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A general asymptotic dynamic model for Lipschitzian curved rods

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Abstract

In this paper we study the asymptotic behaviour of solutions to the linear evolution problem for clamped curved rods with the small thickness ϵ under minimal regularity assumptions on the geometry. In addition, non-constant density of the curved rods is considered.

1 Introduction

The main task of this paper is to relax the regularity assumptions on the shape of the curved elastic rods in the general asymptotic model and to derive this general model from the linear evolution equation of three dimensional elasticity by asymptotic technique.

We use the asymptotic approach presented by Aganovič and Tutek [1] for straight rods, which was modified by Jurak and Tambača [6] and [7] for curved rods. Using the idea from Blouza and Le Dret [2], it was shown by Tiba and Vodák [15] that we can admit in the "limit state $\epsilon = 0$" a unit speed curve with Lipschitzian parametrization and that we can approximate this curve by a suitable sequence of smooth curved rods depending on ϵ, which preserves the explicit form of the constant in the Korn inequality corresponding to the thickness of the domain. An analogous strategy as in [15] enables us to generalize the result given by Tambača [14].

The basic idea is rather simple and natural. If we denote by $\epsilon > 0$ the "thickness" parameter specific to asymptotic methods, we also introduce another small parameter $\delta = \epsilon^r$ $(0 < r < \frac{1}{3})$ associated to a regularization procedure applied to the nonsmooth line of centroids. A careful examination of the convergence properties of the arising smooth coefficients, and sharp estimates in the corresponding weak formulation of the linear elasticity system (after scaling), allows to pass to the limit $\epsilon \to 0$ and to obtain the asymptotic model. In the smooth case, this is similar to the model of Tambača [14].

Let us also mention other related works discussing asymptotic dynamic models: Raoult [11] (for plates) and Li-ming [10] (for shells). Further, we refer the reader to [15] for the detailed construction of the local frame in $L^\infty(0,l)$ and its smooth approximation, and to [13] for a special approach to the dynamic model for curved rods.

Finally, we give a brief outline of the paper. In Section 2, we introduce the basic notations and notions that will be further needed. Section 3 contains auxiliary propositions, which are used throughout the paper. Section 4 is devoted to the formulation of the linear elasticity equation and its transformation. Section 5 deals with the existence and uniqueness of the solution to the transformed equation and basic estimates are derived. Section 6 gives us the basic overview about behaviour of the displacements if $\epsilon \to 0$ and about the qualitative properties of their limit state. In Section 7 the passage to the limit $\epsilon \to 0$ is performed and the main existence and uniqueness result is proved.

2 Basic notation

We denote by $\mathbb{R}^3$ the usual three dimensional Euclidean space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. By " $\times$ " we shall denote the Cartesian product of two spaces and by $\lfloor\cdot,\cdot\rfloor$ any ordered pair. In the text the symbol $|A|$ will also denote the Lebesgue measure of some measurable set A, without danger of confusion. The summation convention with respect to repeated indices will be also used, if not otherwise explicitly stated.

Let $S \subset \mathbb{R}^2$ be a bounded simply connected domain of class C^1 satisfying the "symmetry" condition

$$\int_S x_2 \; dx_2 dx_3 = \int_S x_3 \; dx_2 dx_3 = \int_S x_2 x_3 \; dx_2 dx_3 = 0. \tag{2.1}$$

We denote by $\Omega = (0,l) \times S$, $\Omega_\epsilon = (0,l) \times \epsilon S$ open "cylinders" in $\mathbb{R}^3$, where $l > 0$ and $\epsilon > 0$ "small", are given.

Let $\mathcal{C}$ be a unit speed curve of length l in $\mathbb{R}^3$ defined by its parametrization $\mathbf{\Phi}$: $[0,l] \to \mathbb{R}^3$, and let $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ denote its tangent, normal and binormal vectors. As we shall assume less regularity for $\mathbf{\Phi}$ as for instance in [6], [7] and [14], the local frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is not necessarily the Frenêt one. Alternative ways to construct local frames under low regularity assumptions may be found in [13]. Let $\mathbf{\Phi}_\epsilon : [0,l] \to \mathbb{R}^3$ be a smoothing of $\mathbf{\Phi}$ such that it remains a unit speed curve (i.e. $|\mathbf{\Phi}_\epsilon'(y_1)| = 1, \forall y_1 \in [0,l]$) and $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ be the associated local frame. The regularization parameter will be of the form ϵ^r, $r \in (0, \frac{1}{3})$, and we just write $\mathbf{\Phi}_\epsilon$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ to simplify notation. More details on the construction of the functions $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ and their regularizations can be found in [15]. The most important properties of these regularizations are mentioned in Proposition 3.1 and Corollary 3.2.

Further, we define the auxiliary functions α_ϵ, β_ϵ, γ_ϵ (corresponding to the usual notions of curvature and torsion) by

$$\alpha_\epsilon = (\mathbf{t}_\epsilon', \mathbf{b}_\epsilon), \;\; \beta_\epsilon = (\mathbf{t}_\epsilon', \mathbf{n}_\epsilon), \;\; \gamma_\epsilon = (\mathbf{b}_\epsilon', \mathbf{n}_\epsilon).$$

To obtain these relations, we use the assumed orthonormality of the local basis $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ which gives the orthogonality properties $(\mathbf{t}_\epsilon, \mathbf{t}_\epsilon') = 0$, $(\mathbf{n}_\epsilon, \mathbf{n}_\epsilon') = 0$, $(\mathbf{b}_\epsilon, \mathbf{b}_\epsilon') = 0$, that is, $\mathbf{t}_\epsilon'$ may be expressed via $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$, and so on. We obtain the "laws of motion" of the local frame

$$\begin{aligned} \mathbf{t}_\epsilon' &= \alpha_\epsilon \mathbf{b}_\epsilon + \beta_\epsilon \mathbf{n}_\epsilon, \\ \mathbf{n}_\epsilon' &= -\beta_\epsilon \mathbf{t}_\epsilon - \gamma_\epsilon \mathbf{b}_\epsilon, \\ \mathbf{b}_\epsilon' &= -\alpha_\epsilon \mathbf{t}_\epsilon + \gamma_\epsilon \mathbf{n}_\epsilon. \end{aligned} \tag{2.2}$$

We introduce the mapping $\mathbf{R}_\epsilon$

$$\mathbf{R}_\epsilon : \Omega \to \Omega_\epsilon, \;\; \mathbf{R}_\epsilon(x_1, x_2, x_3) = (x_1, \epsilon x_2, \epsilon x_3), \tag{2.3}$$

and the mapping $\bar{\mathbf{P}}_\epsilon$

$$\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \mathbb{R}^3, \ \bar{\mathbf{P}}_\epsilon(y) = \mathbf{\Phi}_\epsilon(y_1) + y_2\mathbf{n}_\epsilon(y_1) + y_3\mathbf{b}_\epsilon(y_1), \tag{2.4}$$

$(y_1, y_2, y_3) \in (0,l) \times \epsilon S$, which gives the parametrization of the curved rod $\widetilde{\Omega}_\epsilon = \bar{\mathbf{P}}_\epsilon(\Omega_\epsilon)$. Furthermore,

$$\bar{d}_\epsilon(y) = \det(\bar{\nabla}\bar{\mathbf{P}}_\epsilon(y)) = 1 - \beta_\epsilon(y_1)y_2 - \alpha_\epsilon(y_1)y_3 \text{ for all } y \in \overline{\Omega}_\epsilon. \tag{2.5}$$

We can suppose that $\bar{d}_\epsilon(y) > 0$ for all $y \in \overline{\Omega}_\epsilon$ and for ϵ "small" (see Corollary 3.2 in this paper or Corollary 3.3 in [15]). Then $\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \widetilde{\Omega}_\epsilon$ is a C^1-diffeomorphism, Ciarlet [3], Theorem 3.1-1. In the sequel, we shall write $\widetilde{\partial}_i = \frac{\partial}{\partial \widetilde{y}_i}$, where $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) \in \widetilde{\Omega}_\epsilon$, $\bar{\partial}_i = \frac{\partial}{\partial y_i}$, for $y = (y_1, y_2, y_3) \in \Omega_\epsilon$, $\partial_i = \frac{\partial}{\partial x_i}$, where $x = (x_1, x_2, x_3) \in \Omega$, $\partial_t = \frac{\partial}{\partial t}$ and $\partial_{tt} = \frac{\partial^2}{\partial t^2}$. Thus, in (2.5), $\bar{\nabla} = (\bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3)$. In the case that a function v depends only on t or x_1 (or y_1), we denote its first (second) derivation by $\dot{v}$ ($\ddot{v}$) and v' (v''), respectively. Sometimes we use the notation $\frac{d}{dt}v$ instead of $\dot{v}$. In an analogous way as above, we denote by $\widetilde{V}$ a function defined on $\widetilde{\Omega}_\epsilon$, $\bar{V}$ a function defined on Ω_ϵ, and V a function defined on Ω. We suppose throughout this subsection that all needed derivatives exist, which will later follow from Section 3.

The covariant basis at the point $\bar{\mathbf{P}}_\epsilon(y)$, $y \in \Omega_\epsilon$, of the curved rod is defined by $\bar{\mathbf{g}}_{i,\epsilon}(y) = \bar{\partial}_i\bar{\mathbf{P}}_\epsilon(y)$, and (using (2.2)) these vectors are given by

$$\bar{\mathbf{g}}_{1,\epsilon}(y) = (1 - y_2\beta_\epsilon(y_1) - y_3\alpha_\epsilon(y_1))\mathbf{t}_\epsilon(y_1) + y_3\gamma_\epsilon(y_1)\mathbf{n}_\epsilon(y_1) - y_2\gamma_\epsilon(y_1)\mathbf{b}_\epsilon(y_1),$$

$$\bar{\mathbf{g}}_{2,\epsilon}(y) = \mathbf{n}_\epsilon(y_1), \ \bar{\mathbf{g}}_{3,\epsilon}(y) = \mathbf{b}_\epsilon(y_1). \tag{2.6}$$

The vectors $\bar{\mathbf{g}}^{j,\epsilon}$ defined by the relations $(\bar{\mathbf{g}}_{i,\epsilon}, \bar{\mathbf{g}}^{j,\epsilon}) = \delta^{ij}$, constitute the contravariant basis of the curved rod at the point $\bar{\mathbf{P}}_\epsilon(y)$. They have the form

$$\bar{\mathbf{g}}^{1,\epsilon}(y) = \frac{\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)}, \ \bar{\mathbf{g}}^{2,\epsilon}(y) = \frac{-y_3\gamma_\epsilon(y_1)\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{n}_\epsilon(y_1),$$

$$\bar{\mathbf{g}}^{3,\epsilon}(y) = \frac{y_2\gamma_\epsilon(y_1)\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{b}_\epsilon(y_1). \tag{2.7}$$

Further, we define the covariant and contravariant metric tensors $(\bar{g}_{ij,\epsilon})_{i,j=1}^3$ and $(\bar{g}^{ij,\epsilon})_{i,j=1}^3$, where

$$\bar{g}_{ij,\epsilon} = (\bar{\mathbf{g}}_{i,\epsilon}, \bar{\mathbf{g}}_{j,\epsilon}), \ \bar{g}^{ij,\epsilon} = (\bar{\mathbf{g}}^{i,\epsilon}, \bar{\mathbf{g}}^{j,\epsilon}). \tag{2.8}$$

After the substitution $y = \mathbf{R}_\epsilon(x)$, we adopt the notation

$$g^{ij,\epsilon}(x) = \bar{g}^{ij,\epsilon}(\mathbf{R}_\epsilon(x)), \ g_{ij,\epsilon}(x) = \bar{g}_{ij,\epsilon}(\mathbf{R}_\epsilon(x)), \ \mathbf{g}_{i,\epsilon}(x) = \bar{\mathbf{g}}_{i,\epsilon}(\mathbf{R}_\epsilon(x)), \tag{2.9}$$

$$\mathbf{g}^{j,\epsilon}(x) = \bar{\mathbf{g}}^{j,\epsilon}(\mathbf{R}_\epsilon(x)), \ d_\epsilon(x) = \bar{d}_\epsilon(\mathbf{R}_\epsilon(x)), \tag{2.10}$$

where $x \in \Omega$.

In an analogous way, we can derive the covariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$. Thus, $\mathbf{o}_{i,\epsilon}(x) = \partial_i(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $i = 1, 2, 3$, and these vectors are given by

$$\mathbf{o}_{1,\epsilon}(x) = (1 - \epsilon x_2 \beta_\epsilon(x_1) - \epsilon x_3 \alpha_\epsilon(x_1))\mathbf{t}_\epsilon(x_1) + \epsilon x_3 \gamma_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) - \epsilon x_2 \gamma_\epsilon(x_1)\mathbf{b}_\epsilon(x_1),$$
$$\mathbf{o}_{2,\epsilon}(x) = \epsilon \mathbf{n}_\epsilon(x_1), \ \mathbf{o}_{3,\epsilon}(x) = \epsilon \mathbf{b}_\epsilon(x_1). \tag{2.11}$$

The vectors $\mathbf{o}^{j,\epsilon}$ defined by the relations $(\mathbf{o}_{i,\epsilon}, \mathbf{o}^{j,\epsilon}) = \delta^{ij}$, constitute the contravariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$. They have the form

$$\mathbf{o}^{1,\epsilon}(x) = \frac{\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)}, \ \mathbf{o}^{2,\epsilon}(x) = \frac{-x_3\gamma_\epsilon(x_1)\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)} + \frac{\mathbf{n}_\epsilon(x_1)}{\epsilon},$$
$$\mathbf{o}^{3,\epsilon}(x) = \frac{x_2\gamma_\epsilon(x_1)\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)} + \frac{\mathbf{b}_\epsilon(x_1)}{\epsilon}. \tag{2.12}$$

We can define the covariant and contravariant metric tensors $(o_{ij,\epsilon})_{i,j=1}^3$ and $(o^{ij,\epsilon})_{i,j=1}^3$, where

$$o_{ij,\epsilon} = (\mathbf{o}_{i,\epsilon}, \mathbf{o}_{j,\epsilon}), \ o^{ij,\epsilon} = (\mathbf{o}^{i,\epsilon}, \mathbf{o}^{j,\epsilon}). \tag{2.13}$$

These tensors have the form

$$(o_{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} d_\epsilon^2 + \epsilon^2 x_3^2 \gamma_\epsilon^2 + \epsilon^2 x_2^2 \gamma_\epsilon^2 & \epsilon^2 x_3 \gamma_\epsilon & -\epsilon^2 x_2 \gamma_\epsilon \\ \epsilon^2 x_3 \gamma_\epsilon & \epsilon^2 & 0 \\ -\epsilon^2 x_2 \gamma_\epsilon & 0 & \epsilon^2 \end{pmatrix} \tag{2.14}$$

and

$$(o^{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} \frac{1}{d_\epsilon^2} & \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{x_2\gamma_\epsilon}{d_\epsilon^2} \\ \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_3^2\gamma_\epsilon^2}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} \\ \frac{x_2\gamma_\epsilon}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_2^2\gamma_\epsilon^2}{d_\epsilon^2} \end{pmatrix}. \tag{2.15}$$

Now, we can compute

$$o_\epsilon(x) = \sqrt{\det(o_{ij,\epsilon}(x))_{i,j=1}^3} = \epsilon^2 d_\epsilon(x). \tag{2.16}$$

We use for constants the symbols C or C_i, for $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Constant vectors will be denoted by $\mathbf{C}$ or $\mathbf{C}_i$ for $i \in \mathbb{N}_0$.

The symbols $H^1(\Omega)$, $H_0^1(\Omega)$ and $L^p(\Omega)$, $p \in [1, \infty]$, respectively, denote the standard Sobolev and Lebesgue spaces endowed with the norms $\|\cdot\|_{1,2}$ or $\|\cdot\|_p$. We will use the same notation of the norms also for vector or tensor functions in the case that all their components belong to the above mentioned Sobolev or Lebesgue spaces. $H^{-1}(\Omega)$ and X' stand for the dual space to $H_0^1(\Omega)$ or X, respectively. The notation $C^m(\overline{\Omega})$, with $m \in \mathbb{N}_0$, means the usual spaces of continuous functions whose derivatives up to the order m are continuous in $\overline{\Omega}$, and we denote by $C_0^\infty(\Omega)$ the space of all functions which have derivatives of any order on Ω and whose supports are compact subsets of Ω. The symbols $L^p(I; X)$, $p \in [1, \infty)$, $L^\infty(I; X)$ and $C(I; X)$, where X is a Banach space and I is a bounded interval, stand for the Bochner spaces endowed with the norms

$$\|v\|_{L^p(I;X)} = \left(\int_I \|v(z)\|_X^p \, dz\right)^{1/p}, \quad \|v\|_{L^\infty(I;X)} = \operatorname{ess\,sup}_I \|v(z)\|_X$$

and

$$\|v\|_{C(\overline{I};X)} = \max_{z\in\overline{I}} \|v(z)\|_X.$$

We say that $v_n \rightharpoonup v$ in X or in $L^p(I;X)$, $p \in (1,\infty)$, or in $L^2(0,l;H^{-1}(S))$, if

$$_{X'}\langle \psi, v_n - v\rangle_X \to 0 \text{ for any } \psi \in X',$$

$$\int_I {}_{X'}\langle \psi(z), v_n(z) - v(z)\rangle_X \, dz \to 0 \text{ for any } \psi \in L^{p'}(I,X'),\ p' = \frac{p}{p-1},$$

and

$$\int_0^l {}_{H^{-1}(S)}\langle v_n(x_1) - v(x_1), \psi(x_1)\rangle_{H_0^1(S)} \, dx_1 \to 0 \text{ for any } \psi \in L^2(0,l;H_0^1(S)),$$

respectively, where $_{X'}\langle\cdot,\cdot\rangle_X$ denotes the dual pairing of X' and X. Further, we denote $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(I,X')$ if

$$\int_I {}_{X'}\langle v_n - v, \psi\rangle_X \, dz \to 0 \text{ for any } \psi \in L^1(I,X).$$

In the case that $X' = L^2(\Omega)'$ or $X' = H^{-1}(\Omega)$, we write without danger of confusion $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(I,L^2(\Omega))$ or $L^\infty(I,H^1(\Omega))$ if

$$\int_I\int_\Omega (v_n - v)\psi \, dxdz \to 0$$

for any $\psi \in L^1(I,L^2(\Omega))$ or $L^\infty(I,H_0^1(\Omega))$, respectively.

Let $v \in L^1_{loc}(0,T)$ and $\varphi \in C_0^\infty(0,T)$. Then we denote $\overline{v}^\varphi = \int_0^T v(t)\varphi(t)\,dt$.

The definitions of the domains $\widetilde{\Omega}_\epsilon$, Ω_ϵ and Ω enable us to introduce the following notation:

$$V(\widetilde{\Omega}_\epsilon) = \{\widetilde{V} \in H^1(\widetilde{\Omega}_\epsilon) : \widetilde{V}|_{\bar{\mathbf{P}}_\epsilon(\{0\}\times\epsilon S)} = \widetilde{V}|_{\bar{\mathbf{P}}_\epsilon(\{l\}\times\epsilon S)} = 0\},$$
$$V(\Omega) = \{V \in H^1(\Omega) : V|_{(\{0\}\times S)} = V|_{(\{l\}\times S)} = 0\},$$

and further we introduce the space

$$\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l) = \{\lfloor \mathbf{V},\psi\rfloor \in H_0^1(0,l)^3 \times L^2(0,l) :\ (\mathbf{V}',\mathbf{t}) = 0$$
$$\text{and } \mathbf{V}_* = -\psi\mathbf{t} + (\mathbf{V}',\mathbf{b})\mathbf{n} - (\mathbf{V}',\mathbf{n})\mathbf{b} \in H_0^1(0,l)^3\}. \qquad (2.17)$$

3 Auxiliary propositions

Proposition 3.1 [15] *Let $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$ be the parametrization of the unit speed curve $\mathcal{C}$. Then there exist vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$, which belong to $L^\infty(0,l)^3$ and form the local frame corresponding to the curve $\mathcal{C}$, such that*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1,\ \mathbf{t} \perp \mathbf{n} \perp \mathbf{b} \textit{ a.e. in } (0,l). \tag{3.1}$$

In addition, there exist functions

$$\{\mathbf{\Phi}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{t}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{n}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{b}_\epsilon\}_{\epsilon\in(0,1)} \subset C^\infty([0,l])^3$$

such that

$$|\mathbf{t}_\epsilon| = |\mathbf{n}_\epsilon| = |\mathbf{b}_\epsilon| = 1,\ \mathbf{t}_\epsilon \perp \mathbf{n}_\epsilon \perp \mathbf{b}_\epsilon \textit{ on } [0,l] \tag{3.2}$$

for all $\epsilon \in (0,1)$,

$$\mathbf{t}_\epsilon \to \mathbf{t},\ \mathbf{n}_\epsilon \to \mathbf{n},\ \mathbf{b}_\epsilon \to \mathbf{b} \textit{ in measure in } (0,l), \tag{3.3}$$

for $\epsilon \to 0$,

$$\|\mathbf{t}'_\epsilon\|_\infty,\ \|\mathbf{n}'_\epsilon\|_\infty,\ \|\mathbf{b}'_\epsilon\|_\infty \sim O(\frac{1}{\epsilon^r}),\ \|\mathbf{t}''_\epsilon\|_\infty,\ \|\mathbf{n}''_\epsilon\|_\infty,\ \|\mathbf{b}''_\epsilon\|_\infty \sim O(\frac{1}{\epsilon^{2r}}), \tag{3.4}$$

and

$$\|\alpha_\epsilon\|_\infty,\ \|\beta_\epsilon\|_\infty,\ \|\gamma_\epsilon\|_\infty \sim O(\frac{1}{\epsilon^r}),\ \|\alpha'_\epsilon\|_\infty,\ \|\beta'_\epsilon\|_\infty,\ \|\gamma'_\epsilon\|_\infty \sim O(\frac{1}{\epsilon^{2r}}),\ r \in (0, \frac{1}{3}), \tag{3.5}$$

where the functions α_ϵ, β_ϵ, $\gamma_\epsilon \in C^\infty([0,l])$ are determined by (2.2).

Corollary 3.2 [15] *There exist the constants C_j, $j = 0,1,2$, such that the function d_ϵ defined by (2.5) and (2.10) satisfies $0 < C_0 \le d_\epsilon(x) \le C_1$ for all $x \in \overline{\Omega}$, and the function $\epsilon d_\epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j}$ defined by (2.15), where ν_i, $i = 1,2,3$, are the components of the unit outward normal for $(0,l) \times \partial S$, satisfies $0 \le d_\epsilon(x)\epsilon\sqrt{\nu_i(x) o^{ij,\epsilon}(x) \nu_j(x)} \le C_2$ for all $x \in \overline{(0,l) \times \partial S}$ and $\epsilon \in (0,1)$. In addition,*

$$d_\epsilon \to 1 \textit{ in } C(\overline{\Omega}), \tag{3.6}$$

$$\epsilon d_\epsilon(x)\sqrt{\nu_i(x) o^{ij,\epsilon}(x) \nu_j(x)} \to 1 \textit{ in } C(\overline{(0,l) \times \partial S}), \tag{3.7}$$

for $\epsilon \to 0$.

