$$

$$t_j \tau_i^{'} + t_i \tau_j^{'} + b_i B_j + b_j B_i + n_i N_j + n_j N_i = 0\,, \qquad (2.20)$$

for $i,j = \overline{1,3}$. The contribution of the first term in (2.16) is already covered via the last one in (2.19). We consider (2.19), (2.20) as a linear algebraic system with principal unknowns $\tau_1', \tau_2', \tau_3', N_1, N_2, B_1$ and secondary unknowns N_3, B_2, B_3. Its determinant is

$$\begin{vmatrix} t_1 & 0 & 0 & n_1 & 0 & b_1 \\ 0 & t_2 & 0 & 0 & n_2 & 0 \\ 0 & 0 & t_3 & 0 & 0 & 0 \\ t_2 & t_1 & 0 & n_2 & n_1 & b_2 \\ t_3 & 0 & t_1 & n_3 & 0 & b_3 \\ 0 & t_3 & t_2 & 0 & n_3 & 0 \end{vmatrix} = (-t_2 n_3 + n_2 t_3) t_3\,. \qquad (2.21)$$

It may be supposed nonzero (otherwise another determinant may be chosen—this point is explained in Section 4 as well). Therefore, the principal unknowns may be expressed as linear combinations of the secondary unknowns, with coefficients depending on the coefficients in (2.19), (2.20). If $N_3, B_2, B_3 \in H_0^1(0,L)$ are arbitrarily chosen, then N_1, N_2, B_1 remain in $H_0^1(0,L)$. Here, it is important to notice that the coefficients in (2.19), (2.20) belong to $W^{1,\infty}(0,L)$ under the given regularity hypotheses. For $\tau_1', \tau_2', \tau_3'$, we still have to perform an integration with two null boundary conditions for each one (in 0 and in L).

This would impose just three scalar conditions on N_3, B_2, B_3 which shows that G is even infinite dimensional (compare with (2.28) below). Assuming that the determinant (2.21) is nonzero, one can easily solve (2.19), (2.20) to obtain that (we use the notation $[xy]_{ij} = x_i y_j - x_j y_i$ for arbitrary vectors $x, y \in \mathbb{R}^3$):

$$\tau_3' \;=\; -\frac{n_3}{t_3}\, N_3 \;-\; \frac{b_3}{t_3}\, B_3 \;=\; I_3\,, \tag{2.22}$$

$$\tau_2' \;=\; -\frac{n_2}{t_3}\, N_3 \;+\; \frac{[nb]_{23}}{[tn]_{23}}\, B_2 \;-\; \frac{n_2}{t_3}\frac{[tb]_{23}}{[tn]_{23}}\, B_3 \;=\; I_2\,, \tag{2.23}$$

$$\tau_1' \;=\; -\frac{n_1}{t_3}\, N_3 - t_2\Big(\frac{t_1[nb]_{23} - n_1[tb]_{23}}{[tn]_{23}} + b_1\Big) B_2 + \Big(\frac{t_2}{t_3}\cdot\frac{[tn]_{12}\,[tb]_{23}}{[tn]_{23}} + [tb]_{13}\Big) B_3 \;=\; I_1\,, \tag{2.24}$$

$$N_2 \;=\; \frac{t_2}{t_3}\, N_3 \;-\; \frac{[tb]_{23}}{[tn]_{23}}\, B_2 \;+\; \frac{t_2}{t_3}\,\frac{[tb]_{23}}{[tn]_{23}}\, B_3\,, \tag{2.25}$$

$$N_1 \;=\; \frac{t_1}{t_3}\, N_3 + n_2\Big(\frac{n_1\,[tb]_{23} \;-\; t_1\,[nb]_{23}}{[tn]_{23}} + b_1\Big) B_2 + \frac{1}{t_3}\Big(n_2\frac{[tn]_{12}\cdot[tb]_{23}}{[tn]_{23}} + n_3\,[tb]_{13}\Big)\, B_3\,, \tag{2.26}$$

$$B_1 \;=\; -\,b_2\Big(\frac{t_1\,[nb]_{23} \;-\; n_1\,[tb]_{23}}{[tn]_{23}} \;+\; b_1\Big) B_2 \;+\; \frac{1}{t_3}\Big(b_2\,\frac{[tn]_{12}\cdot[tb]_{23}}{[tn]_{23}} \;+\; b_3\,[tb]_{13}\Big) B_3\,. \tag{2.27}$$