Proposition 3.3 [15] *Let the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ be defined by (2.17). Then*

$$\psi = -(\mathbf{V}_*, \mathbf{t}) \textit{ and } \mathbf{V}(x_1) = \int_0^{x_1} [-(\mathbf{V}_*, \mathbf{b})\mathbf{n} + (\mathbf{V}_*, \mathbf{n})\mathbf{b}]\ dz_1 \tag{3.8}$$

for $x_1 \in [0,l]$, where

$$\mathbf{V}(l) = \int_0^l [-(\mathbf{V}_*, \mathbf{b})\mathbf{n} + (\mathbf{V}_*, \mathbf{n})\mathbf{b}]\ dx_1 = 0, \tag{3.9}$$

$\psi \in L^\infty(0,l)$, and $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is a nontrivial Hilbert space endowed with the norm

$$\|\lfloor \mathbf{V}, \psi \rfloor\|^2 = \|\mathbf{V}\|_{1,2}^2 + \|\psi\|_2^2 + \|\mathbf{V}_*\|_{1,2}^2. \tag{3.10}$$

Proposition 3.4 [15] *Let* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ *and* $\mathbf{b}_\epsilon$ *be the functions from Proposition* 3.1 *and let the space* $\mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ *be defined by* (2.17) *using the functions* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ *instead of* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. *Let, further,* $\lfloor \mathbf{V}, \psi \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ *be an arbitrary but fixed couple. Then there exist couples* $\lfloor \mathbf{V}_\epsilon, \psi_\epsilon \rfloor \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ *generating the functions* $\mathbf{V}_{*,\epsilon}$ *such that*

$$\{\mathbf{V}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{V}_{*,\epsilon}\}_{\epsilon\in(0,1)} \subset C_0^\infty(0,l)^3,\ \{\psi_\epsilon\}_{\epsilon\in(0,1)} \subset C_0^\infty(0,l),$$

$$\mathbf{V}_\epsilon \to \mathbf{V},\ \mathbf{V}_{*,\epsilon} \to \mathbf{V}_*\ \text{in}\ H_0^1(0,l)^3, \tag{3.11}$$

$$\psi_\epsilon \to \psi\ \text{in measure in}\ (0,l), \tag{3.12}$$

for $\epsilon \to 0$, *and*

$$\|\mathbf{V}_\epsilon''\|_2 \sim O(\frac{1}{\epsilon^r}),\ \|\psi_\epsilon'\|_2 \sim O(\frac{1}{\epsilon^r}),\ r \in (0, \frac{1}{3}). \tag{3.13}$$

Proposition 3.5 [7] *Let* $w \in H^1(\Omega)$. *Then* $\partial_i \partial_j w \in L^2(0,l;H^{-1}(S))$ *for* $i, j = 1,2,3$ *except for* $i = j = 1$. *If, in addition,* $w|_{x_1=0} = w|_{x_1=l} = 0$, *then* $\partial_j w|_{x_1=0} = \partial_j w|_{x_1=l} = 0$, *for* $j = 2,3$, *in the sense of the space* $C([0,l];H^{-1}(S))$. *Furthermore, if* $v \in L^2(0,l;L^2(S))$, $\partial_1 v \in L^2(0,l;H^{-1}(S))$ *and* $v|_{x_1=0} = v|_{x_1=l} = 0$ *in the sense of the space* $C([0,l];H^{-1}(S))$, *there is a constant* C *independent of* v *such that*

$$\|v\|_{L^2(0,l;L^2(S))} \leq C\|\nabla v\|_{L^2(0,l;H^{-1}(S))}. \tag{3.14}$$

Proposition 3.6 [7] *Let* $\{v_n\}_{n=1}^\infty \subset L^2(0,l;L^2(S))$, $\{\partial_1 v_n\}_{n=1}^\infty \subset L^2(0,l;H^{-1}(S))$ *and let* $v_n|_{x_1=0} = v_n|_{x_1=l} = 0$, *for all* $n \in \mathbb{N}$, *in the sense of the space* $C([0,l];H^{-1}(S))$. *Assume, in addition, that this sequence satisfies*

$$\partial_1 v_n \rightharpoonup \xi,\ \partial_j v_n \rightharpoonup 0,\ \text{in}\ L^2(0,l;H^{-1}(S)),\ j = 2,3, \tag{3.15}$$

where $\xi \in L^2(0,l;H^{-1}(S))$. *Then* $\xi \in L^2(0,l)$, *and there exists a unique function* $v \in H_0^1(0,l)$ *such that* $v' = \xi$ *and*

$$v_n \rightharpoonup v\ \text{in}\ L^2(0,l;L^2(S)), \tag{3.16}$$

$$v_n \to v\ \text{in}\ C([0,l];H^{-1}(S)). \tag{3.17}$$

If the convergences in (3.15) *are strong then the convergence* (3.16) *is also strong.*

Proposition 3.7 *Let* $\varphi \in C_0^\infty(0,T)$. *Let the sequence* $\{v_n\}_{n=1}^\infty \subset L^p(0,T;X)$, $p \in (1,\infty)$, *or* $\{v_n\}_{n=1}^\infty \subset L^\infty(0,T;X')$, *where* X *is a Banach space, be such that* $v_n \rightharpoonup v$ *in* $L^p(0,T;X)$, $p \in (1,\infty)$, *or* $v_n \overset{*}{\rightharpoonup} v$ *in* $L^\infty(0,T;X')$, *respectively. Then* $\overline{v_n}^\varphi \rightharpoonup \overline{v}^\varphi$ *in* X *or* $\overline{v_n}^\varphi \rightharpoonup \overline{v}^\varphi$ *in* X', *respectively.*

P r o o f: We start with the case $v_n \rightharpoonup v$ in $L^p(0,T;X)$, $p \in (1,\infty)$. To prove the first part of the proposition, it is enough to show that

$$\int_0^T {}_{X'}\langle \psi, w(t) \rangle_X\ dt = {}_{X'}\langle \psi, \int_0^T w(t)\ dt \rangle_X \tag{3.18}$$

for all $\psi \in X'$, where $w \in L^p(0,T;X)$, $p \in (1,\infty)$. Since $w \in L^p(0,T;X)$, $p \in (1,\infty)$, then w is Bochner integrable and there exists a sequence of simple functions $\{w_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} \|w_m(t) - w(t)\|_X = 0 \tag{3.19}$$

for a.a. $t \in (0,T)$ and

$$\lim_{m\to\infty} \int_0^T \|w_m(t) - w(t)\|_X \; dt = 0 \tag{3.20}$$

see [9]. The functions w_m, $m = 1, 2, \ldots$, are simple and thus they can be expressed by

$$w_m(t) = \sum_{i=1}^{k(m)} \chi_{B_{i,m}}(t) c_{i,m},$$

where $c_{i,m} \in X$ and $\chi_{B_{i,m}}(t)$ are the characteristic functions to the sets $B_{i,m} \subset (0,T)$, $m = 1, 2, \ldots$ and $i = 1, \ldots, k(m)$. Then we get

$$\begin{aligned}
|\int_0^T {}_{X'}\langle \psi, w(t)\rangle_X \; dt - {}_{X'}\langle \psi, \int_0^T w(t)\; dt\rangle_X| &\le |\int_0^T {}_{X'}\langle \psi, w(t) - w_m(t)\rangle_X \; dt| \\
+|\int_0^T {}_{X'}\langle \psi, w_m(t)\rangle_X \; dt - {}_{X'}\langle \psi, \int_0^T w(t)\; dt\rangle_X| &\le \|\psi\|_{X'} \int_0^T \|w_m(t) - w(t)\|_X \; dt \\
+|\int_0^T {}_{X'}\langle \psi, \sum_{i=1}^{k(m)} \chi_{B_{i,m}}(t) c_{i,m}\rangle_X \; dt &- {}_{X'}\langle \psi, \int_0^T w(t)\; dt\rangle_X| \\
&= \|\psi\|_{X'} \int_0^T \|w_m(t) - w(t)\|_X \; dt \\
+|\int_0^T \sum_{i=1}^{k(m)} \chi_{B_{i,m}}(t)\, {}_{X'}\langle \psi, c_{i,m}\rangle_X \; dt &- {}_{X'}\langle \psi, \int_0^T w(t)\; dt\rangle_X| \\
= \|\psi\|_{X'} \int_0^T \|w_m(t) - w(t)\|_X \; dt + |\sum_{i=1}^{k(m)} |B_{i,m}|\, &{}_{X'}\langle \psi, c_{i,m}\rangle_X - {}_{X'}\langle \psi, \int_0^T w(t)\; dt\rangle_X| \\
= \|\psi\|_{X'} \int_0^T \|w_m(t) - w(t)\|_X \; dt + |{}_{X'}\langle \psi, \sum_{i=1}^{k(m)} &|B_{i,m}| c_{i,m}\rangle_X - {}_{X'}\langle \psi, \int_0^T w(t)\; dt\rangle_X| \\
= \|\psi\|_{X'} \int_0^T \|w_m(t) - w(t)\|_X \; dt &+ |{}_{X'}\langle \psi, \int_0^T w_m(t) - w(t)\; dt\rangle_X| \\
&\le 2\|\psi\|_{X'} \int_0^T \|w_m(t) - w(t)\|_X \; dt \to 0
\end{aligned}$$

for $m \to 0$, as consequence of (3.20). Putting $w(t) = v(t)\varphi(t)$, we finish the proof of the first part of the proposition. The proof in the case that $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(0,T;X')$ proceeds in almost the same way. □

Proposition 3.8 *Let* $\varphi \in C_0^\infty(0,T)$ *and* $v \in L^p(0,T;H^1(\Omega))$, $p \in [1,\infty]$. *Then* $\overline{v}^\varphi \in H^1(\Omega)$ *and* $\partial_i \overline{v}^\varphi = \overline{\partial_i v}^\varphi$, $i = 1,2,3$.

P r o o f: Using the Fubini theorem we derive that

$$\int_\Omega \partial_i \psi \overline{v}^\varphi \, dx = \int_\Omega \partial_i \psi \int_0^T v(t)\varphi(t) \, dt dx$$

$$= \int_0^T \varphi(t) \int_\Omega v(t) \partial_i \psi \, dx dt = -\int_0^T \varphi(t) \int_\Omega \partial_i v(t) \psi \, dx dt$$

$$= -\int_\Omega \psi \int_0^T \varphi(t) \partial_i v(t) \, dt dx = -\int_\Omega \psi \overline{\partial_i v}^\varphi \, dx$$

for all $\psi \in C_0^\infty(\Omega)$ and $i = 1,2,3$. □

Proposition 3.9 *Let* $\varphi \in C_0^\infty(0,T)$ *and* $v \in L^p(0,T;C([0,l];X))$, *for* $p \in (1,\infty]$. *Then* $\overline{v}^\varphi \in C([0,l];X)$.

P r o o f: We know from the definition of Bochner spaces that

$$v(t) \in C([0,l];X) \text{ for a.a. } t \in [0,T],$$

and thus

$$\lim_{x_1 \to \widehat{x}_1} \|v(t,x_1) - v(t,\widehat{x}_1)\|_X = 0 \text{ for a.a. } t \in (0,T) \tag{3.21}$$

and for some $\widehat{x}_1 \in [0,l]$. Then, using the Vitali theorem (see [9] and (3.21)), we find that

$$\lim_{x_1 \to \widehat{x}_1} \|\overline{v(x_1)}^\varphi - \overline{v(\widehat{x}_1)}^\varphi\|_X = \lim_{x_1 \to \widehat{x}_1} \|\overline{v(x_1) - v(\widehat{x}_1)}^\varphi\|_X$$

$$\leq \lim_{x_1 \to \widehat{x}_1} \int_0^T \varphi(t) \|v(t,x_1) - v(t,\widehat{x}_1)\|_X \, dt$$

$$= \int_0^T \varphi(t) \lim_{x_1 \to \widehat{x}_1} \|v(t,x_1) - v(t,\widehat{x}_1)\|_X \, dt = 0.$$

□

Every function $\mathbf{V} \in H^1(\Omega)^3$ may be represented in the local frame generated by the vectors $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$. So,

$$\mathbf{V}(x) = v_{1,\epsilon}(x)\mathbf{t}_\epsilon(x_1) + v_{2,\epsilon}(x)\mathbf{n}_\epsilon(x_1) + v_{3,\epsilon}(x)\mathbf{b}_\epsilon(x_1), \tag{3.22}$$

where the components of the vector $\mathbf{v}_\epsilon = (v_{1,\epsilon}, v_{2,\epsilon}, v_{3,\epsilon}) \in H^1(\Omega)^3$ are defined by

$$(\mathbf{V}, \mathbf{t}_\epsilon) = v_{1,\epsilon}, \;\; (\mathbf{V}, \mathbf{n}_\epsilon) = v_{2,\epsilon}, \;\; (\mathbf{V}, \mathbf{b}_\epsilon) = v_{3,\epsilon}. \tag{3.23}$$

Using (2.2) together with (3.22), we get similar relations for the derivative ∂_1 of $\mathbf{V}$ having the form

$$(\partial_1 \mathbf{V}(x), \mathbf{t}_\epsilon(x_1)) = \partial_1 v_{1,\epsilon}(x) - \alpha_\epsilon(x_1) v_{3,\epsilon}(x) - \beta_\epsilon(x_1) v_{2,\epsilon}(x), \tag{3.24}$$

$$(\partial_1 \mathbf{V}(x), \mathbf{n}_\epsilon(x_1)) = \partial_1 v_{2,\epsilon}(x) + \beta_\epsilon(x_1) v_{1,\epsilon}(x) + \gamma_\epsilon(x_1) v_{3,\epsilon}(x), \tag{3.25}$$

$$(\partial_1 \mathbf{V}(x), \mathbf{b}_\epsilon(x_1)) = \partial_1 v_{3,\epsilon}(x) + \alpha_\epsilon(x_1) v_{1,\epsilon}(x) - \gamma_\epsilon(x_1) v_{2,\epsilon}(x) \tag{3.26}$$

for a.a. $x \in \Omega$. The following proposition shows that the relations (3.24)–(3.26) remain valid under weaker assumptions on the function $\mathbf{V}$.

Proposition 3.10 [15] *Let $\mathbf{V} \in L^2(\Omega)^3$ and the vector function $\mathbf{v}_\epsilon = (v_{1,\epsilon}, v_{2,\epsilon}, v_{3,\epsilon})$ from (3.23) be such that $\partial_1 \mathbf{v}_\epsilon \in L^2(0,l; H^{-1}(S)^3)$. Then the function $\mathbf{V}$ of the form (3.22) is such that $\partial_1 \mathbf{V} \in L^2(0,l; H^{-1}(S)^3)$ and fulfills the relations (3.24)–(3.26) in the sense of the space $L^2(0,l;H^{-1}(S))$ for all $\epsilon \in (0,1)$.*

Proposition 3.11 [15] *Let $\lambda \geq 0$, $\mu > 0$ and*

$$A_\epsilon^{ijkl} = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu (g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}).$$

Then there exists a constant $C_3 > 0$ such that the estimate

$$\sum_{i,j=1}^{3} |t_{ij}|^2 \leq C_3 A_\epsilon^{ijkl}(x) t_{kl} t_{ij} \tag{3.27}$$

holds for all $x \in \overline{\Omega}$, all $\epsilon \in [0,1]$ and all symmetric matrices $(t_{ij})_{i,j=1}^3$, with the constant C_3 being independent of ϵ and x.

Proposition 3.12 [15] *There exist constant $C_4 > 0$ independent of ϵ such that*

$$\|\mathbf{V}\|_{1,2} \leq \frac{C_4}{\epsilon} \|\omega^\epsilon(\mathbf{V})\|_2, \ \forall \mathbf{V} \in V(\Omega)^3 \ \text{and} \ \forall \epsilon \in (0,1). \tag{3.28}$$

4 Weak formulation of the evolution equation for the curved rods and its transformation

We consider $\widetilde{\Omega}_\epsilon$ defined by mapping $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ (see (2.3)–(2.4)) for $\epsilon \in (0,1)$ arbitrary but fixed as a three-dimensional homogeneous and isotropic elastic body with the Lamé constants $\lambda \geq 0$, $\mu > 0$ and with mass density $\widetilde{\rho}_\epsilon$. Let $\widetilde{\mathbf{F}}_\epsilon$ be the body force and $\widetilde{\mathbf{G}}_\epsilon$ the surface traction acting on the curved rod $\widetilde{\Omega}_\epsilon$ such that $\widetilde{\mathbf{F}}_\epsilon \in L^2(0,T; L^2(\widetilde{\Omega}_\epsilon)^3)$ and $\widetilde{\mathbf{G}}_\epsilon \in W^{1,1}(0,T; L^2((\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S))^3)$, for $\epsilon \in (0,1)$. Let $\widetilde{\Omega}_\epsilon$ be clamped on both bases $\bar{\mathbf{P}}_\epsilon(\{0\} \times \epsilon S)$ and $\bar{\mathbf{P}}_\epsilon(\{l\} \times \epsilon S)$. The equilibrium displacement $\widetilde{\mathbf{U}}_\epsilon$ is the (weak) solution of the equation

$$\begin{aligned} &{}_{[V(\widetilde{\Omega}_\epsilon)^3]'}\langle \widetilde{\rho}_\epsilon \partial_{tt} \widetilde{\mathbf{U}}_\epsilon(t), \widetilde{\mathbf{V}} \rangle_{V(\widetilde{\Omega}_\epsilon)^3} + \int_{\widetilde{\Omega}_\epsilon} \widetilde{A}^{ijkl} e_{kl}(\widetilde{\mathbf{U}}_\epsilon(t)) e_{ij}(\widetilde{\mathbf{V}}) \, d\widetilde{y} \\ &= \int_{\widetilde{\Omega}_\epsilon} (\widetilde{\mathbf{F}}_\epsilon(t), \widetilde{\mathbf{V}}) \, d\widetilde{y} + \int_{\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon((0,l) \times \partial S)} (\widetilde{\mathbf{G}}_\epsilon(t), \widetilde{\mathbf{V}}) \, d\widetilde{S}_\epsilon d\widetilde{y}_1 \end{aligned} \tag{4.1}$$

for all $\widetilde{\mathbf{V}} \in V(\widetilde{\Omega}_\epsilon)^3$ and for almost all $t \in (0,T)$, where $\widetilde{S}_\epsilon = (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S)$, $\widetilde{A}^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $(e_{ij}(\widetilde{\mathbf{V}}))_{i,j=1}^3$ stands for the symmetric part of the gradient of the function $\widetilde{\mathbf{V}}$. The solution $\widetilde{\mathbf{U}}_\epsilon$ satisfies the initial state

$$\widetilde{\mathbf{U}}_\epsilon|_{t=0} = \widetilde{\mathbf{Q}}_{0,\epsilon},\ \ \widetilde{\rho}_\epsilon\partial_t\widetilde{\mathbf{U}}_\epsilon|_{t=0} = \widetilde{\rho}_\epsilon\widetilde{\mathbf{Q}}_{1,\epsilon}. \tag{4.2}$$

From (2.3)–(2.4) and from the regularization of the local frame (see Proposition 3.1), it follows that the mapping $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ is the parametrization of the smooth three-dimensional curved rod.

We transform now the equation (4.1). Denoting $\mathbf{U}_\epsilon = \widetilde{\mathbf{U}}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\rho_\epsilon = \widetilde{\rho}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$ and $\mathbf{V}_\epsilon = \widetilde{\mathbf{V}}(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, we get for arbitrary $\psi \in C_0^\infty(0,T)$ that

$$\begin{aligned}\int_0^T \psi(t)\,{}_{[V(\widetilde{\Omega}_\epsilon)^3]'}\langle\widetilde{\rho}_\epsilon\partial_{tt}\widetilde{\mathbf{U}}_\epsilon(t),\widetilde{\mathbf{V}}\rangle_{V(\widetilde{\Omega}_\epsilon)^3}\ dt &= -\int_0^T \partial_t\psi(t)\int_{\widetilde{\Omega}_\epsilon}\widetilde{\rho}_\epsilon(\partial_t\widetilde{\mathbf{U}}_\epsilon(t),\widetilde{\mathbf{V}})\ d\widetilde{y}dt \\ &= -\int_0^T \partial_t\psi(t)\int_\Omega \rho_\epsilon(\partial_t\mathbf{U}_\epsilon(t),\mathbf{V}_\epsilon)\epsilon^2 d_\epsilon\ dxdt \\ &= \epsilon^2\int_0^T \psi(t)\,{}_{[V(\Omega_\epsilon)^3]'}\langle\rho_\epsilon d_\epsilon\partial_{tt}\mathbf{U}_\epsilon(t),\mathbf{V}_\epsilon\rangle_{V(\Omega_\epsilon)^3}\ dt,\end{aligned}$$

and thus

$${}_{[V(\widetilde{\Omega}_\epsilon)^3]'}\langle\widetilde{\rho}_\epsilon\partial_{tt}\widetilde{\mathbf{U}}_\epsilon(t),\widetilde{\mathbf{V}}\rangle_{V(\widetilde{\Omega}_\epsilon)^3} = \epsilon^2\,{}_{[V(\Omega_\epsilon)^3]'}\langle\rho_\epsilon d_\epsilon\partial_{tt}\mathbf{U}_\epsilon(t),\mathbf{V}_\epsilon\rangle_{V(\Omega_\epsilon)^3} \tag{4.3}$$

for a.a. $t \in (0,T)$. Analogously as in [15], we derive that

$$\int_{\widetilde{\Omega}_\epsilon}\widetilde{A}^{ijkl}e_{kl}(\widetilde{\mathbf{U}}_\epsilon(t))e_{ij}(\widetilde{\mathbf{V}})\ d\widetilde{y} = \epsilon^2\int_\Omega A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(t))\omega_{ij}^\epsilon(\mathbf{V}_\epsilon)d_\epsilon\ dx, \tag{4.4}$$

where

$$A_\epsilon^{ijkl} = \lambda g^{ij,\epsilon}g^{kl,\epsilon} + \mu(g^{ik,\epsilon}g^{jl,\epsilon} + g^{il,\epsilon}g^{jk,\epsilon}), \tag{4.5}$$

$$\int_{\widetilde{\Omega}_\epsilon}(\widetilde{\mathbf{F}}_\epsilon(t),\widetilde{\mathbf{V}})\ d\widetilde{y} = \epsilon^2\int_\Omega(\mathbf{F}_\epsilon(t),\mathbf{V}_\epsilon)d_\epsilon\ dx, \tag{4.6}$$

and

$$\int_{(\bar{\mathbf{P}}_\epsilon\circ\mathbf{R}_\epsilon)((0,l)\times\partial S)}(\widetilde{\mathbf{G}}_\epsilon(t),\widetilde{\mathbf{V}})\ d\widetilde{S}_\epsilon d\widetilde{y}_1 = \epsilon^2\int_{(0,l)\times\partial S}(\mathbf{G}_\epsilon(t),\mathbf{V}_\epsilon)d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}dSdx_1. \tag{4.7}$$

The symmetric tensor $\omega^\epsilon(\mathbf{V})$ has the form

$$\omega^\epsilon(\mathbf{V}) = \frac{1}{\epsilon}\theta^\epsilon(\mathbf{V}) + \kappa^\epsilon(\mathbf{V}), \tag{4.8}$$

where the individual nonzero components of the symmetric tensors θ^ϵ and κ^ϵ are defined by

$$\theta_{12}^\epsilon(\mathbf{V}) = \frac{1}{2}(\partial_2\mathbf{V},\mathbf{g}_{1,\epsilon}),\ \ \theta_{22}^\epsilon(\mathbf{V}) = (\partial_2\mathbf{V},\mathbf{n}_\epsilon),\ \ \theta_{33}^\epsilon(\mathbf{V}) = (\partial_3\mathbf{V},\mathbf{b}_\epsilon), \tag{4.9}$$

$$\theta^\epsilon_{13}(\mathbf{V}) = \frac{1}{2}(\partial_3\mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \ \theta^\epsilon_{23}(\mathbf{V}) = \frac{1}{2}\Big((\partial_2\mathbf{V}, \mathbf{b}_\epsilon) + (\partial_3\mathbf{V}, \mathbf{n}_\epsilon)\Big), \tag{4.10}$$

$$\kappa^\epsilon_{11}(\mathbf{V}) = (\partial_1\mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \ \kappa^\epsilon_{12}(\mathbf{V}) = \frac{1}{2}(\partial_1\mathbf{V}, \mathbf{n}_\epsilon),\ \ \kappa^\epsilon_{13}(\mathbf{V}) = \frac{1}{2}(\partial_1\mathbf{V}, \mathbf{b}_\epsilon), \tag{4.11}$$

where $\mathbf{g}_{1,\epsilon} \to \mathbf{t}$ in measure in Ω and $\mathbf{n}_\epsilon \to \mathbf{n}$, $\mathbf{b}_\epsilon \to \mathbf{b}$ in measure in $(0, l)$ for $\epsilon \to 0$. The other components of θ^ϵ and κ^ϵ are equal to zero.