In order to ensure that $\tau_i \in H_0^1(0,L), i = \overline{1,3}$, we have to impose separately

$$\int_0^L I_i\, d\tau \;=\; 0\,, \quad i = \overline{1,3}\,. \tag{2.28}$$

Relations (2.22)–(2.28) give an alternative definition of the subspace $G \subset H_0^1(0,L)^9$, starting with any $N_3, B_2, B_3 \in H_0^1(0,L)$.

The new bilinear functional $\alpha_d : H_0^1(0,L)^3 \times H_0^1(0,L)^3 \to \mathbb{R}$ is obtained from a^d as follows. If $z \in H_0^1(0,L)^3$, we construct $\tilde{z} \in H_0^1(0,L)^9$ by identifying the vector z with (N_3, B_2, B_3) and fixing in $\tilde{z}$ the corresponding N_1, N_2, B_1 as given in (2.25)–(2.27). For the components of $\tilde{z}$ corresponding to τ_1, τ_2, τ_3 we modify (2.22)–(2.24) as below.

$$\tau_i(s) = -\int_0^s I_i \, d\tau + \frac{s}{L}\int_0^L I_i \, d\tau, \quad i = \overline{1,3}. \tag{2.29}$$

Moreover, we add to the functional a penalization of order $\frac{1}{d}$ of the relations (2.28). That is, for any $z, w \in H_0^1(0,L)^3$, we have

$$\alpha_d(z,w) = a^d(\tilde{z}, \tilde{w}) + \frac{1}{d}\sum_{i=1}^3 \int_0^L I_i(z)\, d\tau \cdot \int_0^L I_i(w)\, d\tau. \tag{2.30}$$

Notice that α_d has the same type of "singularity" as a^d (compare with (2.13), (2.17)). The coercivity of α_d is clear,

$$\alpha_d(z,z) \geq a^d(\tilde{z},\tilde{z}) \geq C\,|\tilde{z}|^2_{H_0^1(0,L)^9} \geq C\,|z|^2_{H_0^1(0,L)^3}, \tag{2.31}$$

by the coercivity of a^d. Then, there is a unique $\hat{x}_d \in H_0^1(0,L)^3$ such that

$$\alpha_d(\hat{x}_d, z) = (\ell, \tilde{z})_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9}, \quad \forall\, z \in H_0^1(0,L)^3. \tag{2.32}$$

In (2.32), we also use that the correspondence $z \mapsto \tilde{z}$ as defined above between $H_0^1(0,L)^3$ and $H_0^1(0,L)^9$ is linear and bounded, which is obvious by (2.22)–(2.27).

Let us also define $x^0 \in G \subset H_0^1(0,L)^9$ by

$$a_1(x^0, w) = (\ell, w)_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9}, \quad \forall\, w \in G. \tag{2.33}$$

The existence and uniqueness of $x^0 \in G$ is a consequence of the identity

$$a_1(x,w) = a^d(x,w) - d\, a_2(x,w), \quad \forall\, w \in G, \tag{2.34}$$

as $a_0(x,w) = 0$ for $x \in G$ and $w \in H_0^1(0,L)^3$ (see (2.18)). The boundedness of a_2 (by construction) shows that a_1 is coercive on G, equipped with the same norm as $H_0^1(0,L)^9$, for $d \leq d_0$.

Proposition 2.3 *If $d \searrow 0$ then $x_d \to x^0$ and $\tilde{x}_d \to x^0$ strongly in $H_0^1(0,L)^9$.*

Here $\tilde{x}_d$ is obtained from $\hat{x}_d$ (and not from x_d!) by the mapping $x \mapsto \tilde{x}$ defined from $H_0^1(0,L)^3$ to $H_0^1(0,L)^9$ as in (2.25)–(2.27) and (2.29).

Proof. As both statements are proved in a similar way, we limit our argument to $\tilde{x}_d$. From (2.31), (2.32) it follows that $\{\hat{x}_d\}$ is bounded in $H_0^1(0,L)^3$ and $\{\tilde{x}_d\}$ is bounded in $H_0^1(0,L)^9$. We may assume that $\hat{x}_d \to \hat{x}$ weakly in $H_0^1(0,L)^3$ and $\tilde{x}_d \to \tilde{x}$ weakly in $H_0^1(0,L)^9$, on a subsequence. Moreover, $\tilde{x}$ is obtained from

$\hat{x}$ via (2.25)–(2.27) and (2.29), due to the linearity of these relations and to the fact that all the coefficients appearing there may be assumed in $L^\infty(0,L)$ by our regularity conditions on the geometry of the curved rod.

We multiply (2.32) by d, and we take $d \searrow 0$:

$$a_0(\tilde{x}_d, \tilde{w}) + d\, a_1(\tilde{x}_d, \tilde{w}) + d^2\, a_2(\tilde{x}_d, \tilde{w}) + \sum_{i=1}^{3} \int_0^L I_i(\hat{x}_d)\, d\tau \int_0^L I_i(w)\, d\tau$$
$$= d(\ell, \tilde{w})_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9}, \quad \forall\, w \in H_0^1(0,L)^3.$$

We obtain that

$$a_0(\tilde{x}, \tilde{w}) + \sum_{i=1}^{3} \int_0^L I_i(\hat{x})\, d\tau \int_0^L I_i(w)\, d\tau = 0, \quad \forall\, w \in H_0^1(0,L)^3. \tag{2.35}$$

By fixing $w = \hat{x}$ (and $\tilde{w} = \tilde{x}$ consequently) in (2.35), it follows that $\tilde{x} \in G$. Let us now choose in (2.32) $z \in H_0^1(0,L)^3$ such that $\tilde{z} \in G$. Then $a_0(\tilde{x}_d, \tilde{z}) = 0$ and $I_i(z) = 0$, $i = \overline{1,3}$. We obtain the relation

$$a_1(\tilde{x}_d, \tilde{z}) + d\, a_2(\tilde{x}_d, \tilde{z}) = (\ell, \tilde{z})_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9}. \tag{2.36}$$

Passing to the limit in (2.36), and using that $\tilde{x} \in G$, we infer that $\tilde{x}$ satisfies (2.33), i.e. $\tilde{x} = x^0$, by the uniqueness of the solution in (2.33).

The strong convergence follows again from the coercivity (2.31). By $P_3 : H_0^1(0,L)^9 \to H_0^1(0,L)^3$ we denote the projection on the three components corresponding to N_3, B_2, B_3. We have

$$\begin{aligned}
0 &\le C|\tilde{x}_d - x^0|^2_{H_0^1(0,L)^9} \le a^d(\tilde{x}_d - x^0\,,\, \tilde{x}_d - x^0) \\
&\le \alpha_d(\hat{x}_d - P_3\, x^0\,,\, \hat{x}_d - P_3\, x^0) = \alpha_d(\hat{x}_d, \hat{x}_d) - 2\alpha_d(\hat{x}_d, P_3 x^0) + \alpha_d(P_3 x^0, P_3 x^0) \\
&= (\ell, \tilde{x}_d)_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9} - 2\,(\ell, x^0)_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9} + a_1(x^0, x^0) \\
&\qquad + d\, a_2(x^0, x^0) \to 0.
\end{aligned}$$

Above, we have also repeatedly used (2.32) and (2.30). □

Remark 2.3. Proposition 2.3 shows that, for $d > 0$ "small", the equation (2.32) provides a good approximation for the solution of (2.5) or, equivalently, of (2.4).