It is easy to see that if $\widetilde{\mathbf{V}} \in V(\widetilde{\Omega}_\epsilon)^3$, then $\mathbf{V}_\epsilon \in V(\Omega)^3$. Denoting $\mathbf{Q}_{0,\epsilon} = \widetilde{\mathbf{Q}}_{0,\epsilon}(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\mathbf{Q}_{1,\epsilon} = \widetilde{\mathbf{Q}}_{1,\epsilon}(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\mathbf{F}_\epsilon = \widetilde{\mathbf{F}}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$ and $\mathbf{G}_\epsilon = \widetilde{\mathbf{G}}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$ we can rewrite the model (4.1)–(4.2) using (4.3)–(4.7) as

$$\begin{aligned} &{}_{[V(\Omega)^3]'}\langle \rho_\epsilon d_\epsilon \partial_{tt}\mathbf{U}_\epsilon(t), \mathbf{V}\rangle_{V(\Omega)^3} + \int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{U}_\epsilon(t))\omega^\epsilon_{ij}(\mathbf{V}_\epsilon) d_\epsilon\, dx \\ &= \int_\Omega (\mathbf{F}_\epsilon(t), \mathbf{V}) d_\epsilon\, dx + \int_0^l \int_{\partial S} (\mathbf{G}_\epsilon(t), \mathbf{V}) d_\epsilon \sqrt{\nu_i o^{ij,\epsilon}\nu_j}\, dS dx_1 \end{aligned} \tag{4.12}$$

for all $\mathbf{V} \in V(\Omega)^3$ and for almost all $t \in (0, T)$, where the solution $\mathbf{U}_\epsilon$ satisfies the initial state

$$\mathbf{U}_\epsilon|_{t=0} = \mathbf{Q}_{0,\epsilon},\ \ \rho_\epsilon \partial_t \mathbf{U}_\epsilon|_{t=0} = \rho_\epsilon \mathbf{Q}_{1,\epsilon}. \tag{4.13}$$

Assumptions

The following assumptions will be needed throughout the paper:

1. $\rho_\epsilon = \epsilon^2\rho$, where $\rho \in L^\infty(\Omega)$ and
$$0 < C_5 \le \rho \le C_6 \text{ a.e. in } \Omega. \tag{4.14}$$
2. $\mathbf{F}_\epsilon = \epsilon^2\mathbf{F}$, $\mathbf{F} \in L^2(0, T; L^2(\Omega)^3)$, $\mathbf{G}_\epsilon = \epsilon^3\mathbf{G}$, $\mathbf{G} \in W^{1,1}(0, T; L^2(0, l; L^2(\partial S)^3))$.
3. $\{\mathbf{Q}_{0,\epsilon}\}_{\epsilon\in(0,1)} \subset V(\Omega)^3$, $\{\mathbf{Q}_{1,\epsilon}\}_{\epsilon\in(0,1)} \subset L^2(\Omega)^3$,
$$\frac{1}{\epsilon}\|\omega^\epsilon(\mathbf{Q}_{0,\epsilon})\|_2 \le C, \forall \epsilon \in (0, 1), \tag{4.15}$$
where the constant C is independent of ϵ, and
$$\mathbf{Q}_{0,\epsilon} \rightharpoonup \mathbf{Q}_0 \text{ in } V(\Omega)^3,\ \ \mathbf{Q}_{1,\epsilon} \rightharpoonup \mathbf{Q}_1 \text{ in } L^2(\Omega)^3 \tag{4.16}$$
for $\epsilon \to 0$, where $\mathbf{Q}_0 \in H^1_0(0, l)^3$ and $\mathbf{Q}_1 \in L^2(0, l)^3$.

After the substitution of the above assumptions to (4.12)–(4.13) we get

$$\begin{aligned} &{}_{[V(\Omega)^3]'}\langle \rho d_\epsilon \partial_{tt}\mathbf{U}_\epsilon(t), \mathbf{V}\rangle_{V(\Omega)^3} + \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{V}_\epsilon) d_\epsilon\, dx \\ &= \int_\Omega (\mathbf{F}(t), \mathbf{V}) d_\epsilon\, dx + \int_{(0,l)} \int_{\partial S} (\mathbf{G}(t), \mathbf{V}) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon}\nu_j}\, dS dx_1 \end{aligned} \tag{4.17}$$

for all $\mathbf{V} \in V(\Omega)^3$ and for almost all $t \in (0, T)$, and

$$\mathbf{U}_\epsilon|_{t=0} = \mathbf{Q}_{0,\epsilon},\ \ \rho \partial_t \mathbf{U}_\epsilon|_{t=0} = \rho \mathbf{Q}_{1,\epsilon}. \tag{4.18}$$

The existence of the (weak) solution $\mathbf{U}_\epsilon$ to the problem (4.17)–(4.18) and basic estimates are derived in the next section.

5 On the existence of a unique weak solution to (4.17)–(4.18) and basic estimates

Now, we prove the existence and uniqueness of the weak solution to the problem (4.17)–(4.18) and the appropriate estimates.

Proposition 5.1 *Under the assumptions of Section* 4, *there exists a unique weak solution* $\mathbf{U}_\epsilon$ *to the problem* (4.17)–(4.18) *such that* $\mathbf{U}_\epsilon \in L^\infty(0,T;V(\Omega)^3)$, $\partial_t \mathbf{U}_\epsilon \in L^\infty(0,T;L^2(\Omega)^3)$, $\rho\partial_{tt}\mathbf{U}_\epsilon \in L^2(0,T;[V(\Omega)^3]')$, *where the initial conditions in* (4.18) *are fulfilled in the sense of the space* $C([0,T];L^2(\Omega)^3)$ *or* $C([0,T];[V(\Omega)^3]')$, *respectively. In addition, this solution satisfies for all* $\epsilon \in (0,1)$ *the estimates*

$$\|\sqrt{\rho}\partial_t \mathbf{U}_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \|\frac{1}{\epsilon}\omega(\mathbf{U}_\epsilon)\|^2_{L^\infty(0,T;L^2(\Omega)^9)} \leq C\Big(\|\mathbf{Q}_{1,\epsilon}\|_2^2$$

$$+\|\frac{1}{\epsilon}\omega(\mathbf{Q}_{0,\epsilon})\|_2 + \|\mathbf{F}\|^2_{L^2(0,T;L^2(\Omega)^3)} + \|\mathbf{G}\|^2_{W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))}\Big) \tag{5.1}$$

and

$$\|\rho\partial_{tt}\mathbf{U}_\epsilon\|_{L^2(0,T;[V(\Omega)^3]')} \leq C\Big(\|\mathbf{F}\|_{L^2(0,T;L^2(\Omega)^3)}$$

$$+\|\mathbf{G}\|_{L^2(0,T;L^2(0,l;L^2(\partial S)^3))} + \frac{1}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^2(0,T;L^2(\Omega)^9)}\Big), \tag{5.2}$$

where the constant C *is independent of* ϵ.

Before we start to prove Proposition 5.1, we construct a finite dimensional approximation of the weak solution to our problem using analogous arguments as in [4], and [5], and we prove auxiliary lemmas, which enable us to prove Proposition 5.1.

Let $\epsilon \in (0,1)$ be arbitrary but fixed. Since the space $V(\Omega)$ is a separable Hilbert space with the scalar product $((\cdot,\cdot))_{\rho d_\epsilon,\Omega}$ defined by

$$((V,W))_{\rho d_\epsilon,\Omega} = \int_\Omega \rho V W d_\epsilon \, dx + \int_\Omega \rho(\nabla V, \nabla W) d_\epsilon \, dx,$$

we can select smooth functions $W_k = W_k(x)$, $k = 1,2,\dots$, such that

$$\{W_k\}_{k=1}^\infty \text{ is a basis of } V(\Omega) \tag{5.3}$$

and

$$\{W_k\}_{k=1}^\infty \text{ is an orthonormal basis of } L^2(\Omega) \tag{5.4}$$

in the sense of the scalar product $(\cdot,\cdot)_{\rho d_\epsilon,\Omega}$ defined by

$$(V,W)_{\rho d_\epsilon,\Omega} = \int_\Omega \rho V W d_\epsilon \, dx.$$

The proof that the above mentioned scalar products are well-defined follows from Corollary 3.2 and (4.14).

Now, we fix a positive integer m, and we write

$$U^m_{\epsilon,j}(t,x)=\sum_{k=1}^m d^m_{\epsilon,k,j}(t)W_k(x),\ \ Q^m_{0,\epsilon,j}(x)=\sum_{k=1}^m d^m_{\epsilon,k,j}(0)W_k(x), \tag{5.5}$$

$$Q^m_{1,\epsilon,j}(x)=\sum_{k=1}^m \dot d^m_{\epsilon,k,j}(0)W_k(x),$$

where $j=1,2,3$, $(t,x)\in(0,T)\times\Omega$ and $\mathbf{U}^m_\epsilon=(U^m_{\epsilon,1},U^m_{\epsilon,2},U^m_{\epsilon,3})$. We intend to select the coefficients $d^m_{\epsilon,k,j}(t)$, $j=1,2,3$, to satisfy

$$d^m_{\epsilon,k,j}(0)=\int_\Omega \rho Q_{0,\epsilon,j}W_k d_\epsilon\ dx,\ \ \dot d^m_{\epsilon,k,j}(0)=\int_\Omega \rho Q_{1,\epsilon,j}W_k d_\epsilon\ dx \tag{5.6}$$

for $j=1,2,3$, $k=1,\dots,m$. Using the vectors

$$\mathbf{W}^1_k=(W_k,0,0),\ \ \mathbf{W}^2_k=(0,W_k,0),\ \ \mathbf{W}^3_k=(0,0,W_k),$$

we want to prove the existence of the unique solution to the system of equations

$$\int_\Omega \rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t)W_k d_\epsilon\ dx+\int_\Omega A^{ijkl}_\epsilon\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{W}^{\widehat{i}}_k)d_\epsilon\ dx$$

$$=\int_\Omega F_{\widehat{i}}(t)W_k d_\epsilon\ dx+\int_{(0,l)}\int_{\partial S} G_{\widehat{i}}(t)W_k d_\epsilon\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dSdx_1,\ \widehat{i}=1,2,3, \tag{5.7}$$

completed with the initial states

$$d^m_{\epsilon,k,\widehat{i}}(0)=\int_\Omega \rho Q_{0,\epsilon,\widehat{i}}W_k d_\epsilon\ dx,\ \ \dot d^m_{\epsilon,k,\widehat{i}}(0)=\int_\Omega \rho Q_{1,\epsilon,\widehat{i}}W_k d_\epsilon\ dx \tag{5.8}$$

for $j=1,2,3$, $k=1,\dots,m$.

Lemma 5.2 *There exists a unique solution* $\mathbf{U}^m_\epsilon\in W^{2,2}(0,T;V(\Omega)^3)$ *to the equation* (5.7) *satisfying* (5.8) *for each* $m=1,2,\dots$.

P r o o f: In the first step, we rewrite the equations in (5.7) as a system of ordinary differential equations. We start with the second term. Let us take, for instance, $\omega^\epsilon_{23}(\mathbf{U}^m_\epsilon(t))$. Then (using the summation convention except for ϵ)

$$\omega^\epsilon_{23}(\mathbf{U}^m_\epsilon(t))\overset{(4.8)-(4.11)}{=}\frac{1}{\epsilon}\theta^\epsilon_{23}(\mathbf{U}^m_\epsilon(t))\overset{(4.10)}{=}\frac{1}{2\epsilon}\Big((\partial_2\mathbf{U}^m_\epsilon(t),\mathbf{b}_\epsilon)+(\partial_3\mathbf{U}^m_\epsilon(t),\mathbf{n}_\epsilon)\Big)$$

$$\overset{(5.5)}{=}\frac{1}{2\epsilon}\Big(d^m_{\epsilon,k,\widehat{j}}(t)\partial_2W_k b_{\epsilon,\widehat{j}}+d^m_{\epsilon,k,\widehat{j}}(t)\partial_3W_k n_{\epsilon,\widehat{j}}\Big)$$

$$=\frac{1}{2\epsilon}d^m_{\epsilon,k,\widehat{j}}(t)(\partial_2W_k b_{\epsilon,\widehat{j}}+\partial_3W_k n_{\epsilon,\widehat{j}})=d^m_{\epsilon,k,\widehat{j}}(t)B^\epsilon_{23,\widehat{j}}(W_k),$$

where we denote

$$B^\epsilon_{23,\widehat{j}}(W_k) = \frac{1}{2\epsilon}(\partial_2 W_k b_{\epsilon,\widehat{j}} + \partial_3 W_k n_{\epsilon,\widehat{j}}),\ \widehat{j} = 1,2,3.$$

We express analogously the other components of the symmetric tensor $B^\epsilon_{\widehat{j}}(W_k) = (B^\epsilon_{ij,\widehat{j}}(W_k))^3_{i,j=1}$, $\widehat{j} = 1,2,3$, and thus we get that

$$\omega^\epsilon_{ij}(\mathbf{U}^m_\epsilon) = d^m_{\epsilon,k,\widehat{j}}(t) B^\epsilon_{ij,\widehat{j}}(W_k),\ i,j = 1,2,3,$$

where

$$B^\epsilon_{11,\widehat{j}}(W_k) = \partial_1 W_k [\mathbf{g}_{1,\epsilon}]_{\widehat{j}},\ \text{for } \mathbf{g}_{1,\epsilon} = ([\mathbf{g}_{1,\epsilon}]_1, [\mathbf{g}_{1,\epsilon}]_2, [\mathbf{g}_{1,\epsilon}]_3),$$

$$B^\epsilon_{12,\widehat{j}}(W_k) = \frac{1}{2\epsilon}\partial_2 W_k [\mathbf{g}_{1,\epsilon}]_{\widehat{j}} + \frac{1}{2}\partial_1 W_k n_{\epsilon,\widehat{j}},\ B^\epsilon_{13,\widehat{j}}(W_k) = \frac{1}{2\epsilon}\partial_3 W_k [\mathbf{g}_{1,\epsilon}]_{\widehat{j}} + \frac{1}{2}\partial_1 W_k b_{\epsilon,\widehat{j}},$$

$$B^\epsilon_{22,\widehat{j}}(W_k) = \frac{1}{\epsilon}\partial_2 W_k n_{\epsilon,\widehat{j}},\ B^\epsilon_{33,\widehat{j}}(W_k) = \frac{1}{\epsilon}\partial_3 W_k b_{\epsilon,\widehat{j}}$$

$$B^\epsilon_{23,\widehat{j}}(W_k) = \frac{1}{2\epsilon}(\partial_2 W_k b_{\epsilon,\widehat{j}} + \partial_3 W_k n_{\epsilon,\widehat{j}}),\ \widehat{j} = 1,2,3.$$

Hence

$$\int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t)) \frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{W}^{\widehat{i}}_k) d_\epsilon\ dx = d^m_{\epsilon,k,\widehat{j}}(t) \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon} B^\epsilon_{kl,\widehat{j}}(W_k) \frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{W}^{\widehat{i}}_k) d_\epsilon\ dx$$

$$= d^m_{\epsilon,k,\widehat{j}}(t) D^\epsilon_{\widehat{j}}(B^\epsilon_{\widehat{j}}(W_k), \omega^\epsilon(\mathbf{W}^{\widehat{i}}_k) d_\epsilon), \tag{5.9}$$

where

$$D^\epsilon_{\widehat{j}}(B^\epsilon_{\widehat{j}}(W_k), \omega^\epsilon(\mathbf{W}^{\widehat{i}}_k) d_\epsilon) = \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon} B^\epsilon_{kl,\widehat{j}}(W_k) \frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{W}^{\widehat{i}}_k) d_\epsilon\ dx,$$

$\widehat{i} = 1,2,3$ and $\widehat{j} = 1,2,3$. Further, we denote

$$f^{\widehat{i}}_k(t) = \int_\Omega F_{\widehat{i}}(t) W_k d_\epsilon\ dx,\ g^{\widehat{i}}_k(t) = \int_0^l \int_{\partial S} G_{\widehat{i}}(t) W_k d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dS dx_1, \tag{5.10}$$

for $\widehat{i} = 1,2,3$.

Using the fact that the functions W_k, $k = 1,2,\ldots$, are orthonormal in the sense of the scalar product $(\cdot,\cdot)_{\rho d_\epsilon,\Omega}$, together with (5.5), (5.9)–(5.10), we may rewrite the equation (5.7) as a linear system of ODE's having the form

$$\ddot{d}^m_{\epsilon,k,\widehat{i}}(t) + d^m_{\epsilon,k,\widehat{j}}(t) D^\epsilon_{\widehat{j}}(B^\epsilon_{\widehat{j}}(W_k), \omega^\epsilon(\mathbf{W}^{\widehat{i}}_k) d_\epsilon) = f^{\widehat{i}}_k(t) + g^{\widehat{i}}_k(t), \tag{5.11}$$

for $t \in (0,T)$, with the initial state

$$d^m_{\epsilon,k,\widehat{i}}(0) = \int_\Omega \rho Q_{0,\epsilon,\widehat{i}} W_k d_\epsilon\ dx,\ \dot{d}^m_{\epsilon,k,\widehat{i}}(0) = \int_\Omega \rho Q_{1,\epsilon,\widehat{i}}, W_k d_\epsilon\ dx \tag{5.12}$$

for $\widehat{i} = 1,2,3$, $k = 1,\ldots,m$. Owing to standard ODE theory there exist unique functions $d^m_{\epsilon,k,\widehat{i}}(t) \in W^{2,2}(0,T)$, $\widehat{i} = 1,2,3$ and $k = 1,\ldots m$, that satisfy (5.12) and solve (5.11) for almost all $t \in (0,T)$. $\square$

Lemma 5.3 *Under the assumptions of Section 4, the solution to the problem* (5.7)–(5.8) *satisfies the estimates*

$$\|\sqrt{\rho}\partial_t \mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \|\frac{1}{\epsilon}\omega(\mathbf{U}^m_\epsilon)\|^2_{L^\infty(0,T;L^2(\Omega)^9)} \leq C\Big(\|\mathbf{Q}^m_{1,\epsilon}\|^2_2$$

$$+\|\frac{1}{\epsilon}\omega(\mathbf{Q}^m_{0,\epsilon})\|_2 + \|\mathbf{F}\|^2_{L^2(0,T;L^2(\Omega)^3)} + \|\mathbf{G}\|^2_{W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))}\Big) \tag{5.13}$$

and

$$\|\rho\partial_{tt}\mathbf{U}^m_\epsilon\|_{L^2(0,T;[V(\Omega)^3]')} + \|\rho d_\epsilon \partial_{tt}\mathbf{U}^m_\epsilon\|_{L^2(0,T;[V(\Omega)^3]')} \leq C\Big(\|\mathbf{F}\|_{L^2(0,T;L^2(\Omega)^3)}$$

$$+\|\mathbf{G}\|_{L^2(0,T;L^2(0,l;L^2(\partial S)^3))} + \frac{1}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}^m_\epsilon)\|_{L^2(0,T;L^2(\Omega)^9)}\Big), \tag{5.14}$$

where the constant C is independent of ϵ.

P r o o f: We multiply equation (5.7) by $\dot{d}^m_{\epsilon,k,i}(t)$, $\widehat{i} = 1,2,3$, sum $k = 1,\dots,m$ and recall (5.5) to discover (we do not use the summation convention for index $\widehat{i}$ here) that

$$\int_\Omega \rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t)\partial_t U^m_{\epsilon,\widehat{i}}(t) d_\epsilon\, dx + \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\partial_t \widehat{\mathbf{U}}^{m,\widehat{i}}_\epsilon(t)) d_\epsilon\, dx$$

$$= \int_\Omega F_{\widehat{i}}(t)\partial_t U^m_{\epsilon,\widehat{i}}(t) d_\epsilon\, dx + \int_0^l \int_{\partial S} G_{\widehat{i}}(t)\partial_t U^m_{\epsilon,\widehat{i}}(t) d_\epsilon \epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\; dS dx_1,\; \widehat{i} = 1,2,3, \tag{5.15}$$

where

$$\widehat{\mathbf{U}}^{m,1}_\epsilon = (U^m_{\epsilon,1},0,0),\;\; \widehat{\mathbf{U}}^{m,2}_\epsilon = (0,U^m_{\epsilon,2},0),\;\; \widehat{\mathbf{U}}^{m,3}_\epsilon = (0,0,U^m_{\epsilon,3}).$$

We observe that

$$\int_\Omega \rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t)\partial_t U^m_{\epsilon,\widehat{i}}(t) d_\epsilon\, dx = \frac{d}{dt}(\frac{1}{2}\|\sqrt{\rho d_\epsilon}\partial_t U^m_{\epsilon,\widehat{i}}(t)\|^2_2),\; \widehat{i} = 1,2,3, \tag{5.16}$$

$$\begin{aligned}\sum_{\widehat{i}=1}^3 \int_\Omega & A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\partial_t\widehat{\mathbf{U}}^{m,\widehat{i}}_\epsilon(t)) d_\epsilon\, dx \\ &= \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\partial_t \mathbf{U}^m_\epsilon(t)) d_\epsilon\, dx \\ &= \frac{d}{dt}\Big(\frac{1}{2\epsilon^2}\int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\omega^\epsilon_{ij}(\mathbf{U}^m_\epsilon(t)) d_\epsilon\, dx\Big)\end{aligned} \tag{5.17}$$

for a.a $t \in (0,T)$, because the tensor $(A^{ijkl}_\epsilon)^3_{i,j,k,l=1}$ is symmetric. Summing $\widehat{i} = 1,2,3$ in (5.15) and using (5.16)–(5.17), we get the equality

$$\frac{d}{dt}\Big(\frac{1}{2}\|\sqrt{\rho d_\epsilon}\partial_t \mathbf{U}^m_\epsilon(t)\|^2_2 + \frac{1}{2\epsilon^2}\int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\omega^\epsilon_{ij}(\mathbf{U}^m_\epsilon(t)) d_\epsilon\, dx\Big)$$

$$= \int_\Omega (\mathbf{F}(t), \partial_t \mathbf{U}^m_\epsilon(t)) d_\epsilon \, dx + \int_0^l \int_{\partial S} (\mathbf{G}(t), \partial_t \mathbf{U}^m_\epsilon(t)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1. \quad (5.18)$$

Integrating (5.18) over the interval $[0,t]$, $t \in (0,T)$, yields, together with (5.5) and (5.8),

$$\frac{1}{2} \| \sqrt{\rho d_\epsilon} \partial_t \mathbf{U}^m_\epsilon(t) \|_2^2 + \frac{1}{2\epsilon^2} \int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t)) \omega^\epsilon_{ij}(\mathbf{U}^m_\epsilon(t)) d_\epsilon \, dx$$

$$= \frac{1}{2} \| \sqrt{\rho d_\epsilon} \mathbf{Q}^m_{1,\epsilon} \|_2^2 + \frac{1}{2\epsilon^2} \int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{Q}^m_{0,\epsilon}) \omega^\epsilon_{ij}(\mathbf{Q}^m_{0,\epsilon}) d_\epsilon \, dx + \int_0^t \int_\Omega (\mathbf{F}(s), \partial_t \mathbf{U}^m_\epsilon(s)) d_\epsilon \, dx ds$$

$$+ \int_0^t \int_0^l \int_{\partial S} (\mathbf{G}(s), \partial_t \mathbf{U}^m_\epsilon(s)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1 ds \quad (5.19)$$

for all $t \in [0,T]$. Further, we can estimate the third and fourth term on the right-hand side, using Corollary 3.2 and the Young inequality $|ab| \le a^2/2 + b^2/2$, by

$$\left| \int_0^t \int_\Omega (\mathbf{F}(s), \partial_t \mathbf{U}^m_\epsilon(s)) d_\epsilon \, dx ds \right| \le C_1 \int_0^t \int_\Omega \frac{1}{C_7} |\mathbf{F}(s)| C_7 |\partial_t \mathbf{U}^m_\epsilon(s)| \, dx ds$$

$$\le \frac{C_1}{2C_7^2} \int_0^t \|\mathbf{F}(s)\|_2^2 \, ds + \frac{C_1 C_7^2}{2} \int_0^t \|\partial_t \mathbf{U}^m_\epsilon(s)\|_2^2 \, ds$$

$$= \frac{C_1}{2C_7^2} \|\mathbf{F}\|^2_{L^2(0,T;L^2(\Omega)^3)} + T \frac{C_1 C_7^2}{2} \|\partial_t \mathbf{U}_\epsilon{}^m(t)\|^2_{L^\infty(0,T;L^2(\Omega)^3)}, \quad (5.20)$$

and

$$\left| \int_0^t \int_0^l \int_{\partial S} (\mathbf{G}(s), \partial_t \mathbf{U}^m_\epsilon(s)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1 ds \right|$$

$$= \left| \int_0^t \int_0^l \int_{\partial S} \partial_t (\mathbf{G}(s), \mathbf{U}^m_\epsilon(s)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1 ds \right.$$

$$\left. - \int_0^t \int_0^l \int_{\partial S} (\partial_t \mathbf{G}(s), \mathbf{U}^m_\epsilon(s)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1 ds \right|$$

$$= \left| \int_0^l \int_{\partial S} (\mathbf{G}(t), \mathbf{U}^m_\epsilon(t)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1 \right.$$

$$- \int_0^l \int_{\partial S} (\mathbf{G}(0), \mathbf{Q}^m_{0,\epsilon}) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1$$

$$\left. - \int_0^t \int_0^l \int_{\partial S} (\partial_t \mathbf{G}(s), \mathbf{U}^m_\epsilon(s)) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \, dS dx_1 ds \right|$$

$$\le \frac{C_2}{2C_8^2} \|\mathbf{G}\|^2_{L^\infty(0,T;L^2(0,l;L^2(\partial S)^3))} + \frac{C_2 C_8^2}{2} \|\mathbf{U}^m_\epsilon\|_{L^\infty(0,T;L^2(0,l;L^2(\partial S)^3))}$$

$$+ \frac{C_2}{2C_9^2} \|\mathbf{G}(0)\|^2_{L^2(0,l;L^2(\partial S)^3)} + \frac{C_2 C_9^2}{2} \|\mathbf{Q}^m_{0,\epsilon}\|_{L^2(0,l;L^2(\partial S)^3)}$$

$$+ \|\mathbf{U}^m_\epsilon\|_{L^\infty(0,T;L^2(0,l;L^2(\partial S)^3))} \int_0^t \|\partial_t \mathbf{G}(s) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j}\|_{L^2(0,l;L^2(\partial S)^3)} \, ds$$

$$\leq \frac{C_2}{2C_8^2}\|\mathbf{G}\|^2_{L^\infty(0,T;L^2(0,l;L^2(\partial S)^3))} + \frac{C_2C_8^2}{2}\|\mathbf{U}^m_\epsilon\|_{L^\infty(0,T;L^2(0,l;L^2(\partial S)^3))}$$

$$+\frac{C_2}{2C_9^2}\|\mathbf{G}(0)\|^2_{L^2(0,l;L^2(\partial S)^3)} + \frac{C_2C_9^2}{2}\|\mathbf{Q}^m_{0,\epsilon}\|_{L^2(0,l;L^2(\partial S)^3)}$$

$$+\frac{C_2C_{10}^2}{2}\|\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;L^2(0,l;L^2(\partial S)^3))} + \frac{C_2}{2C_{10}^2}\|\partial_t\mathbf{G}\|^2_{L^1(0,T;L^2(0,l;L^2(\partial S)^3))}. \tag{5.21}$$

Let the constant C_{11} comes from the embbeding

$$W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3)) \hookrightarrow C([0,T];L^2(0,l;L^2(\partial S)^3))$$

and the constant C_{12} from

$$H^1(S) \hookrightarrow L^2(\partial S).$$

Then we deduce from (5.18)–(5.21) using Corollary 3.2 and (4.14) that

$$\frac{C_0}{2}\|\sqrt{\rho}\partial_t\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)} - T\frac{C_1C_7^2}{2}\|\partial_t\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)} +$$

$$\Big(\operatorname*{ess\,sup}_{(0,T)} \frac{1}{2\epsilon^2}\int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\omega^\epsilon_{ij}(\mathbf{U}^m_\epsilon(t))d_\epsilon\; dx$$

$$-\frac{C_2C_{12}^2}{2}(C_8^2+C_{10}^2)\|\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\Big) \leq \frac{C_1C_6}{2}\|\mathbf{Q}^m_{1,\epsilon}\|^2_2 + \frac{C_2C_9^2C_{12}^2}{2}\|\mathbf{Q}^m_{0,\epsilon}\|^2_{1,2}$$

$$+\frac{C_1}{2\epsilon^2}\int_\Omega A^{ijkl}_\epsilon\omega^\epsilon_{kl}(\mathbf{Q}^m_{0,\epsilon})\omega^\epsilon_{ij}(\mathbf{Q}^m_{0,\epsilon})\; dx + \frac{C_1}{2C_7^2}\|\mathbf{F}\|^2_{L^2(0,T;L^2(\Omega)^3)}$$

$$+C_2(\frac{C_{11}^2}{2C_8^2} + \frac{C_{11}^2}{2C_9^2} + \frac{1}{2C_{10}^2})\|\mathbf{G}\|^2_{W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))}, \tag{5.22}$$

where the constants C_j, $j = 0,\ldots,12$, do not depend on ϵ. From (4.14), it follows that

$$T\frac{C_1C_7^2}{2}\|\partial_t\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)} \leq T\frac{C_1C_7^2}{2C_5}\|\sqrt{\rho}\partial_t\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)}. \tag{5.23}$$

Further, the estimates (3.27) and (3.28) together with Corollary 3.2 provide

$$\operatorname*{ess\,sup}_{(0,T)} \frac{1}{2\epsilon^2}\int_\Omega A^{ijkl}_\epsilon\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\omega^\epsilon_{ij}(\mathbf{U}^m_\epsilon(t))d_\epsilon\; dx$$

$$\geq \frac{C_0}{2\epsilon^2C_3}\|\omega^\epsilon(\mathbf{U}^m_\epsilon)\|^2_{L^\infty(0,T;L^2(\Omega)^9)}, \tag{5.24}$$

$$\frac{C_2C_{12}^2}{2}(C_8^2+C_{10}^2)\|\mathbf{U}^m_\epsilon\|^2_{L^\infty(0,T;H^1(\Omega)^3)}$$

$$\leq \frac{C_2C_4^2C_{12}^2}{2\epsilon^2}(C_8^2+C_{10}^2)\|\omega^\epsilon(\mathbf{U}^m_\epsilon)\|^2_{L^\infty(0,T;L^2(\Omega)^9)} \tag{5.25}$$

and

$$\frac{C_2C_9^2C_{12}^2}{2}||\mathbf{Q}^m_{0,\epsilon}||^2_{1,2} \le \frac{C_2C_4^2C_9^2C_{12}^2}{2\epsilon^2}||\omega^\epsilon(\mathbf{Q}^m_{0,\epsilon})||^2_{L^2(\Omega)^9} \tag{5.26}$$

In addition, the estimate

$$||(A^{ijkl}_\epsilon)^3_{i,j,k,l=1}||_{C(\overline{\Omega})} \le C_{13} \tag{5.27}$$

holds with the constant C_{13} being independent of ϵ as a consequence of the relations (2.9), (3.5) and (4.5). Hence

$$\frac{C_1}{2\epsilon^2}\int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{Q}^m_{0,\epsilon})\omega^\epsilon_{ij}(\mathbf{Q}^m_{0,\epsilon})\ dx \le \frac{C_1C_{13}}{2\epsilon^2}||\omega^\epsilon(\mathbf{Q}^m_{0,\epsilon})||^2_{L^2(\Omega)^9}. \tag{5.28}$$

The substitution of the inequalities (5.23)–(5.28) to (5.22) leads to the estimate

$$\begin{aligned}
&\Big(\frac{C_0}{2} - T\frac{C_1C_7^2}{2C_5}\Big)||\sqrt{\rho}\partial_t\mathbf{U}^m_\epsilon||^2_{L^\infty(0,T;L^2(\Omega)^3)} \\
&+\frac{C_0 - C_2C_3C_4^2C_{12}^2(C_8^2+C_{10}^2)}{2\epsilon^2C_3}||\omega^\epsilon(\mathbf{U}^m_\epsilon)||^2_{L^\infty(0,T;L^2(\Omega)^9)} \\
&\le \frac{C_1C_6}{2}||\mathbf{Q}^m_{1,\epsilon}||^2_2 + \frac{C_1C_{13}+C_2C_4^2C_9^2C_{12}^2}{2\epsilon^2}||\omega^\epsilon(\mathbf{Q}^m_{0,\epsilon})||^2_2 \\
&+\frac{C_1}{2C_7^2}||\mathbf{F}||^2_{L^2(0,T;L^2(\Omega)^3)} + C_2\Big(\frac{C_{11}^2}{2C_8^2}+\frac{C_{11}^2}{2C_9^2}+\frac{1}{2C_{10}^2}\Big)||\mathbf{G}||^2_{W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))}.
\end{aligned} \tag{5.29}$$

Putting now

$$C_7 = \sqrt{\frac{C_0C_5}{2TC_1}},\ C_{10} = C_8,\ C_8 = \sqrt{\frac{C_0}{4C_2C_3C_4^2C_8^2C_{12}^2}},$$

we conclude (5.13).

It remains to show (5.14). We fix any $V \in V(\Omega)$ such that $\|V\|_{1,2} \le 1$. The function $\frac{V}{d_\epsilon}$ belongs to $V(\Omega)$ for ϵ sufficiently small as well, which is a consequence of Proposition 3.1, Corollary 3.2 and the definitions (2.5), (2.10) of the function d_ϵ. In addition, $\|\frac{V}{d_\epsilon}\|_{1,2} \le C_{14}$, where the constant C_{14} is independent of ϵ (see (2.5), (2.10) and (3.5)). We can decompose this function as a sum

$$\frac{V}{d_\epsilon} = V_1^\epsilon + V_2^\epsilon,$$

where $V_1^\epsilon \in \mathrm{span}\{W_k\}_{k=1}^m$,

$$\int_\Omega \rho V_2^\epsilon W_k d_\epsilon\ dx + \int_\Omega \rho(\nabla V_2^\epsilon, \nabla W_k)d_\epsilon\ dx = 0,\ k = 1, \ldots, m.$$

We can derive from Corollary 3.2 and (4.14) the estimate

$$\|V\|^2_{1,2} \ge \frac{1}{C_1C_6}\int_\Omega [\rho(V^2 + |\nabla V|^2)d_\epsilon]\ dx = \frac{1}{C_1C_6}\int_\Omega \rho[(V_1^\epsilon)^2 + (V_2^\epsilon)^2 + 2V_1^\epsilon V_2^\epsilon]d_\epsilon\ dx$$

$$+\frac{1}{C_1C_6}\int_\Omega \rho[|\nabla V_1^\epsilon|^2+|\nabla V_2^\epsilon|^2+2(\nabla V_1^\epsilon,\nabla V_2^\epsilon)]d_\epsilon\ dx \geq \frac{C_0C_5}{C_1C_6}\|V_1^\epsilon\|_{1,2},$$

and thus $\|V_1^\epsilon\|_{1,2} \leq C_{15} = \frac{C_{14}C_1C_6}{C_0C_5}$, where C_{15} is independent of ϵ. Then (5.5) and (5.7) imply, after the substitution $W_k = V_1^\epsilon$,

$$_{V(\Omega)'}\langle \rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t), V\rangle_{V(\Omega)} = \int_\Omega \rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t)\frac{V}{d_\epsilon}d_\epsilon\ dx = \int_\Omega \rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t)V_1^\epsilon d_\epsilon\ dx$$

$$= \int_\Omega F_{\widehat{i}}(t)V_1^\epsilon d_\epsilon\ dx + \int_0^l\int_{\partial S} G_{\widehat{i}}(t)V_1^\epsilon d_\epsilon \epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dSdx_1$$

$$-\int_\Omega A^{ijkl}_\epsilon\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\frac{\mathbf{V}_1^{\widehat{i},\epsilon}}{d_\epsilon})d_\epsilon\ dx, \tag{5.30}$$

$\widehat{i} = 1,2,3$, where

$$\mathbf{V}_1^{1,\epsilon} = (V_1^\epsilon,0,0),\ \mathbf{V}_1^{2,\epsilon} = (0,V_1^\epsilon,0),\ \mathbf{V}_1^{3,\epsilon} = (0,0,V_1^\epsilon).$$

Since (5.27) and the estimate $\|V_1^\epsilon\|_{1,2} \leq C_{15}$ imply that

$$\int_\Omega A^{ijkl}_\epsilon\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}^m_\epsilon(t))\frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{V}_1^{\widehat{i},\epsilon})d_\epsilon\ dx \leq \frac{C}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}^m_\epsilon(t))\|_2,\ \text{for a.a. } t\in(0,T),$$

where the constant C is independent of ϵ, we get the estimate

$$|\sum_{\widehat{i}=1}^3 {}_{V(\Omega)'}\langle\rho\partial_{tt}U^m_{\epsilon,\widehat{i}}(t),V\rangle_{V(\Omega)}| \leq C\Big(\|\mathbf{F}(t)\|_2$$

$$+\|\mathbf{G}(t)\|_{L^2(0,l;L^2(\partial S)^3)} + \frac{1}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}^m_\epsilon(t))\|_2\Big),\ \text{for a.a. } t\in(0,T), \tag{5.31}$$

where the constant C is independent of ϵ. Taking the function V instead of $\frac{V}{d_\epsilon}$, and using the same procedure as above for the term $\rho d_\epsilon\partial_{tt}\mathbf{U}^m_\epsilon$, we derive (5.14). □

P r o o f of Proposition 5.1: Using (4.14), (5.3)–(5.6), we can easily derive that

$$\mathbf{Q}^m_{0,\epsilon}\to\mathbf{Q}_{0,\epsilon}\ \text{in } V(\Omega)^3,\ \rho\mathbf{Q}^m_{1,\epsilon}\to\rho\mathbf{Q}_{1,\epsilon}\ \text{in } L^2(\Omega)^3. \tag{5.32}$$

From the estimates (3.28) and (5.13), it follows (passing to a subsequence if necessary) that

$$\mathbf{U}^m_\epsilon \overset{*}{\rightharpoonup} \mathbf{U}_\epsilon\ \text{in } L^\infty(0,T;H^1(\Omega)^3), \tag{5.33}$$

$$\partial_t\mathbf{U}^m_\epsilon \overset{*}{\rightharpoonup} \partial_t\mathbf{U}_\epsilon\ \text{in } L^\infty(0,T;L^2(\Omega)^3), \tag{5.34}$$

$$\rho\partial_t\mathbf{U}^m_\epsilon \overset{*}{\rightharpoonup} \rho\partial_t\mathbf{U}_\epsilon\ \text{in } L^\infty(0,T;L^2(\Omega)^3), \tag{5.35}$$

$$\rho\partial_{tt}\mathbf{U}^m_\epsilon \rightharpoonup \mathbf{W}_\epsilon\ \text{in } L^2(0,T;H^{-1}(\Omega)^3) \tag{5.36}$$

$$\rho d_\epsilon\partial_{tt}\mathbf{U}^m_\epsilon \rightharpoonup \widehat{\mathbf{W}}_\epsilon\ \text{in } L^2(0,T;H^{-1}(\Omega)^3) \tag{5.37}$$

for $m \to \infty$. It remains to show that $\mathbf{W}_\epsilon = \rho\partial_{tt}\mathbf{U}_\epsilon$ and $\widehat{\mathbf{W}}_\epsilon = \rho d_\epsilon \partial_{tt}\mathbf{U}_\epsilon$. From (5.35), it follows that

$$\rho\partial_{tt}\mathbf{U}_\epsilon^m \rightharpoonup \rho\partial_{tt}\mathbf{U}_\epsilon,\ \rho d_\epsilon\partial_{tt}\mathbf{U}_\epsilon^m \rightharpoonup \rho d_\epsilon\partial_{tt}\mathbf{U}_\epsilon,\ \text{in } W^{-1,2}(0,T;L^2(\Omega)^3)$$

for $m \to \infty$, which leads to the desired conclusion. The estimates (5.1)–(5.2) immediately follow from (5.13)–(5.14) and (5.32)–(5.37). Using the standard theorems about compact imbeddings in Bochner's spaces, see [12] together with (5.33)–(5.34) and (5.35)–(5.36), we can deduce that

$$\mathbf{U}_\epsilon^m \to \mathbf{U}_\epsilon \text{ in } C^\infty([0,T];L^2(\Omega)^3)$$

and

$$\rho\partial_t\mathbf{U}_\epsilon^m \to \rho\partial_t\mathbf{U}_\epsilon \text{ in } C([0,T];[V(\Omega)^3]')$$

for $m \to \infty$. The uniqueness of the solution follows from the linearity of the equation (4.17) and the estimate (5.1). □

Corollary 5.4 *Under the assumptions of Proposition* 5.1, *there exists a sequence* $\{\epsilon_n\}_{n=1}^\infty \subset (0,1)$ *such that* $\epsilon_n \to 0$ *and*

$$\mathbf{U}_{\epsilon_n} \overset{*}{\rightharpoonup} \mathbf{U} \text{ in } L^\infty(0,T;H^1(\Omega)^3), \tag{5.38}$$

$$\partial_t\mathbf{U}_{\epsilon_n} \overset{*}{\rightharpoonup} \partial_t\mathbf{U} \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{5.39}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(0,T;L^2(\Omega)^9) \tag{5.40}$$

for $\epsilon_n \to 0$.

P r o o f: The proof follows immediately from the estimate (5.1). □

Corollary 5.5 *Let* $\varphi \in C_0^\infty(0,T)$ *and the assumptions of Proposition* 5.1 *be fulfilled. Then*

$$\overline{\mathbf{U}_{\epsilon_n}}^\varphi \rightharpoonup \overline{\mathbf{U}}^\varphi \text{ in } H^1(\Omega)^3, \tag{5.41}$$

$$\overline{\partial_t\mathbf{U}_{\epsilon_n}}^\varphi \rightharpoonup \overline{\partial_t\mathbf{U}}^\varphi \text{ in } L^2(\Omega)^3, \tag{5.42}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\overline{\mathbf{U}_{\epsilon_n}}^\varphi) = \frac{1}{\epsilon_n}\overline{\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n})}^\varphi \rightharpoonup \overline{\zeta}^\varphi \text{ in } L^2(\Omega)^9, \tag{5.43}$$

for $\epsilon_n \to 0$.

P r o o f: The proof is a consequence of (5.38)–(5.40) and Proposition 3.7. □

6 Qualitative properties of the limit displacement

Proposition 6.1 *Suppose that* $\{\epsilon_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\epsilon_n \to 0$. *Let, in addition, a sequence* $\{\mathbf{U}_{\epsilon_n}\}_{n=1}^{\infty} \subset L^{\infty}(0,T;V(\Omega)^3)$ *be such that*

$$\mathbf{U}_{\epsilon_n} \overset{*}{\rightharpoonup} \mathbf{U} \text{ in } L^{\infty}(0,T;H^1(\Omega)^3), \tag{6.1}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \overset{*}{\rightharpoonup} \zeta \text{ in } L^{\infty}(0,T;L^2(\Omega)^9), \tag{6.2}$$

for $\epsilon_n \to 0$. *Then the couple* $\lfloor \mathbf{U}, \phi \rfloor \in L^{\infty}(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ *(in the sense* $\partial_j \mathbf{U} = 0$, $j = 2,3$*), where the function* ϕ *is such that*

$$\frac{1}{2\epsilon_n}\Big((\partial_2 \mathbf{U}_{\epsilon_n}, \mathbf{b}_{\epsilon_n}) - (\partial_3 \mathbf{U}_{\epsilon_n}, \mathbf{n}_{\epsilon_n})\Big) \overset{*}{\rightharpoonup} \phi \tag{6.3}$$

in $L^{\infty}(0,T;L^2(\Omega))$ *for* $\epsilon_n \to 0$. *In addition, the couple* $\lfloor \mathbf{U}, \phi \rfloor$ *generates a function* $\mathbf{U}_* \in L^{\infty}(0,T;H_0^1(0,l)^3)$ *which together with the function* $\mathbf{U}$ *satisfies the relations*

$$(\partial_1 \mathbf{U}, \mathbf{t}) = 0 \text{ a.e. in } (0,T)\times(0,l), \tag{6.4}$$

$$(\partial_1 \mathbf{U}_*, \mathbf{t}) = \partial_3\zeta_{12} - \partial_2\zeta_{13} \text{ in } L^{\infty}(0,T;L^2(0,l;H^{-1}(S))), \tag{6.5}$$

$$(\partial_1 \mathbf{U}_*, \mathbf{n}) = -\partial_3\zeta_{11} \text{ a.e. in } (0,T)\times(0,l), \tag{6.6}$$

$$(\partial_1 \mathbf{U}_*, \mathbf{b}) = \partial_2\zeta_{11} \text{ a.e. in } (0,T)\times(0,l). \tag{6.7}$$

Remark 6.2 Since $\overline{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n})}^{\varphi} = \frac{1}{\epsilon_n}\omega^{\epsilon_n}(\overline{\mathbf{U}_{\epsilon_n}}^{\varphi})$ (see (4.8)–(4.11)), we can use (5.41), (5.43) and Proposition 7.2 from [15] to derive the existence of the pair $\lfloor \overline{\mathbf{U}}^{\varphi}, \phi_{\varphi} \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (in the sense $\partial_j \overline{\mathbf{U}}^{\varphi} = 0$, $j = 2,3$) for arbitrary $\varphi \in C_0^{\infty}(0,T)$, where the function ϕ_{φ} is such that

$$\frac{1}{2\epsilon_n}\Big((\partial_2 \overline{\mathbf{U}_{\epsilon_n}}^{\varphi}, \mathbf{b}_{\epsilon_n}) - (\partial_3 \overline{\mathbf{U}_{\epsilon_n}}^{\varphi}, \mathbf{n}_{\epsilon_n})\Big) \rightharpoonup \phi_{\varphi} \tag{6.8}$$

in $L^2(\Omega)$ for $\epsilon_n \to 0$ and for arbitrary $\varphi \in C_0^{\infty}(0,T)$. In addition, the couple $\lfloor \overline{\mathbf{U}}^{\varphi}, \phi_{\varphi} \rfloor$ generates the function $\mathbf{U}_{*,\varphi} \in H_0^1(0,l)^3$ which together with the function $\overline{\mathbf{U}}^{\varphi}$ satisfies the relations

$$(\partial_1 \overline{\mathbf{U}}^{\varphi}, \mathbf{t}) = 0 \text{ a.e. in } (0,l), \tag{6.9}$$

$$(\partial_1 \mathbf{U}_{*,\varphi}, \mathbf{t}) = \overline{\partial_3\zeta_{12} - \partial_2\zeta_{13}}^{\varphi} \text{ in } L^2(0,l;H^{-1}(S)), \tag{6.10}$$

$$(\partial_1 \mathbf{U}_{*,\varphi}, \mathbf{n}) = \overline{-\partial_3\zeta_{11}}^{\varphi} \text{ a.e. in } (0,l), \tag{6.11}$$

$$(\partial_1 \mathbf{U}_{*,\varphi}, \mathbf{b}) = \overline{\partial_2\zeta_{11}}^{\varphi} \text{ a.e. in } (0,l), \tag{6.12}$$

for arbitrary $\varphi \in C_0^{\infty}(0,T)$. If the sequence $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\overline{\mathbf{U}_{\epsilon_n}}^{\varphi})\}_{n=1}^{\infty}$ converges strongly in $L^2(\Omega)^9$, then the convergence of the sequence $\{\overline{\mathbf{U}_{\epsilon_n}}^{\varphi}\}_{n=1}^{\infty}$ is strong as well for arbitrary $\varphi \in C_0^{\infty}(0,T)$.

Remark 6.3 From Remark 6.2, it follows that to prove Proposition 6.1 we must check that

$$\phi_\varphi(x_1) = \overline{\phi}^\varphi(x_1) \text{ and } \mathbf{U}_{*,\varphi}(x_1) = \overline{\mathbf{U}_*}^\varphi(x_1) \tag{6.13}$$

for all $\varphi \in C_0^\infty(0,T)$ and for a.a. $x_1 \in (0,l)$.

The proof of Proposition 6.1 is decomposed into the following lemmas and corollaries.