3 Discretization and uniform approximation

We first define the subspace $G_\alpha \subset H_0^1(0,L)^3$ given by

$$G_\alpha = \left\{ \bar{x} \in H_0^1(0,L)^3\,;\, \int_0^L I_i(\bar{x})\, d\tau = 0, \quad i = \overline{1,3} \right\}. \tag{3.1}$$

Clearly, G_α has codimension three in $H_0^1(0,L)^3$, and $G_\alpha \neq \{0\}$. It plays the same role as the subspace G defined in Section 2, here applied to α_d instead of to a_d.

We denote by $V_h \subset H_0^1(0,L)$, $h > 0$, the usual discretization space of piecewise linear and continuous functions. Clearly, $\bigcup_{h>0} (V_h)^3$ is dense in $H_0^1(0,L)^3$, in this norm, Ciarlet [6]. We also denote $G_h = G_\alpha \cap (V_h)^3$. We then have

$$G_h \subset (V_h)^3 \subset H_0^1(0,L)^3, \quad G_h \subset G_\alpha, \quad \forall\, h > 0. \tag{3.2}$$

Proposition 3.1 $\bigcup_{h>0} G_h$ *is dense in* G_α *in the norm of* $H_0^1(0,L)^3$.

Proof. Take any $\bar v \in G_\alpha \subset H_0^1(0,L)^3$. Then $I_i(\bar v) = 0$, $i = \overline{1,3}$. Take $\bar v_h \in (V_h)^3$ such that $\bar v_h \to \bar v$ strongly in $H_0^1(0,L)^3$, which is always possible.

Consider now some $\tilde v \in H_0^1(0,L)^3$ whose three components attain the value 1 in $L/2$ and are linear in both $\left[0, \frac{L}{2}\right]$ and $\left[\frac{L}{2}, 0\right]$.

Clearly, we may assume that $\tilde v \in (V_h)^3, \forall\, h > 0$, that is, any subdivision of $[0,L]$ that we construct has to contain the point $L/2$.

Denote by $c_i = I_i(\tilde v)$, and assume (without loss of generality) that $c_i \neq 0$, $i = \overline{1,3}$. If this is not fulfilled, one may choose another example of $\tilde v$ such that $\tilde v \in (V_h)^3$ remains true. Denote as well $c_i^h = I_i(\bar v_h)$, $i = \overline{1,3}$. Clearly, $c_i^h \to 0$, as $\bar v_h \to \bar v$ strongly in $H_0^1(0,L)^3$ and $I_i(\bar v) = 0$, $i = \overline{1,3}$.

Define $\hat v_h^i = \frac{c_i^h}{c_i} v_i$, $i = \overline{1,3}$. Then

$$\hat v_h \;=\; (\hat v_h^1, \hat v_h^2, \hat v_h^3) \in (V_h)^3, \quad \forall\, h > 0,$$

$$\hat v_h \to 0 \text{ in } H_0^1(0,L)^3,$$

as $c_i^h \to 0$. Moreover, $\bar v_h - \hat v_h \in G_h$, as $\bar v_h - \hat v_h \in G_\alpha$. This follows from the relation

$$I_i(\bar v_h - \hat v_h) = c_i^h - \frac{c_i^h}{c_i} I_i(v_i) = 0, \quad i = \overline{1,3}.$$

This concludes the proof of the assertion. □

Remark 3.1 The definition of G_α via integral conditions makes the result of Proposition 3.1 possible. In the case of the subspace G, defined pointwisely, this property is not valid (cf. Chenais and Paumier [4]), and other methods have to be used.

We introduce now the discretized problem

$$\alpha_d(x_d^h, v_h) \;=\; (\ell, \tilde v_h)_{H_0^1(0,L)^9 \times H_0^{-1}(0,L)^9}, \quad \forall\, v_h \in (V_h)^3. \tag{3.3}$$

The existence of a unique solution $x_d^h \in (V_h)^3$ for (3.3) follows from (2.30) and the Lax–Milgram lemma.