Lemma 6.4 *Under the assumptions in Proposition* 6.1 *the following convergences hold true:*

$$\frac{1}{\epsilon^q}\theta^\epsilon(\mathbf{U}_\epsilon) \to 0 \textit{ in } L^\infty(0,T;L^2(\Omega)^9),\ q \in [0,1), \tag{6.14}$$

$$\left(\frac{1}{\epsilon^2}\theta^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\kappa^\epsilon(\mathbf{U}_\epsilon)\right) \stackrel{*}{\rightharpoonup} \zeta \textit{ in } L^\infty(0,T;L^2(\Omega)^9). \tag{6.15}$$

P r o o f: We can observe that the $*$–weak convergences (6.1) and (6.2) together with (4.8)–(4.11) imply the boundedness of the set of the tensors $\{\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\}_{\epsilon\in(0,1)}$ and $\{\kappa^\epsilon(\mathbf{U}_\epsilon)\}_{\epsilon\in(0,1)}$ in $L^\infty(0,T;L^2(\Omega)^9)$. Using these facts, we can easily deduce (6.14). (6.15) immediately follows from (6.2) and (4.8). □

Corollary 6.5 *Under hypotheses* (6.1)–(6.2) *we have:*

$$\frac{1}{\epsilon^q}(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to 0,\ (\partial_2\mathbf{U}, \mathbf{t}) = 0, \tag{6.16}$$

$$\frac{1}{\epsilon^q}(\partial_3\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to 0,\ (\partial_3\mathbf{U}, \mathbf{t}) = 0, \tag{6.17}$$

$$\frac{1}{\epsilon^q}(\partial_1\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to 0,\ (\partial_1\mathbf{U}, \mathbf{t}) = 0, \tag{6.18}$$

$$\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\right) \to 0, \tag{6.19}$$

$$\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\right) \to 0, \tag{6.20}$$

in $L^\infty(0,T;L^2(\Omega))$ *for* $q \in [0,1)$ *and* $\epsilon \to 0$,

$$\partial_j\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\right) \to 0,\ j = 2,3, \tag{6.21}$$

$$\partial_j\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\right) \to 0,\ j = 2,3, \tag{6.22}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)))$ *for* $\epsilon \to 0$ *and* $q \in [0,1)$,

$$\frac{1}{\epsilon^{q_1}}(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \to 0,\ (\partial_2\mathbf{U}, \mathbf{n}) = 0, \tag{6.23}$$

$$\frac{1}{\epsilon^{q_1}}(\partial_3 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) \to 0, \ (\partial_3 \mathbf{U}, \mathbf{b}) = 0, \tag{6.24}$$

$$\frac{1}{\epsilon^{q_1}}\Big((\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big) \to 0, \ (\partial_2 \mathbf{U}, \mathbf{b}) + (\partial_3 \mathbf{U}, \mathbf{n}) = 0, \tag{6.25}$$

in $L^\infty(0,T; L^2(\Omega))$ *for* $q_1 \in [0,2)$ *and* $\epsilon \to 0$, *and*

$$\frac{1}{\epsilon^{q_2}}(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \to 0 \ \textit{in} \ L^\infty(0,T; L^2(\Omega)), \ j = 2,3, \tag{6.26}$$

$$\frac{1}{\epsilon^{q_2}}(\partial_1 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \to 0 \ \textit{in} \ L^\infty(0,T; L^2(\Omega)), \tag{6.27}$$

for $q_2 \in [0, 1-r)$, $r \in (0, \frac{1}{3})$, *and* $\epsilon \to 0$.

P r o o f: We can easily derive from (6.14)–(6.15) and (4.8)–(4.11) the convergences (6.16)–(6.20) and (6.23)–(6.25). It remains to prove the associated equalities. Since from Corollary 7.4 in [15] it follows that

$$0 = (\partial_2 \overline{\mathbf{U}}^\varphi, \mathbf{t}) = \overline{(\partial_2 \mathbf{U}, \mathbf{t})}^\varphi \text{ in } \Omega, \ \forall \varphi \in C_0^\infty(0,T),$$

we get $(\partial_2 \mathbf{U}, \mathbf{t}) = 0$ a.e. in $(0,T) \times \Omega$. The convergence (6.21) follows from the estimate

$$\|\partial_j \frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\right)\|_{L^2(0,l;H^{-1}(S))}$$

$$= \left(\int_0^l |\sup_{\psi \in H_0^1(S), \|\psi\|_{1,2} \le 1} \langle \partial_j \frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\right), \psi\rangle|^2 \, dx_1\right)^{\frac{1}{2}}$$

$$\le \|\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\right)\|_2, \text{ for a.a. } t \in (0,T),$$

and from (6.19). The convergence (6.22) can be obtained analogously from (6.20).

Further, we can derive from (2.6) that

$$(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) = (\partial_j \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + \epsilon\beta_\epsilon x_2(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + \epsilon\alpha_\epsilon x_3(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)$$

$$-\epsilon\gamma_\epsilon x_3(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + \epsilon\gamma_\epsilon x_2(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon), \ j = 2,3, \quad \text{in } (0,T) \times \Omega.$$

Hence, and from (3.5), we get the estimate

$$(1 - C\epsilon^{1-r})\|(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} \le \|(\partial_j \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\|_{L^\infty(0,T;L^2(\Omega))}$$

$$+C\epsilon^{1-r}(\|(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))}),$$

which together with (6.1), (6.16)–(6.17) and the fact that $r \in (0, \frac{1}{3})$ lead to (6.26). The convergence (6.27) can be proved analogously, and we omit its proof. □

Lemma 6.6 *Under the assumptions of Proposition* 6.1, *we have that* $\mathbf{U} \in L^\infty(0,T; H_0^1(0,l)^3)$ (*in the sense* $\partial_j \mathbf{U} = 0$, $j = 2,3$) *and satisfies the relation* (6.4).

P r o o f: We know from (6.1) that $\mathbf{U} \in L^\infty(0,T;V(\Omega)^3)$, and from Remark 6.2 and (2.17) that $\overline{\mathbf{U}}^\varphi \in H_0^1(0,l)^3$ for all $\varphi \in C_0^\infty(0,T)$. Let us suppose that there exist two points $\lfloor x_2^j, x_3^j \rfloor \in S$, $j = 1,2$, such that for $x_1 \in I_{x_1} \subset (0,l)$ and $t \in I_t \subset (0,T)$, where $|I_{x_1}| \neq 0$ and $|I_t| \neq 0$,

$$\mathbf{U}(t,x_1,x_2^1,x_3^1) \neq \mathbf{U}(t,x_1,x_2^2,x_3^2).$$

Then

$$\begin{aligned} 0 &= \overline{\mathbf{U}(x_1,x_2^1,x_3^1)}^\varphi - \overline{\mathbf{U}(x_1,x_2^2,x_3^2)}^\varphi \\ &= \int_0^T (\mathbf{U}(t,x_1,x_2^1,x_3^1) - \mathbf{U}(t,x_1,x_2^2,x_3^2))\varphi(t)\ dt \end{aligned}$$

for all $\varphi \in C_0^\infty(0,T)$ and for a.a. $x_1 \in I_{x_1}$, which implies that

$$\mathbf{U}(t,x_1,x_2^1,x_3^1) = \mathbf{U}(t,x_1,x_2^2,x_3^2)$$

for a.a. $t \in I_t$ and $x_1 \in I_{x_1}$, a contradiction. (6.4) can be derived from the relation (6.27) for $q_2 = 0$. □

In the following lemmas and corollaries, we construct the function ϕ from Proposition 6.1, we show that $\lfloor \mathbf{U}, \phi \rfloor \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)^3)$, and we derive the equations (6.5)–(6.7). But first we introduce the following notation. Let the functions $\mathbf{U}_\epsilon \in L^\infty(0,T;V(\Omega)^3)$, $\epsilon \in (0,1)$, be the functions from Proposition 6.1. We define auxiliary functions ϕ_ϵ, $\epsilon \in (0,1)$, by the relation

$$\phi_\epsilon = \frac{1}{2\epsilon}\Big((\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - (\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big). \tag{6.28}$$

Further, we define the vector functions $\mathbf{u}_{*,\epsilon} = (u_{*,1}^\epsilon, u_{*,2}^\epsilon, u_{*,3}^\epsilon)$ by

$$u_{*,1}^\epsilon = -\phi_\epsilon,\ u_{*,2}^\epsilon = -\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}),\ u_{*,3}^\epsilon = \frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}), \tag{6.29}$$

and the vector functions $\mathbf{U}_{*,\epsilon}$, $\epsilon \in (0,1)$, by

$$\mathbf{U}_{*,\epsilon} = -\phi_\epsilon \mathbf{t}_\epsilon - \frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{n}_\epsilon + \frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{b}_\epsilon. \tag{6.30}$$

Lemma 6.7 *We have*

$$\partial_j \phi_\epsilon \to 0 \ \textit{in}\ L^\infty(0,T;L^2(0,l;H^{-1}(S))),\ j = 2,3, \tag{6.31}$$

for $\epsilon \to 0$, *and* $\phi_\epsilon(t)|_{x_1=0} = \phi_\epsilon(t)|_{x_1=l} = 0$ *for all* $\epsilon \in (0,1)$ *and for almost all* $t \in (0,T)$ *in the sense of the space* $C([0,l];H^{-1}(S))$.

P r o o f: Since $\mathbf{U}_\epsilon(t) \in V(\Omega)^3$ for almost all $t \in (0,T)$, then Proposition 3.5 and (6.28) together with the fact that $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon \in C^\infty([0,l])^3$ imply that $\phi_\epsilon(t)|_{x_1=0} = \phi_\epsilon(t)|_{x_1=l} = 0$ for all $\epsilon \in (0,1)$ and for almost all $t \in (0,T)$ in the sense of the space $C([0,l];H^{-1}(S))$.

Further, we can express the functions $\partial_2 \phi_\epsilon(t)$ for almost all $t \in (0,T)$ in this way:

$$\partial_2 \phi_\epsilon(t) = \frac{1}{2\epsilon}\Big(\partial_2(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon) - \partial_2(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\Big)$$

$$= \frac{1}{2\epsilon}\Big(\partial_2(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon) + \partial_2(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\Big) - \frac{1}{\epsilon}\partial_3(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)$$

in $L^2(0,l;H^{-1}(S))$ (see Proposition 3.5). Since the estimate

$$\|\partial_j v(t)\|_{L^2(0,l;H^{-1}(S))} = \left(\int_0^l \sup_{\psi \in H_0^1(S), \|\psi\|_{1,2} \le 1} \Big| \int_S v(t) \partial_j \psi \; dx_2 dx_3 \Big|^2 \; dx_1\right)^{\frac{1}{2}} \le \|v(t)\|_2$$

holds for almost all $t \in (0,T)$ and $j = 2,3$, we can apply (6.23)–(6.25) for $q_1 = 1$, $v(t) = \frac{1}{2\epsilon}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon) + (\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)$, $v(t) = \frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)$, and we obtain the convergence (6.31) for $j = 2$. The proof of the convergence (6.31) for $j = 3$ proceeds in almost the same way. □

Lemma 6.8 *Let the assumptions of Proposition* 6.1 *be fulfilled. Then*

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) \overset{*}{\rightharpoonup} \partial_3 \zeta_{12} - \partial_2 \zeta_{13} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S))), \tag{6.32}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon) \overset{*}{\rightharpoonup} \partial_2 \zeta_{11} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S))), \tag{6.33}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon) \overset{*}{\rightharpoonup} -\partial_3 \zeta_{11} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S))), \tag{6.34}$$

and thus

$$\partial_1 \mathbf{U}_{*,\epsilon} \overset{*}{\rightharpoonup} (\partial_3 \zeta_{12} - \partial_2 \zeta_{13})\mathbf{t} - \partial_3 \zeta_{11} \mathbf{n} + \partial_2 \zeta_{11} \mathbf{b} \tag{6.35}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)^3))$ *for* $\epsilon \to 0$.

P r o o f: From (6.15) and (4.8)–(4.11), it follows that

$$\frac{1}{\epsilon^2}\partial_3 \theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3 \kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2 \theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2 \kappa_{13}^\epsilon(\mathbf{U}_\epsilon) \overset{*}{\rightharpoonup} \partial_3 \zeta_{12} - \partial_2 \zeta_{13} \tag{6.36}$$

and

$$\frac{\partial_j \kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon} \overset{*}{\rightharpoonup} \partial_j \zeta_{11}, \; j = 2,3, \tag{6.37}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)))$ for $\epsilon \to 0$. Thus to prove (6.32)–(6.34) it is enough to check that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) - \Big(\frac{1}{\epsilon^2}\partial_3 \theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3 \kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2 \theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2 \kappa_{13}^\epsilon(\mathbf{U}_\epsilon)\Big) \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{6.38}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon) - \frac{\partial_2 \kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon} \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{6.39}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon) + \frac{\partial_3 \kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon} \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{6.40}$$

First, we find expressions for the terms $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)$, $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon)$ and $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon)$. Using the definition (4.8)–(4.11) of the tensors θ^ϵ and κ^ϵ, we can derive analogously as in [15] Lemma 7.7 that

$$\frac{1}{\epsilon^2}\partial_3\theta^\epsilon_{12}(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa^\epsilon_{12}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta^\epsilon_{13}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa^\epsilon_{13}(\mathbf{U}_\epsilon)$$
$$= \frac{1}{2\epsilon}\Big(\partial_1(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) - \partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big)$$
$$+\frac{1}{\epsilon}\Bigg(\beta_\epsilon(\partial_3\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - \alpha_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + \gamma_\epsilon\Big((\partial_3\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)\Bigg)$$

in $L^\infty(0,T;H^{-1}(\Omega))$. By rewriting the above mentioned expression in such a way that it involves the terms $\frac{1}{\epsilon}\beta_\epsilon(\partial_3\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})$ and $\frac{1}{\epsilon}\alpha_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})$ instead of $\frac{1}{\epsilon}\beta_\epsilon(\partial_3\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)$ and $\frac{1}{\epsilon}\alpha_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)$, we conclude that

$$\frac{1}{\epsilon^2}\partial_3\theta^\epsilon_{12}(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa^\epsilon_{12}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta^\epsilon_{13}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa^\epsilon_{13}(\mathbf{U}_\epsilon)$$
$$= \Big(-\partial_1\phi_\epsilon + \frac{1}{\epsilon}\beta_\epsilon(\partial_3\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) - \frac{1}{\epsilon}\alpha_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\Big)$$
$$+\Big((\beta_\epsilon^2 x_2 + \alpha_\epsilon\beta_\epsilon x_3)(\partial_3\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - (\alpha_\epsilon\beta_\epsilon x_2 + \alpha_\epsilon^2 x_3)(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\Big)$$
$$+\Big((\beta_\epsilon\gamma_\epsilon x_2 + \frac{\gamma_\epsilon}{\epsilon})(\partial_3\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\alpha_\epsilon\gamma_\epsilon x_3 + \frac{\gamma_\epsilon}{\epsilon})(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)$$
$$-\Big(\beta_\epsilon\gamma_\epsilon x_3(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + \alpha_\epsilon\gamma_\epsilon x_2(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big) \tag{6.41}$$

in $L^\infty(0,T;H^{-1}(\Omega))$. In addition, since all terms except $\partial_1\phi_\epsilon$ belong to the space $L^\infty(0,T;L^2(0,l;H^{-1}(S)))$, then $\partial_1\phi_\epsilon \in L^\infty(0,T;L^2(0,l;H^{-1}(S)))$, as well. From (6.30), (6.41), it follows that

$$(\partial_1\mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) = \Big(\frac{1}{\epsilon^2}\partial_3\theta^\epsilon_{12}(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa^\epsilon_{12}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta^\epsilon_{13}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa^\epsilon_{13}(\mathbf{U}_\epsilon)\Big)$$
$$-\Big((\beta_\epsilon^2 x_2 + \alpha_\epsilon\beta_\epsilon x_3)(\partial_3\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - (\alpha_\epsilon\beta_\epsilon x_2 + \alpha_\epsilon^2 x_3)(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\Big)$$
$$-\Big((\beta_\epsilon\gamma_\epsilon x_2 + \frac{\gamma_\epsilon}{\epsilon})(\partial_3\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\alpha_\epsilon\gamma_\epsilon x_3 + \frac{\gamma_\epsilon}{\epsilon})(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)$$
$$+\Big(\beta_\epsilon\gamma_\epsilon x_3(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + \alpha_\epsilon\gamma_\epsilon x_2(\partial_2\mathbf{U}_\epsilon \mathbf{b}_\epsilon)\Big) \tag{6.42}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)))$. We can use the same procedure as in [15] Lemma 7.7 for the derivation of the relations

$$(\partial_1\mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon) = \Big(\frac{\partial_2\kappa^\epsilon_{11}(\mathbf{U}_\epsilon)}{\epsilon} + \frac{\alpha_\epsilon}{\epsilon}\Big(\frac{(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)}{2}\Big)\Big)$$
$$+\gamma_\epsilon\Big(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big)$$

$$-\Big((-\frac{\beta_\epsilon}{\epsilon}+\beta_\epsilon^2 x_2+\alpha_\epsilon\beta_\epsilon x_3-\gamma_\epsilon' x_3+\gamma_\epsilon^2 x_2)(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)$$
$$-\Big((\beta_\epsilon' x_2+\alpha_\epsilon' x_3+\beta_\epsilon\gamma_\epsilon x_3-\alpha_\epsilon\gamma_\epsilon x_2)(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big)$$
$$-\Big((\alpha_\epsilon\beta_\epsilon x_2+\alpha_\epsilon^2 x_3+\gamma_\epsilon^2 x_3+\gamma_\epsilon' x_2)(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-\beta_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big) \tag{6.43}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)))$, and

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{n}_\epsilon)=\Big(-\frac{\partial_3\kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon}+\frac{\beta_\epsilon}{\epsilon}\Big(\frac{(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)}{2}\Big)\Big)$$
$$+\gamma_\epsilon\Big(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)$$
$$+\Big((-\frac{\alpha_\epsilon}{\epsilon}+\alpha_\epsilon^2 x_3+\alpha_\epsilon\beta_\epsilon x_2+\gamma_\epsilon' x_2+\gamma_\epsilon^2 x_3)(\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\Big)$$
$$+\Big((\beta_\epsilon' x_2+\alpha_\epsilon' x_3+\beta_\epsilon\gamma_\epsilon x_3-\alpha_\epsilon\gamma_\epsilon x_2)(\partial_3\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big)$$
$$+\Big((\alpha_\epsilon\beta_\epsilon x_3+\beta_\epsilon^2 x_2+\gamma_\epsilon^2 x_2-\gamma_\epsilon' x_3)(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)-\alpha_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big) \tag{6.44}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)))$.

Now, we check the convergence (6.38). The convergences (6.39)–(6.40) can be proved analogously. From (6.42) and the facts that $\mathbf{U}_\epsilon\in L^\infty(0,T;V(\Omega)^3)$, α_ϵ, β_ϵ, $\gamma_\epsilon\in C^\infty([0,l])$, $\mathbf{g}_{1,\epsilon}\in C^\infty(\overline{\Omega})^3$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon\in C^\infty([0,l])^3$, it follows that the difference

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)-\Big(\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon)+\frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon)-\frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon)-\frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon)\Big)$$

is well-defined in $L^\infty(0,T;L^2(\Omega))$ for all $\epsilon\in(0,1)$ and satisfies for $r\in(0,\frac{1}{3})$ and for a.a. $t\in(0,T)$ the estimate

$$\|(\partial_1\mathbf{U}_{*,\epsilon}(t),\mathbf{t}_\epsilon)-(\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon(t))+\frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon(t))$$
$$-\frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon(t))-\frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon(t)))\|_2\overset{(6.42)}{\le}\|(\beta_\epsilon^2 x_2+\alpha_\epsilon\beta_\epsilon x_3)(\partial_3\mathbf{U}_\epsilon(t),\mathbf{t}_\epsilon)\|_2$$
$$+\|(\alpha_\epsilon\beta_\epsilon x_2+\alpha_\epsilon^2 x_3)(\partial_2\mathbf{U}_\epsilon(t),\mathbf{t}_\epsilon)\|_2+\|(\beta_\epsilon\gamma_\epsilon x_2+\frac{\gamma_\epsilon}{\epsilon})(\partial_3\mathbf{U}_\epsilon(t),\mathbf{b}_\epsilon)\|_2$$
$$+\|(\alpha_\epsilon\gamma_\epsilon x_3+\frac{\gamma_\epsilon}{\epsilon})(\partial_2\mathbf{U}_\epsilon(t),\mathbf{n}_\epsilon)\|_2+\|\beta_\epsilon\gamma_\epsilon x_3(\partial_3\mathbf{U}_\epsilon(t),\mathbf{n}_\epsilon)\|_2$$
$$+\|\alpha_\epsilon\gamma_\epsilon x_2(\partial_2\mathbf{U}_\epsilon(t),\mathbf{b}_\epsilon)\|_2\overset{(3.5)}{\le}C\Big(\frac{1}{\epsilon^{2r}}\|(\partial_3\mathbf{U}_\epsilon(t),\mathbf{t}_\epsilon)\|_2+\frac{1}{\epsilon^{2r}}\|(\partial_2\mathbf{U}_\epsilon(t),\mathbf{t}_\epsilon)\|_2$$
$$+\frac{1}{\epsilon^{1+r}}\|(\partial_3\mathbf{U}_\epsilon(t),\mathbf{b}_\epsilon)\|_2+\frac{1}{\epsilon^{1+r}}\|(\partial_2\mathbf{U}_\epsilon(t),\mathbf{n}_\epsilon)\|_2+\frac{1}{\epsilon^{2r}}\|(\partial_3\mathbf{U}_\epsilon(t),\mathbf{n}_\epsilon)\|_2$$
$$+\frac{1}{\epsilon^{2r}}\|(\partial_2\mathbf{U}_\epsilon(t),\mathbf{b}_\epsilon)\|_2\Big)$$

$$= C(t,\epsilon) + \frac{1}{\epsilon^{2r}}\Big(\|\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon\|_2 + \|\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon\|_2\Big), \tag{6.45}$$

where $C(t,\epsilon) \to 0$ in $L^\infty(0,T)$ for $\epsilon \to 0$ as a consequence of (6.23)–(6.24), (6.26). It remains to study the behaviour of the terms

$$\frac{1}{\epsilon^{2r}}\|(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\|_2,\ \ \frac{1}{\epsilon^{2r}}\|(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon)\|_2,\ \ r \in (0, \frac{1}{3}).$$

The estimate

$$\begin{aligned}
&\|\frac{1}{\epsilon^{2r}}(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\|_2 + \|\frac{1}{\epsilon^{2r}}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon)\|_2 \\
&\leq \|\frac{1}{\epsilon^{2r}}((\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon) + (\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon))\|_2 + \|\frac{1}{\epsilon^{2r}}((\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon) - (\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon))\|_2 \\
&\overset{(6.28)}{=} C_1(t,\epsilon) + 2\epsilon^{1-2r}\|\phi_\epsilon(t)\|_2 \\
&\overset{(3.14),\ \text{Lemma 6.7}}{\leq} C_1(t,\epsilon) + C\epsilon^{1-2r}\sum_{j=1}^{3}\|\partial_j \phi_\epsilon(t)\|_{L^2(0,l;H^{-1}(S))} \\
&\overset{(6.31)}{=} \sum_{j=1}^{2} C_j(t,\epsilon) + C\epsilon^{1-2r}\|\partial_1 \phi_\epsilon(t)\|_{L^2(0,l;H^{-1}(S))} \\
&\overset{(6.41),(3.5)}{\leq} \sum_{j=1}^{2} C_j(t,\epsilon) + \frac{C}{\epsilon^{2r}}\Big(\|\frac{1}{\epsilon}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon(t)) + \partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon(t))\|_{L^2(0,l;H^{-1}(S))} \\
&\quad + \|\frac{1}{\epsilon}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon(t)) + \partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon(t))\|_{L^2(0,l;H^{-1}(S))}\Big) \\
&\quad + \frac{C}{\epsilon^{3r}}\Big(\|(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{g}_{1,\epsilon})\|_2 + \|(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{g}_{1,\epsilon})\|_2\Big) \\
&\quad + C\epsilon^{1-4r}\Big(\|(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{t}_\epsilon)\|_2 + \|(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{t}_\epsilon)\|_2\Big) \\
&\quad + \frac{C}{\epsilon^{3r}}\Big(\|(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon)\|_2 + \|(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\|_2\Big) \\
&\quad + C\epsilon^{1-2r}\Big(\|\frac{1}{\epsilon^{2r}}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon)\|_2 + \|\frac{1}{\epsilon^{2r}}(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\|_2\Big) \\
&= \sum_{j=1}^{6} C_j(t,\epsilon) + C\epsilon^{1-2r}\Big(\|\frac{1}{\epsilon^{2r}}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon)\|_2 + \|\frac{1}{\epsilon^{2r}}(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\|_2\Big),
\end{aligned}$$

for a.a. $t \in (0,T)$, leads to the estimate

$$\|\frac{1}{\epsilon^{2r}}(\partial_3 \mathbf{U}_\epsilon(t), \mathbf{n}_\epsilon)\|_2 + \|\frac{1}{\epsilon^{2r}}(\partial_2 \mathbf{U}_\epsilon(t), \mathbf{b}_\epsilon)\|_2 \leq C\sum_{j=1}^{6} C_j(t,\epsilon)$$

for $\epsilon \in (0,1)$, where $C_1(t,\epsilon) \to 0$ in $L^\infty(0,T)$ (see (6.25)), $C_2(t,\epsilon) \to 0$ in $L^\infty(0,T)$ as a consequence of (6.31), $C_3(t,\epsilon) \to 0$ in $L^\infty(0,T)$ (see (6.21)–(6.22)), because $r \in (0, \frac{1}{3})$, $C_4(t,\epsilon) \to 0$ and $C_6(t,\epsilon) \to 0$ in $L^\infty(0,T)$ as a result of (6.16)–(6.17), (6.23)–(6.24) and the fact that $r \in (0, \frac{1}{3})$, $C_5(t, \epsilon) \to 0$ in $L^\infty(0, T)$ as a consequence of (6.26), because $4r - 1 < 1 - r$ for $r \in (0, \frac{1}{3})$. Hence, we can conclude that

$$\frac{1}{\epsilon^{2r}}\Big(\|(\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|(\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))}\Big) \to 0 \tag{6.46}$$

for $r \in (0, \frac{1}{3})$, which together with (6.45) imply (6.38) and thus (using (6.36)) (6.32).

Now, it remains to prove (6.35). Since

$$\partial_1 \mathbf{U}_{*,\epsilon} = (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon,$$

it is enough to show that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon \overset{*}{\rightharpoonup} (\partial_3\zeta_{12} - \partial_2\zeta_{13})\mathbf{t} \text{ in } L^\infty(0,T; L^2(0,l; H^{-1}(S)^3)), \tag{6.47}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon \overset{*}{\rightharpoonup} -\partial_3\zeta_{11}\mathbf{n} \text{ in } L^\infty(0,T; L^2(0,l; H^{-1}(S)^3)), \tag{6.48}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon \overset{*}{\rightharpoonup} \partial_2\zeta_{11}\mathbf{n} \text{ in } L^\infty(0,T; L^2(0,l; H^{-1}(S)^3)), \tag{6.49}$$

for $\epsilon \to 0$. We only check (6.47). The convergences (6.48) and (6.49) can be proved in almost the same way. Since $\mathbf{t}$ is a bounded function depending only on x_1, then (6.32) yields

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t} \overset{*}{\rightharpoonup} (\partial_3\zeta_{12} - \partial_2\zeta_{13})\mathbf{t} \text{ in } L^\infty(0,T; L^2(0,l; H^{-1}(S)^3)).$$

It remains to show that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon - (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t} \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(0,T; L^2(0,l; H^{-1}(S)^3))$$

for $\epsilon \to 0$, which follows from the estimate

$$\left|\int_0^T \int_\Omega (\partial_1 \mathbf{U}_{*,\epsilon}(t), \mathbf{t}_\epsilon)(\mathbf{t}_\epsilon - \mathbf{t})\varphi(t)\, dxdt\right|$$

$$\leq C \int_0^T \left(\int_0^l |\mathbf{t}_\epsilon(x_1) - \mathbf{t}(x_1)|^2 \|\varphi(t, x_1)\|^2_{1,2,S}\, dx_1\right)^{\frac{1}{2}} dt \to 0, \tag{6.50}$$

for $\epsilon \to 0$ and for arbitrary but fixed function $\varphi \in L^p(0,T; L^2(0,l; H^1_0(S)))$, $p > 1$, because $|\mathbf{t}_\epsilon| = |\mathbf{t}| = 1$, $\forall \epsilon \in (0,1)$, $\mathbf{t}_\epsilon \to \mathbf{t}$ in measure in $(0,l)$ and the term $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon - (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}$ is bounded in $L^\infty(0,T; L^2(0,l; H^{-1}(S)^3))$. □

To derive the equations (6.5)–(6.7), we must describe more precisely the limit state of the functions $\mathbf{U}_{*,\epsilon}$ for $\epsilon \to 0$. This will be done in the following lemma and corollary.

Lemma 6.9 *Let the assumptions of Proposition* 6.1 *be fulfilled. Then*

$$\partial_j \mathbf{U}_{*,\epsilon} \overset{*}{\rightharpoonup} \mathbf{0} \text{ in } L^\infty(0,T; L^2(0,l; H^{-1}(S)^3)),\ j = 2, 3, \tag{6.51}$$

and $\mathbf{U}_{*,\epsilon}(t)|_{x_1=0} = \mathbf{U}_{*,\epsilon}(t)|_{x_1=l} = \mathbf{0}$ *for almost all* $t \in (0,T)$ *in the sense of the space* $C([0,l]; H^{-1}(S)^3)$.

P r o o f: Since $\phi_\epsilon(t)|_{x_1=0} = \phi_\epsilon(t)|_{x_1=l} = 0$ for all $\epsilon \in (0,1)$ and a.a. $t \in (0,T)$ in the sense of the space $C([0,l];H^{-1}(S)^3)$ (see Lemma 6.7), $\mathbf{U}_\epsilon(t) \in V(\Omega)^3$ for a.a. $t \in (0,T)$ and since the functions $\mathbf{g}_{1,\epsilon}$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ belong to $C^\infty(\overline{\Omega})^3$, we can use the definition (6.30) of the function $\mathbf{U}_{*,\epsilon}$, and applying Proposition 3.5, we get that $\mathbf{U}_{*,\epsilon}(t)|_{x_1=0} = \mathbf{U}_{*,\epsilon}(t)|_{x_1=l} = \mathbf{0}$ for a.a. $t \in (0,T)$ in the sense of the space $C([0,l];H^{-1}(S)^3)$.

It remains to show (6.51). Using the definition (6.30) of the function $\mathbf{U}_{*,\epsilon}$, we obtain the identity

$$\begin{aligned}\partial_j \mathbf{U}_{*,\epsilon} &= -\partial_j\phi_\epsilon \mathbf{t}_\epsilon - \partial_j \frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{n}_\epsilon + \partial_j \frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{b}_\epsilon \\ &= -\partial_j\phi_\epsilon \mathbf{t}_\epsilon + \partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon - \partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \\ &\quad -\partial_j\left(\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\right)\mathbf{n}_\epsilon + \partial_j\left(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\right)\mathbf{b}_\epsilon \end{aligned} \tag{6.52}$$

in $L^\infty(0,T;L^2(0,l;H^{-1}(S)^3))$, $j = 2,3$. From (6.21), (6.22), (6.31) and from the fact that the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ are bounded in $L^\infty(0,l)^3$, it follows that

$$\partial_j\phi_\epsilon \mathbf{t}_\epsilon \to \mathbf{0} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)), \tag{6.53}$$

$$\partial_j\left(\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\right)\mathbf{n}_\epsilon \to \mathbf{0} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)), \tag{6.54}$$

$$\partial_j\left(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\right)\mathbf{b}_\epsilon \to \mathbf{0} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)), \tag{6.55}$$

for $\epsilon \to 0$ and $j = 2,3$. We can see from (6.52) that it remains to prove that

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon \overset{*}{\rightharpoonup} \mathbf{0} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)), \tag{6.56}$$

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \overset{*}{\rightharpoonup} \mathbf{0} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)), \tag{6.57}$$

for $\epsilon \to 0$ and $j = 2,3$. From the convergence (6.1), it follows that $(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}) \overset{*}{\rightharpoonup} (\partial_1 \mathbf{U}, \mathbf{n})$ in $L^\infty(0,T;L^2(\Omega))$, because $\mathbf{n}$ is a bounded function. Further, we have the estimate

$$\begin{aligned}&\left|\int_0^T \int_\Omega (\partial_1 \mathbf{U}_\epsilon(t), (\mathbf{n}_\epsilon - \mathbf{n}))\varphi(t)\, dxdt\right| \\ &\le C\int_0^T \left(\int_0^l |\mathbf{n}_\epsilon(x_1) - \mathbf{n}(x_1)|^2 \|\varphi(t,x_1)\|_{2,S}^2\, dx_1\right)^{\frac{1}{2}} dt \to 0,\end{aligned}$$

where $\varphi \in L^p(0,T;L^2(\Omega))$, $p > 1$, is arbitrary but fixed and $\mathbf{n}_\epsilon \to \mathbf{n}$ in measure in $(0,l)$ for $\epsilon \to 0$. Hence we can deduce that

$$(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \overset{*}{\rightharpoonup} (\partial_1 \mathbf{U}, \mathbf{n}) \text{ in } L^\infty(0,T;L^2(\Omega))$$

(compare with (6.50)). The proof that

$$(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \overset{*}{\rightharpoonup} (\partial_1 \mathbf{U}, \mathbf{n})\mathbf{b} \text{ in } L^\infty(0,T;L^2(\Omega)^3)$$

is almost the same as the proof that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon \stackrel{*}{\rightharpoonup} (\partial_3\zeta_{12} - \partial_2\zeta_{13})\mathbf{t} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)),$$

because we take only $\varphi \in L^p(0,T;L^2(\Omega))$ instead of $\varphi \in L^p(0,T;L^2(0,l;H^{-1}(S)))$, $p > 1$, in the estimate (6.50) modified for the functions $(\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon$. The analogous result can be obtained for $(\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon$. Hence we get that

$$\partial_j(\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \stackrel{*}{\rightharpoonup} \partial_j(\partial_1\mathbf{U}, \mathbf{n})\mathbf{b} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)),\ j = 2,3,$$

$$\partial_j(\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon \stackrel{*}{\rightharpoonup} \partial_j(\partial_1\mathbf{U}, \mathbf{b})\mathbf{n} \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)),\ j = 2,3.$$

In Lemma 6.6 we have proved that the function $\mathbf{U}$ depends only on x_1, and hence

$$\partial_j(\partial_1\mathbf{U}, \mathbf{n})\mathbf{b} = \mathbf{0},\ \partial_j(\partial_1\mathbf{U}, \mathbf{b})\mathbf{n} = \mathbf{0},\ \text{in } (0,T)\times(0,l),\ j = 2,3.$$

Thus we have proved (6.56) and (6.57). □

Corollary 6.10 *Let the assumptions of Proposition* 6.1 *be fulfilled. Then*

$$\partial_i\mathbf{U}_{*,\epsilon} \stackrel{*}{\rightharpoonup} \partial_i\mathbf{U}_* \text{ in } L^\infty(0,T;L^2(0,l;H^{-1}(S)^3)),\ i = 1,2,3, \tag{6.58}$$

$$\mathbf{U}_{*,\epsilon} \stackrel{*}{\rightharpoonup} \mathbf{U}_* \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{6.59}$$

for $\epsilon \to 0$, *and* $\mathbf{U}_* \in L^\infty(0,T;H_0^1(0,l)^3)$, *where*

$$\begin{aligned}\mathbf{U}_*(t,x_1) = \int_0^{x_1} &[(\partial_3\zeta_{12}(t,z_1,x_2,x_3) - \partial_2\zeta_{13}(t,z_1,x_2,x_3))\mathbf{t}(z_1) \\ &-\partial_3\zeta_{11}(t,z_1,x_2,x_3)\mathbf{n}(z_1) + \partial_2\zeta_{11}(t,z_1,x_2,x_3)\mathbf{b}(z_1)]\ dz_1\end{aligned} \tag{6.60}$$

for $(t,x_1,x_2,x_3) \in (0,T)\times\Omega$. *In addition,*

$$\phi_\epsilon \stackrel{*}{\rightharpoonup} \phi = (\mathbf{U}_*, \mathbf{t}) \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{6.61}$$

for $\epsilon \to 0$ *and* $\phi \in L^\infty((0,T)\times(0,l))$.

P r o o f: Lemmas 6.8 and 6.9 enable us to use Proposition 3.5 to prove (6.58)–(6.59). Proposition 3.7 and 3.8 provide

$$\overline{\mathbf{U}_{*,\epsilon}}^\varphi \rightharpoonup \overline{\mathbf{U}_*}^\varphi \stackrel{Remark\ 6.2}{=} \mathbf{U}_{*,\varphi},$$

$$\partial_j\overline{\mathbf{U}_{*,\epsilon}}^\varphi = \overline{\partial_j\mathbf{U}_{*,\epsilon}}^\varphi \rightharpoonup \overline{\partial_j\mathbf{U}_*}^\varphi = \partial_j\overline{\mathbf{U}_*}^\varphi,$$

in $L^2(\Omega)$ for $j = 1,2,3$, which together with Proposition 3.6 and 3.9 give $\overline{\mathbf{U}_*}^\varphi \in H_0^1(0,l)^3$ for all $\varphi \in C_0^\infty(0,T)$. We leave it to the reader to prove that then $\mathbf{U}_* \in L^\infty(0,T;H^1(0,l)^3)$. From compact imbbeding, it follows that the function $\mathbf{U}_*$ belongs to $L^\infty(0,T;C([0,l])^3)$, which together with the limit

$$0 = \lim_{x_1\to 0,l}\int_0^T \mathbf{U}_*(t,x_1)\varphi(t)\ dt = \int_0^T \lim_{x_1\to 0,l}\mathbf{U}_*(t,x_1)\varphi(t)\ dt$$

for all $\varphi \in C_0^\infty(0,T)$, yields that $\mathbf{U}_* \in L^\infty(0,T;H_0^1(0,l))$. From (6.30), it follows that $\phi_\epsilon = -(\mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)$. Then (6.61) easily follows from (6.59) and from the convergence in measure in $(0,l)$ of the functions $\mathbf{t}_\epsilon$. □

Lemma 6.11 *Let the assumptions of Proposition* 6.1 *be fulfilled. Let the function* $\mathbf{U}$ *be determined by* (6.1) *and the function* ϕ *by* (6.61). *Then* $\lfloor \mathbf{U}, \phi \rfloor \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$.

P r o o f: To prove that $\lfloor \mathbf{U}, \phi \rfloor \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$, it is enough to check that $\mathbf{U} = \widehat{\mathbf{U}}$, where

$$\widehat{\mathbf{U}}(t,x_1) = \int_0^{x_1} [-(\mathbf{U}_*(t),\mathbf{b})\mathbf{n} + (\mathbf{U}_*(t),\mathbf{n})\mathbf{b}]\ dz_1,\ x_1 \in [0,l],\ t \in (0,T),$$

(see Proposition 3.3). We define the function $\widehat{\mathbf{U}}_\epsilon$ by

$$\widehat{\mathbf{U}}_\epsilon(t,x_1,x_2,x_3) = \int_0^{x_1} [-(\mathbf{U}_{*,\epsilon}(t,z_1,x_2,x_3),\mathbf{b}_\epsilon(z_1))\mathbf{n}_\epsilon(z_1) + (\mathbf{U}_{*,\epsilon}(t,z_1,x_2,x_3),\mathbf{n}_\epsilon(z_1))\mathbf{b}_\epsilon(z_1)]\ dz_1, \tag{6.62}$$

for $(t,x_1,x_2,x_3) \in (0,T)\times\Omega$. The definition (6.30) of the function $\mathbf{U}_{*,\epsilon}$ together with (6.62), enables us to express the function $\widehat{\mathbf{U}}_\epsilon$ by

$$\widehat{\mathbf{U}}_\epsilon = -\int_0^{x_1} \left[\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\mathbf{n}_\epsilon + \frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\mathbf{b}_\epsilon\right]\ dz_1, \tag{6.63}$$

where we omit to write the points (t,z_1,x_2,x_3) and (t,z_1) in the right-hand side to simplify the notation. Using (6.63), we can deduce that

$$\begin{aligned}\mathbf{U}_\epsilon &= \int_0^{x_1} \partial_1\mathbf{U}_\epsilon\ dz_1 = \int_0^{x_1} [(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\mathbf{b}_\epsilon]\ dz_1\\ &= \widehat{\mathbf{U}}_\epsilon + \int_0^{x_1}\Bigg[(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\mathbf{t}_\epsilon + \Big(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)\mathbf{n}_\epsilon\\ &\qquad + \Big(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) + (\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\Big)\mathbf{b}_\epsilon\Bigg]\ dz_1.\end{aligned} \tag{6.64}$$

As a result of (6.64) and (6.19)–(6.20), (6.27), we get

$$\partial_1\widehat{\mathbf{U}}_\epsilon - \partial_1\mathbf{U}_\epsilon \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)^3)$$

and

$$\widehat{\mathbf{U}}_\epsilon - \mathbf{U}_\epsilon \to 0 \text{ in } L^\infty(0,T;C([0,l];L^2(S)^3))$$

for $\epsilon \to 0$. Since $\mathbf{U}_\epsilon \overset{*}{\rightharpoonup} \mathbf{U}$ in $L^\infty(0,T;H^1(\Omega)^3)$ and $\mathbf{U} \in L^\infty(0,T;H_0^1(0,l)^3)$, we can conclude that $\mathbf{U} = \widehat{\mathbf{U}}$ a.e. in $(0,T)\times(0,l)$, and thus

$$\mathbf{U}(t,x_1) = \int_0^{x_1} [-(\mathbf{U}_*(t),\mathbf{b})\mathbf{n} + (\mathbf{U}_*(t),\mathbf{n})\mathbf{b}]\ dz_1,\ x_1 \in [0,l],$$

and

$$\mathbf{U}(t,l) = \int_0^{l} [-(\mathbf{U}_*(t),\mathbf{b})\mathbf{n} + (\mathbf{U}_*(t),\mathbf{n})\mathbf{b}]\ dx_1 = 0$$

for almost all $t \in (0,T)$. Hence, and from Proposition 3.3, we get that $\lfloor \mathbf{U}, \phi \rfloor \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$. □

Corollary 6.12 *Let the function* $\mathbf{U}_*$ *be defined by* (6.60). *Then the function* $\mathbf{U}_*$ *satisfies the equations* (6.5)–(6.7).

P r o o f: The proof immediately follows from (6.60). □

7 The main result

In this section, we pass from the three-dimensional model to the asymptotic model, and our main result is stated and proved.

Let us mention for the reader's convenience that we have proved in Corollary 5.4 that

$$\mathbf{U}_{\epsilon_n} \overset{*}{\rightharpoonup} \mathbf{U} \text{ in } L^\infty(0,T;H^1(\Omega)^3),\ \partial_t\mathbf{U}_\epsilon \overset{*}{\rightharpoonup} \partial_t\mathbf{U} \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{7.1}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(0,T;L^2(\Omega)^9), \tag{7.2}$$

for $\epsilon_n \to 0$, where $\mathbf{U} \in L^\infty(0,T;H_0^1(0,l)^3)$ according to Proposition 6.1.

To find the form of the tensor ζ, we must obtain the corresponding equations for its components.

Proposition 7.1 *Let the tensor* ζ *be the limit determined by* (7.2). *Then it satisfies the equation*

$$\int_\Omega A_0^{ijkl}\zeta_{kl}(t)\theta_{ij}^0(\mathbf{V})\ dx = 0 \tag{7.3}$$

for all $\mathbf{V} \in L^2(0,l;H^1(S)^3)$ *and for a.a.* $t \in (0,T)$, *where the tensor* $\theta^0(\mathbf{V})$ *is defined by*

$$\theta^0(\mathbf{V}) = \begin{pmatrix} 0 & \frac{(\partial_2\mathbf{V},\mathbf{t})}{2} & \frac{(\partial_3\mathbf{V},\mathbf{t})}{2} \\ \frac{(\partial_2\mathbf{V},\mathbf{t})}{2} & (\partial_2\mathbf{V},\mathbf{n}) & \frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2} \\ \frac{(\partial_3\mathbf{V},\mathbf{t})}{2} & \frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2} & (\partial_3\mathbf{V},\mathbf{b}). \end{pmatrix}. \tag{7.4}$$

P r o o f: In the proof, we will use ϵ instead of ϵ_n to simplify the notation. Multiplying (4.17) by ϵ^2 and using an arbitrary function $\varphi \in C_0^\infty(0,T)$ as a test function, we get, after integration by parts, the equation

$$-\epsilon^2\int_\Omega \rho(\overline{\partial_t\mathbf{U}_\epsilon}^{\dot\varphi},\mathbf{V})d_\epsilon\ dx + \int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\overline{\mathbf{U}_\epsilon}^{\varphi})\epsilon\omega_{ij}^\epsilon(\mathbf{V})d_\epsilon\ dx$$

$$= \epsilon^2\int_\Omega(\overline{\mathbf{F}}^{\varphi},\mathbf{V})d_\epsilon\ dx + \epsilon^2\int_0^l\int_{\partial S}(\overline{\mathbf{G}}^{\varphi},\mathbf{V})d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dS dx_1,\ \forall\mathbf{V}\in V(\Omega)^3. \tag{7.5}$$

Letting $\epsilon \to 0$, we want to pass from the above equation to the equation

$$\int_\Omega A_0^{ijkl}\overline{\zeta_{kl}}^{\varphi}\theta_{ij}^0(\mathbf{V})\ dx = 0,\ \forall\mathbf{V}\in V(\Omega)^3, \tag{7.6}$$

where the tensor $\theta^0(\mathbf{V})$ is defined by (7.5). To prove that it is enough to show analogously as in [15] Proposition 8.1 that $\theta^\epsilon(\mathbf{V}) + \epsilon\kappa^\epsilon(\mathbf{V}) \to \theta^0(\mathbf{V})$ in $L^2(\Omega)^9$ for $\epsilon \to 0$.

Using the definition (see (2.9) and (4.5)) of the tensor $(A_\epsilon^{ijkl})_{i,j,k,l=1}^3$, we can easily check that

$$A_\epsilon^{ijkl} \to A_0^{ijkl} \text{ in } C(\overline{\Omega}), \text{ where } A_0^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{7.7}$$

for $i, j, k, l = 1, 2, 3$. The rest of the proof follows from density of the space $V(\Omega)^3$ in $L^2(0, l; H^1(S)^3)$ and from (7.4) and (7.6), because the equation (7.6) is fulfilled for all $\varphi \in C_0^\infty(0, T)$ and

$$\int_0^T \varphi(t) \int_\Omega A_0^{ijkl}\zeta_{kl}(t)\theta_{ij}^0(\mathbf{V})\, dxdt = \int_\Omega A_0^{ijkl}\overline{\zeta_{kl}}^\varphi \theta_{ij}^0(\mathbf{V})\, dx = 0$$

□

Now, we introduce the following notation:

$$\zeta_{22}^H = \zeta_{22} + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11},\ \zeta_{33}^H = \zeta_{33} + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11},\ \zeta_{23}^H = \zeta_{23}. \tag{7.8}$$

Corollary 7.2 *We have*

$$\int_S \zeta_{12}(t, x_1)\, dx_2dx_3 = \int_S \zeta_{13}(t, x_1)\, dx_2dx_3 = \int_S \zeta_{12}(t, x_1)x_2\, dx_2dx_3$$

$$= \int_S \zeta_{13}(t, x_1)x_3\, dx_2dx_3 = \int_S [\zeta_{12}(t, x_1)x_3 + \zeta_{13}(t, x_1)x_2]\, dx_2dx_3 = 0, \tag{7.9}$$

$$\int_S \zeta_{23}^H(t, x_1)\, dx_2dx_3 = \int_S \zeta_{23}^H(t, x_1)x_2\, dx_2dx_3 = \int_S \zeta_{23}^H(t, x_1)x_3\, dx_2dx_3 = 0, \tag{7.10}$$

$$\int_S (\zeta_{22}^H(t, x_1) + \zeta_{33}^H(t, x_1))\, dx_2dx_3 = \int_S (\zeta_{22}^H(t, x_1) + \zeta_{33}^H(t, x_1))x_2\, dx_2dx_3$$

$$= \int_S (\zeta_{22}^H(t, x_1) + \zeta_{33}^H(t, x_1))x_3\, dx_2dx_3 = 0, \tag{7.11}$$

for a.a. $(t, x_1) \in (0, T) \times (0, l)$.

P r o o f: Let $v \in L^2(0, l)$ be an arbitrary, but fixed function, and let $\mathbf{V} = v\mathbf{t}$. Testing equation (7.3) with the functions $\mathbf{V}x_2$, $\mathbf{V}x_3$, $\mathbf{V}x_2^2/2$, $\mathbf{V}x_3^2/2$ and $\mathbf{V}x_2x_3$, we can derive (7.9).