Proposition 3.2 *For any $\delta > 0$ there are $d(\delta) > 0$ and $h(\delta) > 0$ such that for any $h \in]0, h(\delta)[$ and any $d \in]0, d(\delta)[$, it holds*

$$\left|x_d^h - \hat{x}_d\right|_{H_0^1(0,L)^3} < \delta\,. \tag{3.4}$$

Proof. By (2.31), we have

$$\begin{aligned} c\left|\hat{x}_d - x_d^h\right|^2_{H_0^1(0,L)^3} &\leq \alpha_d(\hat{x}_d - x_d^h, \hat{x}_d - x_d^h) \\ &= \alpha_d(\hat{x}_d - x_d^h, \hat{x}_d) = \alpha_d(\hat{x}_d - x_d^h, \hat{x}_d - \bar{v}_h)\,, \end{aligned} \tag{3.5}$$

for any $\bar{v}_h \in (V_h)^3$, by the known orthogonality property $\alpha_d(\hat{x}_d - x_d^h, \bar{v}_h) = 0\,,\ \forall\ \bar{v}_h \in (V_h)^3$. Then

$$\begin{aligned} & c\left|\hat{x}_d - x_d^h\right|^2_{H_0^1(0,L)^3} \\ \leq\ & \alpha_d(\hat{x}_d - \bar{v}_h, \hat{x}_d - \bar{v}_h) + \alpha_d(\bar{v}_h - x_d^h, \hat{x}_d - x_d^h) + \alpha_d(\bar{v}_h - x_d^h, x_d^h - \bar{v}_h) \\ \leq\ & \alpha_d(\hat{x}_d - \bar{v}_h, \hat{x}_d - \bar{v}_h) + \alpha_d(\bar{v}_h - x_d^h, \hat{x}_d - x_d^h) \\ =\ & \alpha_d(\hat{x}_d - \bar{v}_h, \hat{x}_d - \bar{v}_h)\,, \end{aligned} \tag{3.6}$$

again by the above orthogonality property.

If $\bar{v}_h = \bar{w}_h \in G_h \subset (V_h)^3 \subset H_0^1(0,L)^3$, then $\bar{v}_h \in G_\alpha$ by (3.2) and $I_i(\bar{w}_h) = 0\,,\ i = \overline{1,3}$. Denoting by $\tilde{w}_h \in H_0^1(0,L)^9$, the usual "extension" of $\bar{w}_h$ as defined by (2.22)–(2.27), then $\tilde{w}_h \in G$ by (2.22) - (2.28), and we can write

$$\begin{aligned} \alpha_d(\hat{x}_d - \bar{w}_h, \hat{x}_d - \bar{w}_h) &= a^d(\tilde{x}_d - \tilde{w}_h, \tilde{x}_d - \tilde{w}_h) = \frac{1}{d}\, a_0(\tilde{x}_d, \tilde{x}_d) \\ + \frac{1}{d}\sum_{i=1}^{3}\left(\int_0^L I_i(\hat{x}_d)\,d\tau\right)^2 &+ a_1(\tilde{x}_d - \tilde{w}_h, \tilde{x}_d - \tilde{w}_h) + d\, a_2(\tilde{x}_d - \tilde{w}_h, \tilde{x}_d - \tilde{w}_h), \end{aligned} \tag{3.7}$$

where $\tilde{x}_d$ was defined in Proposition 2.3. Moreover, from (2.32) we get that

$$\begin{aligned} & \frac{1}{d}\, a_0(\tilde{x}_d, \tilde{x}_d) + \frac{1}{d}\sum_{i=1}^{3}\left(\int_0^L I_i(\hat{x}_d)\,d\tau\right)^2 \\ =\ & (\ell, \tilde{x}_d)_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9} - a_1(\tilde{x}_d, \tilde{x}_d) - d\, a_2(\tilde{x}_d, \tilde{x}_d) \\ \to\ & (\ell, x^0)_{H_0^1(0,L)^9 \times H^{-1}(0,L)^9} - a_1(x^0, x^0) = 0\,, \end{aligned} \tag{3.8}$$

by Propostion 2.3 and (2.33). Relation (3.8) shows that there is some $d_1(\delta) > 0$ such that

$$0 \leq \frac{1}{d}\, a_0(\tilde{x}_d, \tilde{x}_d) + \frac{1}{d}\sum_{i=1}^{3}\left(\int_0^L I_i(\hat{x}_d)\,d\tau\right)^2 < \frac{\delta}{3}\,, \quad \text{for } d < d_1(\delta)\,. \tag{3.9}$$

By the triangle inequality, we obtain that

$$a_i(\tilde{x}_d - \tilde{w}_h, \tilde{x}_d - \tilde{w}_h) \le C_i \left|\tilde{x}_d - \tilde{w}_h\right|^2_{H_0^1(0,L)^9}$$
$$\le C_i\Big[\left|\tilde{x}_d - x^0\right|_{H_0^1(0,L)^9} + \left|x^0 - \tilde{w}_h\right|_{H_0^1(0,L)^9} \Big]^2, \quad i = 1,2\,. \tag{3.10}$$

The constants $C_i > 0$, $i = 1,2$, in (3.10) are the boundedness constants for the bilinear functionals $a_i(\cdot,\cdot)$, $i = 1,2$.

Proposition 2.3 gives $|\tilde{x}_d - x^0|_{H_0^1(0,L)^9} < \delta/3$ if $d < d_2(\delta)$, and Proposition 3.1 allows to choose $\bar{w}_h \in G_h$ such that $|\tilde{w}_h - x^0|_{H_0^1(0,L)^9} < \delta/3$ if $h < h(\delta)$. By (3.6)–(3.10), we get (3.4) with the same $h(\delta)$ and with $d(\delta) = \min\{d_1(\delta), d_2(\delta), d_0\}$. □

Proposition 3.3 *For any $\hat{d} > 0$, we have*

$$\lim_{h\to 0} \sup_{\hat{d}\le d\le d_0} \left|x_d^h - \hat{x}_d\right|_{H_0^1(0,L)^3} = 0\,. \tag{3.11}$$

Proof. By (3.5), (3.6), we get

$$c\left|\hat{x}_d - x_d^h\right|^2_{H_0^1(0,L)^3} \le \alpha_d(\hat{x}_d - x_d^h, \hat{x}_d - x_d^h) \le \alpha_d(\hat{x}_d - \bar{v}_h, \hat{x}_d - \bar{v}_h)\,, \quad \forall\ \bar{v}_h \in (V_h)^3\,.$$

Using (2.30) and the continuity properties of the bilinear functionals a_0, a_1, a_2 and of the linear functionals I_1, I_2, I_3, we can write

$$c\left|\hat{x}_d - x_d^h\right|^2_{H_0^1(0,L)^3} \le \frac{M}{d}\left|\tilde{x}_d - \tilde{v}_h\right|^2_{H_0^1(0,L)^9}$$
$$\le \frac{M_1}{d}\left|\hat{x}_d - \bar{v}_h\right|^2_{H_0^1(0,L)^3} \le \frac{M_1}{\hat{d}}\left|\hat{x}_d - \bar{v}_h\right|^2_{H_0^1(0,L)^3}\,. \tag{3.12}$$

As $\bigcup_{h>0}(V_h)^3$ is dense in $H_0^1(0,L)^3$, we may choose in (3.12) $\bar{v}_h(d) \to \hat{x}_d$ in $H_0^1(0,L)^3$ and (3.11) follows. □

Remark 3.2 By combining Proposition 3.2 and Proposition 3.3 we see that $\lim_{h\to 0}(x_d^h - \hat{x}_d) = 0$ uniformly with respect to $d > 0$ in $H_0^1(0,L)^3$.

Remark 3.3 A similar reduction method may be applied in many problems. For instance, in the arch equation considered in Chenais and Paumier [4], Chenais and Zerner [5], one can eliminate w_1 via the Proposition 4 and also obtain uniform convergence properties for the discretization for nonconstant curvature c.