Let us take now some arbitrary function $\mathbf{V} \in L^2(0, l; H^1(S)^3)$ such that $(\mathbf{V}, \mathbf{t}) = (\mathbf{V}, \mathbf{b}) = 0$. Then we can derive from (7.3) and (7.4) that

$$\int_\Omega [(\lambda(\zeta_{11}(t) + \zeta_{22}(t) + \zeta_{33}(t)) + 2\mu\zeta_{22}(t))(\partial_2\mathbf{V}, \mathbf{n})$$

$$+2\mu\zeta_{23}(t)(\partial_3\mathbf{V},\mathbf{n})]\ dx = 0, \tag{7.12}$$

for a.a. $t \in (0,T)$. Analogously we find for arbitrary functions $\mathbf{V} \in L^2(0,l;H^1(S)^3)$, which satisfy $(\mathbf{V},\mathbf{t}) = (\mathbf{V},\mathbf{n}) = 0$, that

$$\int_\Omega [(\lambda(\zeta_{11}(t)+\zeta_{22}(t)+\zeta_{33}(t)) + 2\mu\zeta_{33}(t))(\partial_3\mathbf{V},\mathbf{b})$$

$$+2\mu\zeta_{23}(t)(\partial_2\mathbf{V},\mathbf{b})]\ dx = 0. \tag{7.13}$$

After substitution of (7.8), we can transform (7.12) and (7.13) as

$$\int_\Omega [(\lambda(\zeta_{22}^H(t)+\zeta_{33}^H(t)) + 2\mu\zeta_{22}^H(t))(\partial_2\mathbf{V},\mathbf{n}) + 2\mu\zeta_{23}^H(t)(\partial_3\mathbf{V},\mathbf{n})]\ dx = 0 \tag{7.14}$$

and

$$\int_\Omega [(\lambda(\zeta_{22}^H(t)+\zeta_{33}^H(t)) + 2\mu\zeta_{33}^H(t))(\partial_3\mathbf{V},\mathbf{b}) + 2\mu\zeta_{23}^H(t)(\partial_2\mathbf{V},\mathbf{b})]\ dx = 0, \tag{7.15}$$

respectively, for a.a. $t \in (0,T)$. Taking $\mathbf{V}x_3$, $\mathbf{V}x_3^2/2$ and $\mathbf{V}x_2^2/2$, where $\mathbf{V} = v\mathbf{n}$ or $\mathbf{V} = v\mathbf{b}$, as test functions in (7.14) and (7.15), respectively, yields (7.10). In an analogous way, we substitute the functions $\mathbf{V}x_2$, $\mathbf{V}x_3$, $\mathbf{V}x_2^2/2$, $\mathbf{V}x_2x_3$ and $\mathbf{V}x_2x_3$, $\mathbf{V}x_3^2/2$, where $\mathbf{V} = v\mathbf{n}$ or $\mathbf{V} = v\mathbf{b}$, to (7.14) and (7.15), respectively, to derive (7.11). □

If we define the vector $\boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)^2)$ by $\boldsymbol{\eta} = \lfloor \zeta_{12}, \zeta_{13} \rfloor$, then the equation (7.3), after putting $\mathbf{V} = \varphi\mathbf{t}$, $\varphi \in L^2(0,l;H^1(S))$, and (6.5) can be rewritten in the form

$$\int_\Omega (\boldsymbol{\eta}(t), \nabla_{23}\varphi)_2\ dx = 0,\ \ \forall\varphi \in L^2(0,l;H^1(S)), \tag{7.16}$$

$$\int_\Omega (\boldsymbol{\eta}(t), \mathrm{rot}_{23}\psi)_2\ dx = \int_\Omega (\mathbf{U}'_*(t),\mathbf{t})\psi\ dx,\ \ \forall\psi \in H_0^1(\Omega), \tag{7.17}$$

for a.a. $t \in (0,T)$, where we have denoted $\nabla_{23}\varphi = \lfloor \partial_2\varphi, \partial_3\varphi \rfloor$, $\mathrm{rot}_{23}\psi = \lfloor -\partial_3\psi, \partial_2\psi \rfloor$, and where $(\cdot,\cdot)_2$ means the scalar product in the usual two dimensional Euclidean space $\mathbb{R}^2$.

Lemma 7.3 *Let S be a simply connected domain, and let $\partial S \in C^1$. Then the system (7.16), (7.17) has unique solution in $L^\infty(0,T;L^2(\Omega)^2)$, given by*

$$\boldsymbol{\eta} = \lfloor \zeta_{12}, \zeta_{13} \rfloor = -\frac{1}{2}(\mathbf{U}'_*,\mathbf{t})\lfloor \partial_2 p - x_3, \partial_3 p + x_2 \rfloor, \tag{7.18}$$

where the function $p \in H^1(S)$ is the unique solution to the Neumann problem

$$\int_S [(\partial_2 p - x_3)\partial_2 r + (\partial_3 p + x_2)\partial_3 r]\ dx_2 dx_3 = 0,\ \int_S p\ dx_2 dx_3 = 0, \tag{7.19}$$

for all $r \in H^1(S)$.

P r o o f: The proof is analogous as that of Lemma 8.3 in [15], and we omit it. □

Now, we derive the asymptotic model. First we introduce some constants:

$$I_{x_2^2} = \int_S x_2^2 \, dx_2 dx_3, \;\; I_{x_3^2} = \int_S x_3^2 \, dx_2 dx_3, \tag{7.20}$$

$$E = \mu \frac{3\lambda + 2\mu}{\lambda + \mu}, \;\; K = \int_S [(\partial_2 p - x_3)^2 + (\partial_3 p + x_2)^2] \, dx_2 dx_3, \tag{7.21}$$

where $p \in H^1(S)$ is the unique solution to the Neumann problem (7.19).

Lemma 7.4 *Let* $\{\mathbf{U}_{\epsilon_n}\}_{n=1}^{\infty}$, $\epsilon_n \to 0$, *be a subsequence of the weak solutions to the problem* (4.17)–(4.18) *satisfying* (5.1), (7.1)–(7.2). *Then the limit* $\lfloor \mathbf{U}, \phi \rfloor \in L^\infty(0, T; \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l))$ *obtained in Proposition* 6.1 *generates a function* $\mathbf{U}_*$ *which satisfies the equation*

$$-\int_0^l \check{\rho}(\overline{\partial_t \mathbf{U}}^{\dot{\varphi}}, \mathbf{V}) \, dx_1 + \int_0^l E[I_{x_2^2}(\overline{\partial_1 \mathbf{U}_*}^{\varphi}, \mathbf{b})(\mathbf{V}_*', \mathbf{b}) + I_{x_3^2}(\overline{\partial_1 \mathbf{U}_*}^{\varphi}, \mathbf{n})(\mathbf{V}_*', \mathbf{n})] \, dx_1$$

$$+ \int_0^l \mu K (\overline{\partial_1 \mathbf{U}_*}^{\varphi}, \mathbf{t})(\mathbf{V}_*', \mathbf{t}) \, dx_1 = \int_0^l (\overline{\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}}^{\varphi}, \mathbf{V}) \, dx_1 \tag{7.22}$$

for all functions $\mathbf{V}_* \in H_0^1(0, l)^3$ *generated by an arbitrary couple* $\lfloor \mathbf{V}, \psi \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$ *(see* (2.17)) *and for all* $\varphi \in C_0^\infty(0, T)$, *where* $\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}(t, x_1) = \int_S \mathbf{F}(t, x_1) \, dx_2 dx_3 + \int_{\partial S} \mathbf{G}(t, x_1) \, dS$ *and* $\check{\rho}(x_1) = \int_S \rho(x_1) \, dx_2 dx_3$, $(t, x_1) \in (0, T) \times (0, l)$.

P r o o f: In the proof, we will use ϵ instead of ϵ_n to simplify the notation. Let $\lfloor \mathbf{V}, \psi \rfloor$ be an arbitrary couple from the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$. Proposition 3.4 enables us to approximate the couple $\lfloor \mathbf{V}, \psi \rfloor$ with couples $\lfloor \mathbf{V}_\epsilon, \psi_\epsilon \rfloor \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0, l)$ satisfying $\mathbf{V}_\epsilon \in C_0^\infty(0, l)^3$ and $\psi_\epsilon \in C_0^\infty(0, l)$. We define the functions $\mathbf{W}_\epsilon \in C^\infty(\overline{\Omega})^3$ by

$$\mathbf{W}_\epsilon(x_1, x_2, x_3) = -\Big((\mathbf{V}_\epsilon'(x_1), \mathbf{n}_\epsilon(x_1))x_2 + (\mathbf{V}_\epsilon'(x_1), \mathbf{b}_\epsilon(x_1))x_3\Big)\mathbf{t}_\epsilon(x_1)$$

$$-x_3 \psi_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) + x_2 \psi_\epsilon(x_1)\mathbf{b}_\epsilon(x_1) \tag{7.23}$$

for $(x_1, x_2, x_3) \in \Omega$.

Let us define the function $\widehat{\mathbf{V}}_\epsilon$ by

$$\widehat{\mathbf{V}}_\epsilon = \mathbf{V}_\epsilon + \epsilon \mathbf{W}_\epsilon \in C^\infty(\overline{\Omega})^3 \cap V(\Omega)^3. \tag{7.24}$$

Denoting $B_\epsilon = (B_\epsilon^{ij})_{i,j=1}^3$, where $B_\epsilon^{ij} = 0$ except for $i = j = 1$ and

$$B_\epsilon^{11} = \epsilon^2 \Big((\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{V}_\epsilon', \mathbf{n}_\epsilon)' + x_3(\mathbf{V}_\epsilon', \mathbf{b}_\epsilon)' - \beta_\epsilon x_3 \psi_\epsilon + \alpha_\epsilon x_2 \psi_\epsilon)$$

$$+\gamma_\epsilon x_3(\partial_1 \mathbf{W}_\epsilon, \mathbf{n}_\epsilon) - \gamma_\epsilon x_2(\partial_1 \mathbf{W}_\epsilon, \mathbf{b}_\epsilon)\Big), \tag{7.25}$$

we deduce analogously as in [15] Lemma 8.4 that

$$\omega^\epsilon(\widehat{\mathbf{V}}_\epsilon) = \epsilon\Upsilon(\mathbf{V}_{*,\epsilon}) + B_\epsilon, \tag{7.26}$$

where

$$\Upsilon_{11}(\mathbf{V}_{*,\epsilon}) = -(\mathbf{V}'_{*,\epsilon}, \mathbf{n}_\epsilon)x_3 + (\mathbf{V}'_{*,\epsilon}, \mathbf{b}_\epsilon)x_2, \tag{7.27}$$

$$\Upsilon_{12}(\mathbf{V}_{*,\epsilon}) = \Upsilon_{21}(\mathbf{V}_{*,\epsilon}) = \frac{x_3}{2}(\mathbf{V}'_{*,\epsilon}, \mathbf{t}_\epsilon), \tag{7.28}$$

$$\Upsilon_{13}(\mathbf{V}_{*,\epsilon}) = \Upsilon_{31}(\mathbf{V}_{*,\epsilon}) = -\frac{x_2}{2}(\mathbf{V}'_{*,\epsilon}, \mathbf{t}_\epsilon) \tag{7.29}$$

and

$$\Upsilon_{ij}(\mathbf{V}_{*,\epsilon}) = 0, \ \ i,j = 2,3. \tag{7.30}$$

Since we know that $\mathbf{t}_\epsilon \to \mathbf{t}$, $\mathbf{n}_\epsilon \to \mathbf{n}$, $\mathbf{b}_\epsilon \to \mathbf{b}$ in measure in $(0,l)$, we can use (3.11) to prove that

$$\Upsilon_{ij}(\mathbf{V}_{*,\epsilon}) \to \Upsilon_{ij}(\mathbf{V}_*) \text{ in } L^2(\Omega), \ \ i,j = 1,2,3. \tag{7.31}$$

Moreover, using (3.4)–(3.5), (3.11), (3.13) and (7.23) we can easily check that

$$\|B_\epsilon\|_2 = \|B^{11}_\epsilon\|_2 \le C\epsilon^{2(1-r)}, \ \ r \in (0, \frac{1}{3}), \tag{7.32}$$

and

$$\widehat{\mathbf{V}}_\epsilon \to \mathbf{V} \text{ in } H^1(\Omega)^3 \tag{7.33}$$

for $\epsilon \to 0$.

These convergences and estimates, together with (3.6)–(3.7), (7.1)–(7.2) and (7.7), enable us to pass to the limit in the equation (since $\widehat{\mathbf{V}}_\epsilon \in C^\infty(\Omega)^3 \cap V(\Omega)^3$)

$$-\int_\Omega \rho(\overline{\partial_t \mathbf{U}_\epsilon}^{\dot\varphi}, \widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dx + \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\overline{\mathbf{U}_\epsilon}^\varphi)\frac{1}{\epsilon}\omega^\epsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dx = \int_\Omega (\overline{\mathbf{F}}^\varphi, \widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dx$$
$$+ \int_0^l \int_{\partial S} (\overline{\mathbf{G}}^\varphi, \widehat{\mathbf{V}}_\epsilon)d_\epsilon \epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j} \, dS dx_1$$

and to establish that

$$-\int_0^l \check\rho(\overline{\partial_t \mathbf{U}}^{\dot\varphi}, \mathbf{V}) \, dx_1 + \int_\Omega A^{ijkl}_0 \overline{\zeta_{kl}}^\varphi \Upsilon_{ij}(\mathbf{V}_*) \, dx = \int_\Omega (\overline{\mathbf{F}}^\varphi, \mathbf{V}) \, dx$$
$$+ \int_0^l \int_{\partial S} (\overline{\mathbf{G}}^\varphi, \mathbf{V}) \, dS dx_1 \tag{7.34}$$

for all $\lfloor \mathbf{V}, \psi \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, which generate the functions $\mathbf{V}_*$ (see (2.17)).

From (6.6) and (6.7), it follows the existence of the function $Q_0 \in L^\infty(0,T; L^2(0,l))$ such that

$$\zeta_{11} = Q_0 + (\partial_1 \mathbf{U}_*, \mathbf{b})x_2 - (\partial_1 \mathbf{U}'_*, \mathbf{n})x_3 \text{ in } (0,T) \times \Omega. \tag{7.35}$$

By the form of the tensor $(A_0^{ijkl})_{i,j,k,l=1}^3$ (see (7.7)), we have after the substitution (7.27)–(7.30) to (7.34)

$$\int_\Omega A_0^{ijkl}\overline{\zeta_{kl}}^\varphi \Upsilon_{ij}(\mathbf{V}_*)\ dx =$$

$$\int_\Omega [\lambda(\overline{\zeta_{11}}^\varphi + \overline{\zeta_{22}}^\varphi + \overline{\zeta_{33}}^\varphi) + 2\mu\overline{\zeta_{11}}^\varphi]\Upsilon_{11}(\mathbf{V}_*)\ dx + \int_\Omega [4\mu(\overline{\zeta_{12}}^\varphi\Upsilon_{12}(\mathbf{V}_*) + \overline{\zeta_{13}}^\varphi\Upsilon_{13}(\mathbf{V}_*))]\ dx.$$

Hence, using (7.27)–(7.29), we can write

$$\int_\Omega A_0^{ijkl}\overline{\zeta_{kl}}^\varphi \Upsilon_{ij}(\mathbf{V}_*)\ dx = \mathcal{I}_{1,\varphi} + \mathcal{I}_{2,\varphi}, \tag{7.36}$$

where

$$\mathcal{I}_{1,\varphi} = \int_\Omega [\lambda(\overline{\zeta_{11}}^\varphi + \overline{\zeta_{22}}^\varphi + \overline{\zeta_{33}}^\varphi) + 2\mu\overline{\zeta_{11}}^\varphi][(\mathbf{V}'_*, \mathbf{b})x_2 - (\mathbf{V}'_*, \mathbf{n})x_3]\ dx,$$

$$\mathcal{I}_{2,\varphi} = 2\mu\int_\Omega [\overline{\zeta_{12}}^\varphi(\mathbf{V}'_*, \mathbf{t})x_3 - \overline{\zeta_{13}}^\varphi(\mathbf{V}'_*, \mathbf{t})x_2]\ dx.$$

Using (7.8), we find that

$$\lambda(\overline{\zeta_{11}}^\varphi + \overline{\zeta_{22}}^\varphi + \overline{\zeta_{33}}^\varphi) + 2\mu\overline{\zeta_{11}}^\varphi = (\lambda + 2\mu - \frac{\lambda^2}{\lambda+\mu})\overline{\zeta_{11}}^\varphi + \lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi).$$

Hence, using (7.21), we can rewrite the integral $\mathcal{I}_{1,\varphi}$ in the form

$$\mathcal{I}_{1,\varphi} = \int_\Omega [E\overline{\zeta_{11}}^\varphi + \lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)][(\mathbf{V}'_*, \mathbf{b})x_2 - (\mathbf{V}'_*, \mathbf{n})x_3]\ dx \tag{7.37}$$

for all $\varphi \in C_0^\infty(0,T)$. The terms involving function $\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi$ disappear from (7.37) because of (7.11), and the dependence of the terms $(\mathbf{V}'_*, \mathbf{b})$ and $(\mathbf{V}'_*, \mathbf{n})$ only on x_1. After the substitution (7.35) to (7.37), we can conclude, using (2.1) and (7.20)–(7.21), that

$$\mathcal{I}_{1,\varphi} = \int_0^l E[I_{x_2^2}(\overline{\partial_1\mathbf{U}_*}^\varphi, \mathbf{b})(\mathbf{V}'_*, \mathbf{b}) + I_{x_3^2}(\partial_1\overline{\mathbf{U}_*}^\varphi, \mathbf{n})(\mathbf{V}'_*, \mathbf{n})]\ dx_1. \tag{7.38}$$

After the substitution $\boldsymbol{\eta} = \lfloor \zeta_{12}, \zeta_{13} \rfloor$ from (7.17) to $\mathcal{I}_{2,\varphi}$, we obtain

$$\mathcal{I}_{2,\varphi} = \int_\Omega \mu\left(-(\partial_2 p - x_3)x_3 + (\partial_3 p + x_2)x_2\right)(\partial_1\overline{\mathbf{U}_*}^\varphi, \mathbf{t})(\mathbf{V}'_*, \mathbf{t})\ dx, \tag{7.39}$$

where p is the unique solution to the Neumann problem (7.19) and it is easy to verify from (7.39) (using (7.21) and (7.19) with the test function $r = p$) that

$$\begin{aligned}\mathcal{I}_{2,\varphi} &\overset{(7.19)}{=} \int_\Omega \mu(-\partial_2 p x_3 + x_3^2 + \partial_3 p x_2 + x_2^2)(\overline{\partial_1\mathbf{U}_*}^\varphi, \mathbf{t})(\mathbf{V}'_*, \mathbf{t})\ dx \\ &+ \int_0^l (\overline{\partial_1\mathbf{U}_*}^\varphi, \mathbf{t})(\mathbf{V}'_*, \mathbf{t})\int_S \mu[(\partial_2 p)^2 - \partial_2 p x_3 + (\partial_3 p)^2 + \partial_3 p x_2]\ dx_2 dx_3 dx_1 \\ &= \int_0^l \mu K(\overline{\partial_1\mathbf{U}_*}^\varphi, \mathbf{t})(\mathbf{V}'_*, \mathbf{t})\ dx_1.\end{aligned} \tag{7.40}$$

Thus after adding (7.38) to (7.40) we obtain (7.22). □

Lemma 7.5 *It holds* $Q_0 = \zeta^H_{22} = \zeta^H_{23} = \zeta^H_{33} = 0$ *in* $(0,T)\times\Omega$.

P r o o f: In the proof, we will write ϵ instead of ϵ_n to simplify the notation. Let us define

$$\Lambda_{\epsilon,\varphi} = \int_\Omega A^{ijkl}_\epsilon \left(\frac{1}{\epsilon}\omega^\epsilon_{kl}(\overline{\mathbf{U}_\epsilon}^\varphi) - \overline{\zeta_{kl}}^\varphi\right)\left(\frac{1}{\epsilon}\omega^\epsilon_{ij}(\overline{\mathbf{U}_\epsilon}^\varphi) - \overline{\zeta_{ij}}^\varphi\right) d_\epsilon\, dx$$

for all $\varphi \in C^\infty_0(0,T)$.

According to Proposition 3.11, there exists a constant $C > 0$ independent of ϵ and φ such that

$$\left\|\frac{1}{\epsilon}\omega^\epsilon(\overline{\mathbf{U}_\epsilon}^\varphi) - \overline{\zeta}^\varphi\right\|_2^2 \le C\Lambda_{\epsilon,\varphi} \tag{7.41}$$

for all $\varphi \in C^\infty_0(0,T)$. Equation (4.17) implies that

$$\Lambda_{\epsilon,\varphi} = \int_\Omega (\overline{\mathbf{F}}^\varphi, \overline{\mathbf{U}_\epsilon}^\varphi) d_\epsilon\, dx + \int_0^l \int_{\partial S} (\overline{\mathbf{G}}^\varphi, \overline{\mathbf{U}_\epsilon}^\varphi) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon}\nu_j}\, dS dx_1 +$$

$$\int_\Omega A^{ijkl}_\epsilon \left(\left(\overline{\zeta_{kl}}^\varphi - \frac{1}{\epsilon}\omega^\epsilon_{kl}(\overline{\mathbf{U}_\epsilon}^\varphi)\right)\overline{\zeta_{ij}}^\varphi - \overline{\zeta_{kl}}^\varphi \frac{1}{\epsilon}\omega^\epsilon_{ij}(\overline{\mathbf{U}_\epsilon}^\varphi)\right) d_\epsilon\, dx + \int_\Omega \rho(\overline{\partial_t \mathbf{U}_\epsilon}^{\dot\varphi}, \overline{\mathbf{U}_\epsilon}^\varphi) d_\epsilon\, dx.$$

As a result of (7.1)–(7.2) and (7.7), we obtain the convergence of the sequence $\Lambda_{\epsilon,\varphi}$, i.e.

$$\Lambda_\varphi = \lim_{\epsilon\to 0}\Lambda_{\epsilon,\varphi} = \int_0^l (\overline{\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}}^\varphi, \overline{\mathbf{U}}^\varphi)\, dx_1 - \int_\Omega A^{ijkl}_0 \overline{\zeta_{kl}}^\varphi \overline{\zeta_{ij}}^\varphi\, dx$$
$$+\int_0^l \check\rho(\overline{\partial_t \mathbf{U}}^{\dot\varphi}, \overline{\mathbf{U}}^\varphi)\, dx_1. \tag{7.42}$$

Using (7.7) leads after substitution of (7.8) to the identity (see [15], Lemma 8.5, for a detailed proof)

$$\int_\Omega A^{ijkl}_0 \overline{\zeta_{kl}}^\varphi \overline{\zeta_{ij}}^\varphi\, dx = \int_\Omega [E(\overline{\zeta_{11}}^\varphi)^2 + 4\mu((\overline{\zeta_{12}}^\varphi)^2 + (\overline{\zeta_{13}}^\varphi)^2)$$
$$+\lambda(\overline{\zeta^H_{22}}^\varphi + \overline{\zeta^H_{33}}^\varphi)^2 + 2\mu((\overline{\zeta^H_{22}}^\varphi)^2 + (\overline{\zeta^H_{33}}^\varphi)^2 + 2(\overline{\zeta^H_{23}}^\varphi)^2)]\, dx. \tag{7.43}$$