4 Numerical experiments

We have considered the three-dimensional curve (spiral or helix) parametrized by

$$\bar{\theta}(t) = \left(\cos\frac{t}{\sqrt{2}}, \sin\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right), \quad t \in \left[0, \frac{\pi}{2}\right].$$

By Arnăutu, Sprekels and Tiba [1], § 8, if we choose the functions $\varphi(t) = \frac{\pi}{4}$ and $\varphi(t) = \frac{\pi}{2} + \frac{t}{\sqrt{2}}$, then it is possible to show that the tangent and the normal vectors to $\bar{\theta}(t)$ are given by $(\sin\varphi\cos\psi, \sin\varphi\sin\psi, \cos\varphi)$ and $(\cos\varphi\cos\psi, \cos\varphi\sin\psi, -\sin\psi)$, respectively. In particular, assumption (2.21) is fulfilled.

The cross section of the curved rod is assumed to be a disk of radius $R > 0$, and the parameter $d = \pi R^2$ is the area of the cross section. In the numerical experiments, we have used the values 0.3, 0.1, 0.05, 0.01 and 0.001 for R, and d varies from $2.827433 \cdot 10^{-1}$ to $3.1415927 \cdot 10^{-6}$.

The finite element method was applied by dividing the interval $[0, \frac{\pi}{2}]$ by an equidistant grid with 100 subintervals, giving $h = 0.157$. For the integrals over the cross section, the usual change of variables to polar coordinates was applied. This allows the computation of iterated integrals by numerical integration methods corresponding to the discrete grid. For the bilinear functionals a_d, α_d it was possible to compute them exactly. See (2.15), (2.22)–(2.24) and the definition of C_{ij} in Section 2.

The obtained algebraic linear system was solved by the Gauss algorithm. The bilinear functional a_d has been numerically generated by using (2.14).

Example 4.1 We fix the force in the right-hand side to be of the form $f = (0, 0, f_3)$ with

$$f_3(z) = \begin{cases} 10 & z \in \left[0, \frac{\pi}{4}\right], \\ -10 & z \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]. \end{cases}$$

The displacements (for R between 0.3 and 0.01) are shown in Figure 1. In order to give a clear representation of the displacement vector, we have used the scaling factor 3 in the first three cases and 0.15 in the last one. The figure has been produced with Matlab.

Example 4.2 We choose a "torsional"-type force

$$f(x, y, z) = \begin{cases} 50(-y, x, 0), & z \in \left[0, \frac{\pi}{4}\right], \\ 50(y, -x, 0), & z \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]. \end{cases}$$

In Figure 2 the obtained displacement is represented with a scaling factor 100. The notations are as in Fig. 1.

It should be noticed that for $R = 0.3, 0.1, 0.05, 0.01$ the three bilinear forms A_d, a_d and α_d produce numerical results that are very close and therefore we did not specify the used bilinear form in the two examples above.

However, for $R = 0.001$ (which corresponds to $d \approx 10^{-6}$ and is the critical case) there is a relevant difference between the results obtained with α_d and the results obtained with A_d or a_d (which remain very close).

In Table 1, the error and the relative error in the ℓ^2 norm are listed, between the solutions obtained by A_d, a_d, respectively α_d, for $R = 0.001$ and for $\bar{\tau} = (\tau_1, \tau_2, \tau_3)$.

	A_d versus a_d		a_d versus α_d	
	abs. error	rel. error	abs. error	rel. error
τ_1	$6.191 \cdot 10^{-15}$	$1.003 \cdot 10^{-6}$	$1.743 \cdot 10^{-11}$	$2.831 \cdot 10^{-3}$
τ_2	$5.431 \cdot 10^{-15}$	$5.785 \cdot 10^{-7}$	$3.062 \cdot 10^{-11}$	$3.260 \cdot 10^{-3}$
τ_3	$2.365 \cdot 10^{-15}$	$2.162 \cdot 10^{-7}$	$3.103 \cdot 10^{-11}$	$2.839 \cdot 10^{-3}$

Table 1

According to the theoretical results from Section 3, the bilinear functional α_d has to be taken into account. The experiments show that this is also important for small values of the parameter d.

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