The expressions for ζ_{11}, ζ_{12} and ζ_{13}, i.e (7.35) and (7.18), imply (together with (7.22) and (2.1)) after substitution to (7.43) that

$$\int_\Omega A^{ijkl}_0 \overline{\zeta_{kl}}^\varphi \overline{\zeta_{ij}}^\varphi\, dx = \int_\Omega \Big[E(\overline{\zeta_{11}}^\varphi)^2 + 4\mu((\overline{\zeta_{12}}^\varphi)^2 + (\overline{\zeta_{13}}^\varphi)^2) + \lambda(\overline{\zeta^H_{22}}^\varphi + \overline{\zeta^H_{33}}^\varphi)^2$$

$$+2\mu((\overline{\zeta^H_{22}}^\varphi)^2 + (\overline{\zeta^H_{33}}^\varphi)^2 + 2(\overline{\zeta^H_{23}}^\varphi)^2)\Big]\, dx = \int_\Omega \Big[E\Big(\overline{Q_0}^\varphi + (\overline{\partial_1 \mathbf{U}_*}^\varphi, \mathbf{b})x_2 - (\overline{\partial_1 \mathbf{U}_*}^\varphi, \mathbf{n})x_3\Big)^2$$

$$+4\mu\Big(-\frac{1}{2}(\overline{\partial_1 \mathbf{U}_*}^\varphi, \mathbf{t})(\partial_2 p - x_3)\Big)^2 + 4\mu\Big(-\frac{1}{2}(\overline{\partial_1 \mathbf{U}_*}^\varphi, \mathbf{t})(\partial_3 p + x_2)\Big)^2 + \lambda(\overline{\zeta^H_{22}}^\varphi + \overline{\zeta^H_{33}}^\varphi)^2$$

$$+2\mu((\overline{\zeta_{22}^H}^\varphi)^2 + (\overline{\zeta_{33}^H}^\varphi)^2 + 2(\overline{\zeta_{23}^H}^\varphi)^2)\Bigg] \, dx \overset{(7.22),(2.1)}{=} \int_0^l [(\overline{\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}}^\varphi, \overline{\mathbf{U}}^\varphi) + E(\overline{Q_0}^\varphi)^2] \, dx_1$$

$$\int_0^l \check{\rho}(\overline{\partial_t \mathbf{U}}^{\dot\varphi}, \overline{\mathbf{U}}^\varphi) \, dx_1 + \int_\Omega \left[\lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)^2 + 2\mu\left((\overline{\zeta_{22}^H}^\varphi)^2 + (\overline{\zeta_{33}^H}^\varphi)^2 + 2(\overline{\zeta_{23}^H}^\varphi)^2\right)\right] \, dx,$$

and substituting to (7.42) leads to

$$\Lambda_\varphi = -\int_\Omega \left[\frac{E(\overline{Q_0}^\varphi)^2}{|S|} + \lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)^2 + 2\mu\left((\overline{\zeta_{22}^H}^\varphi)^2 + (\overline{\zeta_{33}^H}^\varphi)^2 + 2(\overline{\zeta_{23}^H}^\varphi)^2\right)\right] \, dx$$

for all $\varphi \in C_0^\infty(0,T)$. The sequence $\Lambda_{\epsilon,\varphi}$ for all $\varphi \in C_0^\infty(0,T)$ consists of non-negative numbers by (7.41) and $\Lambda_\varphi = 0$ for all $\varphi \in C_0^\infty(0,T)$. Hence $Q_0 = \zeta_{22}^H = \zeta_{23}^H = \zeta_{33}^H = 0$ in $(0,T) \times \Omega$. □

Since we have denoted $\boldsymbol{\eta} = \lfloor \zeta_{12}, \zeta_{13} \rfloor$, we obtain from Lemma 7.5 that

$$\begin{aligned}
\zeta_{11} &\overset{(7.35)}{=} (\partial_1 \mathbf{U}_*, \mathbf{b})x_2 - (\partial_1 \mathbf{U}_*, \mathbf{n})x_3, \\
\zeta_{12} &\overset{(7.18)}{=} \zeta_{21} = -\frac{1}{2}(\partial_1 \mathbf{U}_*, \mathbf{t})(\partial_2 p - x_3), \\
\zeta_{13} &\overset{(7.18)}{=} \zeta_{31} = -\frac{1}{2}(\partial_1 \mathbf{U}_*, \mathbf{t})(\partial_3 p + x_2), \\
\zeta_{22} &\overset{(7.8)}{=} -\frac{1}{2}\frac{\lambda}{\lambda+\mu}\Big((\partial_1 \mathbf{U}_*, \mathbf{b})x_2 - (\partial_1 \mathbf{U}_*, \mathbf{n})x_3\Big), \\
\zeta_{23} &= \zeta_{32} = 0, \\
\zeta_{33} &\overset{(7.8)}{=} -\frac{1}{2}\frac{\lambda}{\lambda+\mu}\Big((\partial_1 \mathbf{U}_*, \mathbf{b})x_2 - (\partial_1 \mathbf{U}_*, \mathbf{n})x_3\Big).
\end{aligned} \tag{7.44}$$

Lemma 7.6 *Let the function* $\mathbf{U}$ *be determined by* (7.1) *and the functions* $\mathbf{Q}_0$ *and* $\mathbf{Q}_1$ *by* (4.16). *Then* $\mathbf{U}|_{t=0} = \mathbf{Q}_0$ *and* $\check\rho \partial_t \mathbf{U}|_{t=0} = \check\rho \mathbf{Q}_1$ *in the sense of the space* $C([0,T]; L^2(\Omega)^3)$ *or* $C([0,T]; [\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, *respectively.*

P r o o f: The first initial condition follows easily from (4.16), (4.18) and (7.1). Let $\lfloor \mathbf{V}, \psi \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ be an arbitrary but fixed pair. Proposition 3.4 enables us to approximate this pair by a couple of smooth functions $\lfloor \mathbf{V}_\epsilon, \psi_\epsilon \rfloor \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ satisfying (3.11)–(3.13). Analogously as in the proof of Lemma 7.4, we establish the functions $\widehat{\mathbf{V}}_\epsilon$ (see (7.24)) which satisfy (7.33).

Let $\varphi \in C_0^\infty(0,T)$ be an arbitrary but fixed function. Taking $\varphi \widehat{\mathbf{V}}_\epsilon$ as a test function in (4.17), and using (7.26)–(7.30), leads to the equation

$$-\int_0^T \dot\varphi(t) \int_\Omega \rho(\partial_t \mathbf{U}_\epsilon(t), \widehat{\mathbf{V}}_\epsilon) d_\epsilon \, dxdt + \int_0^T \varphi(t) \int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon} \omega_{kl}^\epsilon(\mathbf{U}_\epsilon(t)) \Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon) d_\epsilon \, dxdt$$

$$+\int_0^T \varphi(t)\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(t))\frac{1}{\epsilon}B_\epsilon^{ij}d_\epsilon\ dxdt = \int_0^T \varphi(t)\int_\Omega(\mathbf{F}(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dxdt$$

$$+\int_0^T \varphi(t)\int_0^l\int_{\partial S}(\mathbf{G}(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dSdx_1dt. \qquad (7.45)$$

From (5.1) and (5.2), it follows that $\int_\Omega \rho(\partial_t\mathbf{U}_\epsilon(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx \in C([0,T])$ for all $\epsilon \in (0,1)$. The equation (7.45) yields that the function $\int_\Omega \rho(\partial_t\mathbf{U}_\epsilon(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx$ belongs to $W^{1,\infty}(0,T)$, which enables us to rewrite the equation (7.45) as

$$\frac{d}{dt}\int_\Omega \rho(\partial_t\mathbf{U}_\epsilon(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx + \int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(t))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx$$

$$+\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(t))\frac{1}{\epsilon}B_\epsilon^{ij}d_\epsilon\ dx = \int_\Omega(\mathbf{F}(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx$$

$$+\int_0^l\int_{\partial S}(\mathbf{G}(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dSdx_1, \text{ for a.a. } t\in(0,T). \qquad (7.46)$$

Integrating the equation (7.46) on the interval $[0,t]$, and using (4.18) and Proposition 5.1, we get

$$\int_\Omega \rho(\partial_t\mathbf{U}_\epsilon(t),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx - \int_\Omega \rho(\mathbf{Q}_{1,\epsilon},\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dx$$

$$= -\int_0^t\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dxds - \int_0^t\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\frac{1}{\epsilon}B_\epsilon^{ij}d_\epsilon\ dxdt$$

$$+\int_0^t\int_\Omega(\mathbf{F}(s),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\ dxds + \int_0^t\int_0^l\int_{\partial S}(\mathbf{G}(s),\widehat{\mathbf{V}}_\epsilon)d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dSdx_1ds. \qquad (7.47)$$

Further, we have that

$$\|\int_0^\cdot\int_\Omega(\mathbf{F}(s),\widehat{\mathbf{V}}_\epsilon d_\epsilon - \mathbf{V})\ dxds\|_{C([0,T])} \le C\|\mathbf{F}\|_{L^2(0,T;L^2(\Omega)^3)}(\|\widehat{\mathbf{V}}_\epsilon - \mathbf{V}\|_2$$

$$+\|\widehat{\mathbf{V}}_\epsilon\|_2\|d_\epsilon - 1\|_{C(\overline{\Omega})}) \overset{(3.6),(7.33)}{\to} 0,$$

$$\|\int_0^\cdot\int_0^l\int_{\partial S}[(\mathbf{G}(s),\widehat{\mathbf{V}}_\epsilon d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j} - \mathbf{V})]\ dSdx_1ds\|_{C([0,T])}$$

$$\le C\|\mathbf{G}\|_{L^2(0,l;L^2(\partial S)^3)}(\|\widehat{\mathbf{V}}_\epsilon - \mathbf{V}\|_{L^2(0,l;L^2(\partial S)^3)}$$

$$+\|\widehat{\mathbf{V}}_\epsilon\|_{L^2(0,l;L^2(\partial S)^3)}\|d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j} - 1\|_{C(\overline{(0,l)\times\partial S})})$$

$$\le \|\mathbf{G}\|_{L^2(0,l;L^2(\partial S)^3)}(\|\widehat{\mathbf{V}}_\epsilon - \mathbf{V}\|_{L^2(0,l;H^1(S)^3)}$$

$$+\|\widehat{\mathbf{V}}_\epsilon\|_{L^2(0,l;H^1(S))}\|d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j} - 1\|_{C(\overline{(0,l)\times\partial S})}) \overset{(3.7),(7.33)}{\to} 0,$$

$$\|\int_0^\cdot\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\frac{1}{\epsilon}B_\epsilon^{ij}d_\epsilon\ dxdt\|_{C([0,T])}$$

$$\leq C\|\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega)^9)}\frac{1}{\epsilon}\|B_\epsilon\|_2 \overset{(7.2),(7.32)}{\to} 0$$

for $\epsilon \to 0$. Since

$$|\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(t))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dx - \int_\Omega A_0^{ijkl}\zeta_{kl}(t)\Upsilon(\mathbf{V})\, dx|$$

$$\leq C\|\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega)^9)}\|\Upsilon(\widehat{\mathbf{V}}_\epsilon)\|_2\|(A_\epsilon^{ijkl})_{i,j,k,l=1}^3 - (A_0^{ijkl})_{i,j,k,l=1}^3\|_{C(\overline{\Omega})^{81}}$$

$$+C\|\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega)^9)}\|\Upsilon(\widehat{\mathbf{V}}_\epsilon)\|_2\|d_\epsilon - 1\|_{C(\overline{\Omega})}$$

$$+C\|\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^\infty(0,T;L^2(\Omega)^9)}\|\Upsilon(\widehat{\mathbf{V}}_\epsilon) - \Upsilon(\mathbf{V})\|_2 \overset{(3.6),(7.8),(7.31)}{\to} 0$$

for $\epsilon \to 0$ and a.a. $t \in (0,T)$, we get (using the estimate above) that

$$-\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dx \to -\int_\Omega A_0^{ijkl}\zeta_{kl}(s)\Upsilon(\mathbf{V})\, dx \text{ in } L^\infty(0,T),$$

which implies that $\int_0^t\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dxds$ converges pointwisely for all $t \in [0,T]$, and that the term $\int_0^t\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dxds$ is moreover bounded in $L^\infty(0,T)$. Passing from the pointwise convergence to the convergence in measure, we obtain that

$$-\int_0^t\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dxds \rightharpoonup -\int_0^t\int_\Omega A_0^{ijkl}\zeta_{kl}(s)\Upsilon(\mathbf{V})\, dxds$$

in $W^{1,p}(0,T)$, $p \in [1,\infty)$, and, from compact imbedding,

$$-\int_0^t\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon(s))\Upsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dxds \to -\int_0^t\int_\Omega A_0^{ijkl}\zeta_{kl}(s)\Upsilon(\mathbf{V})\, dxds$$

in $C([0,T])$. We have proved that the terms on the the right-hand side of the equation (7.47) converge strongly in $C([0,T])$ for $\epsilon \to 0$, which implies that the left-hand side must also converge in $C([0,T])$ and we get from the second convergence in (7.1), from (7.33) and (4.16) that

$$\int_\Omega \rho(\partial_t\mathbf{U}_\epsilon(t) - \mathbf{Q}_{1,\epsilon}, \widehat{\mathbf{V}}_\epsilon)d_\epsilon \, dx \to \int_0^l \check{\rho}(\partial_t\mathbf{U}(t) - \mathbf{Q}_1, \mathbf{V})\, dx_1 \text{ in } C([0,T]). \quad (7.48)$$

The rest of the proof is obvious. □

We have proved that the asymptotic dynamic model for the curved rod has the form:

$$-\int_0^T \dot{\varphi}(t)\int_0^l \check{\rho}(\partial_t\mathbf{U}(t), \mathbf{V})\, dx_1dt + \int_0^T \varphi(t)\int_0^l E[I_{x_2^2}(\partial_1\mathbf{U}_*(t), \mathbf{b})(\mathbf{V}_*', \mathbf{b})$$

$$+I_{x_3^2}(\partial_1\mathbf{U}_*(t), \mathbf{n})(\mathbf{V}_*', \mathbf{n})]\, dx_1dt + \int_0^T \varphi(t)\int_0^l \mu K(\partial_1\mathbf{U}_*(t), \mathbf{t})(\mathbf{V}_*', \mathbf{t})\, dx_1dt$$

$$= \int_0^T \varphi(t) \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}(t), \mathbf{V})\, dx_1 dt \tag{7.49}$$

for all functions $\mathbf{V}_* \in H_0^1(0,l)^3$ generated by an arbitrary couple $\lfloor \mathbf{V}, \psi \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. The function $\mathbf{U}$, which together with the function ϕ generate the function $\mathbf{U}_*$ (see(2.17)), satisfies the initial state

$$\mathbf{U}|_{t=0} = \mathbf{Q}_0 \text{ and } \check{\rho}\partial_t \mathbf{U}|_{t=0} = \check{\rho}\mathbf{Q}_1 \tag{7.50}$$

in the sense of the space $C([0,T]; L^2(0,l)^3)$ and $C([0,T]; [\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, respectively.

Lemma 7.7 *There exists the unique solution to the equation* (7.49) *satisfying* (7.50).

P r o o f: Suppose that there exist two solutions $\lfloor \mathbf{U}_j, \phi_j \rfloor \in L^\infty(0,T; \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ and $\partial_t \mathbf{U}_j \in L^\infty(0,T; L^2(0,l)^3)$, $j = 1,2$. Let us denote $\widehat{\mathbf{U}} = \mathbf{U}_1 - \mathbf{U}_2$ and $\widehat{\phi} = \phi_1 - \phi_2$. Then the couple $\lfloor \widehat{\mathbf{U}}, \widehat{\phi} \rfloor \in L^\infty(0,T; \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ generates the function $\widehat{\mathbf{U}}_* \in L^\infty(0,T; H_0^1(0,l)^3)$ (see (2.17)), $\partial_t \widehat{\mathbf{U}} \in L^\infty(0,T; L^2(0,l)^3)$,

$$- \int_0^T \dot{\varphi}(t) \int_0^l \check{\rho}(\partial_t \widehat{\mathbf{U}}(t), \mathbf{V})\, dx_1 dt + \int_0^T \varphi(t) \int_0^l E[I_{x_2^2}(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{b})(\mathbf{V}_*', \mathbf{b})$$

$$+ I_{x_3^2}(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{n})(\mathbf{V}_*', \mathbf{n}) + \mu K(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{t})(\mathbf{V}_*', \mathbf{t})]\, dx_1 dt = 0 \tag{7.51}$$

and the function $\widehat{\mathbf{U}}$ satisfies the initial state

$$\widehat{\mathbf{U}}|_{t=0} = 0 \text{ and } \check{\rho}\partial_t \widehat{\mathbf{U}}|_{t=0} = 0 \tag{7.52}$$

in the sense of the space $C([0,T]; L^2(0,l)^3)$ and $C([0,T]; [\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, respectively. From the equation (7.51), it follows that the term $\int_0^l \check{\rho}(\partial_t \widehat{\mathbf{U}}(t), \mathbf{V})\, dx dt \in W^{1,\infty}(0,T)$ for all but fixed $\mathbf{V} \in H_0^1(0,l)^3$ such that the couple $\lfloor \mathbf{V}, \psi \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. This fact enables us to rewrite (7.51) as

$$\frac{d}{dt} \int_0^l \check{\rho}(\partial_t \widehat{\mathbf{U}}(t), \mathbf{V})\, dx_1 + \int_0^l E[I_{x_2^2}(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{b})(\mathbf{V}_*', \mathbf{b}) + I_{x_3^2}(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{n})(\mathbf{V}_*', \mathbf{n})]\, dx_1$$

$$+ \int_0^l \mu K(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{t})(\mathbf{V}_*', \mathbf{t})\, dx_1 = 0. \tag{7.53}$$

Integrating on the interval $[0,t]$ and using (7.52), we get

$$\int_0^l \check{\rho}(\partial_t \widehat{\mathbf{U}}(t), \mathbf{V})\, dx_1 + \int_0^t \int_0^l E[I_{x_2^2}(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b})(\mathbf{V}_*', \mathbf{b}) + I_{x_3^2}(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{n})(\mathbf{V}_*', \mathbf{n})$$

$$+ \mu K(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{t})(\mathbf{V}_*', \mathbf{t})]\, dx_1 ds = 0 \tag{7.54}$$

for all $t \in [0, T]$. Since $\lfloor \widehat{\mathbf{U}}(t), \widehat{\phi}(t) \rfloor \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$ for a.a. $t \in (0, T)$, we can use this couple as a test function in (7.54), and we get that

$$\int_0^l \check{\rho}(\partial_t \widehat{\mathbf{U}}(t), \widehat{\mathbf{U}}(t)) \, dx_1 + \int_0^t \int_0^l E[I_{x_2^2}(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b})(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{b})$$

$$+ I_{x_3^2}(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{n})(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{n}) + \mu K(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{t})(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{t})] \, dx_1 ds = 0 \qquad (7.55)$$

for all $t \in [0, T]$.

We immediately see that

$$\int_0^l \check{\rho}(\partial_t \widehat{\mathbf{U}}(t), \widehat{\mathbf{U}}(t)) \, dx_1 = \int_0^l \partial_t (\frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2}) \, dx_1, \qquad (7.56)$$

$$\int_0^t \int_0^l E I_{x_2^2}(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b})(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{b}) \, dx_1 ds$$

$$= \int_0^l \frac{E I_{x_2^2}}{2} \partial_t (\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b}) \, ds)^2 \, dx_1, \qquad (7.57)$$

$$\int_0^t \int_0^l E I_{x_3^2}(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{n})(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{n}) \, dx_1 ds$$

$$= \int_0^l \frac{E I_{x_3^2}}{2} \partial_t (\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{n}) \, ds)^2 \, dx_1, \qquad (7.58)$$

$$\int_0^t \int_0^l \mu K(\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{t})(\partial_1 \widehat{\mathbf{U}}_*(t), \mathbf{t}) \, dx_1 ds$$

$$= \int_0^l \frac{\mu K}{2} \partial_t (\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b}) \, ds)^2 \, dx_1. \qquad (7.59)$$

Since $\widehat{\mathbf{U}} \in W^{1,\infty}(0, T; L^2(0, l)^3)$, we have for arbitrary but fixed $\varphi \in C_0^\infty(0, T)$,

$$\int_0^T \varphi(t) \int_0^l \partial_t (\frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2}) \, dx_1 dt = \int_0^l \int_0^T \varphi(t) \partial_t (\frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2}) \, dt dx_1$$

$$= - \int_0^l \int_0^T \dot{\varphi}(t) \frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2} \, dt dx_1 = - \int_0^T \dot{\varphi}(t) \int_0^l \frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2} \, dx_1 dt,$$

which implies that

$$\int_0^l \partial_t (\frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2}) \, dx_1 = \frac{d}{dt} \int_0^l \frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2} \, dx_1. \qquad (7.60)$$

We change analogously the integral and the derivative ∂_t in (7.57)–(7.59), and thus we can rewrite the equation (7.55) as

$$\frac{d}{dt} \int_0^l \frac{\check{\rho} |\widehat{\mathbf{U}}(t)|^2}{2} \, dx_1 + \frac{d}{dt} \int_0^l \frac{E I_{x_2^2}}{2} (\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b}) \, ds)^2 \, dx_1$$

$$+\frac{d}{dt}\int_0^l \frac{EI_{x_3^2}}{2}(\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{n})\ ds)^2\ dx_1$$

$$+\frac{d}{dt}\int_0^l \frac{\mu K}{2}(\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{t})\ ds)^2\ dx_1 = 0 \tag{7.61}$$

for all $t \in [0,T]$. From the assumptions on the functions $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{U}}_*$ it follows that the functions $\widehat{\mathbf{U}}$ and $\int_0^{\cdot} \partial_1 \widehat{\mathbf{U}} \in C([0,T];L^2(0,l)^3)$, which enables us to integrate (7.61) on the interval $[0,t]$, and we get from (7.52) that

$$\int_0^l \frac{\check{\rho}|\widehat{\mathbf{U}}(t)|^2}{2}\ dx_1 + \int_0^l \frac{EI_{x_2^2}}{2}(\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{b})\ ds)^2\ dx_1$$

$$+\int_0^l \frac{EI_{x_3^2}}{2}(\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{n})\ ds)^2\ dx_1 + \int_0^l \frac{\mu K}{2}(\int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s), \mathbf{t})\ ds)^2\ dx_1 = 0 \tag{7.62}$$

for all $t \in [0,T]$. Hence $\widehat{\mathbf{U}} \equiv \mathbf{0}$ as a consequence of the non-negativity of all terms in (7.62) and (4.14). Further, (7.62) yields that

$$\int_0^t \partial_1 \widehat{\mathbf{U}}_*(s,x_1)\ ds = \int_0^t (\partial_1 \widehat{\mathbf{U}}_*(s,x_1), \mathbf{t}(x_1))\mathbf{t}(x_1)$$

$$+(\partial_1 \widehat{\mathbf{U}}_*(s,x_1), \mathbf{n}(x_1))\mathbf{n}(x_1) + (\partial_1 \widehat{\mathbf{U}}_*(s,x_1), \mathbf{b}(x_1))\mathbf{b}(x_1)\ ds = 0$$

for all $t \in [0,T]$ and for arbitrary but fixed $x_1 \in (0,l)$. Then

$$\partial_1 \mathbf{U}_*(t,x_1) = \lim_{h\to 0} \frac{1}{2h}\int_{t-h}^{t+h} \partial_1 \widehat{\mathbf{U}}_*(s,x_1)\ ds$$

$$= \lim_{h\to 0} \frac{1}{2h}\left(\int_0^{t+h} \partial_1 \widehat{\mathbf{U}}_*(s,x_1)\ ds - \int_0^{t-h} \partial_1 \widehat{\mathbf{U}}_*(s,x_1)\ ds\right) = \mathbf{0}$$

for a.a $(t,x_1) \in (0,T)\times(0,l)$. Since $\mathbf{U}_* \in H_0^1(0,l)$ then also $\mathbf{U}_* \equiv \mathbf{0}$ and $\phi = -(\mathbf{U}_*, \mathbf{t}) = 0$, a contradiction. □

The proof of the main theorem of this article is now complete and we can state it:

Theorem 7.8 *Let the function* $\mathbf{\Phi}$ *be the parametrization of a unit speed curve such that* $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$. *Let, further,* $\mathbf{F} \in L^2(0,T;L^2(\Omega)^3)$, $\mathbf{G} \in W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))$ *and* $\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}$ *be defined as in Lemma 7.4. Then, there is a unique pair* $\langle \mathbf{U}, \phi\rangle \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ *such that* $\partial_t \mathbf{U} \in L^\infty(0,T;L^2(0,l)^3)$ *and satisfying the problem (7.49)–(7.50). Moreover, the constant extension to* $\Omega = (0,l)\times S$ *of* $\langle \mathbf{U}, \phi\rangle$ *may be approximated by the solutions* $\mathbf{U}_\epsilon \in L^\infty(0,T;V(\Omega)^3)\cap W^{1,\infty}(0,T;L^2(\Omega)^3)$ *of the problem (4.17)–(4.18) as follows:*

$$\mathbf{U} = \lim_{\epsilon\to 0}\mathbf{U}_\epsilon \text{ weakly in } L^\infty(0,T;H^1(\Omega)^3),$$

$$\partial_t \mathbf{U} = \lim_{\epsilon\to 0}\partial_t \mathbf{U}_\epsilon \text{ weakly in } L^\infty(0,T;L^2(\Omega)^3),$$

$$\phi = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\Big((\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - (\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big) \text{ weakly in } L^\infty(0,T;L^2(\Omega)).$$

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