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Mott law as lower bound for a random walk in a random environment

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Abstract. We consider a random walk on the support of a stationary simple point process on $\mathbb{R}^d$, $d \geq 2$ which satisfies a mixing condition w.r.t. the translations or has a strictly positive density uniformly on large enough cubes. Furthermore the point process is furnished with independent random bounded energy marks. The transition rates of the random walk decay exponentially in the jump distances and depend on the energies through a factor of the Boltzmann-type. This is an effective model for the phonon-induced hopping of electrons in disordered solids within the regime of strong Anderson localisation. We show that the rescaled random walk converges to a Brownian motion whose diffusion coefficient is bounded below by Mott's law for the variable range hopping conductivity at zero frequency. The proof of the lower bound involves estimates for the supercritical regime of an associated site percolation problem.

1. Result, motivation and overview

Let us directly describe the model and the result in a rough manner, leaving a more precise formulation of hypothesis and statements for later. Suppose given an infinite set of random points $\{x_j\} \subset \mathbb{R}^d$ distributed according to some stationary simple point process with a bounded mean density ρ. To each x_j is associated a random energy mark $E_j \in [-1, 1]$. All marks are drawn independently and identically according to a probability measure ν satisfying $\nu([-E, E]) \geq c_0 E^{1+\alpha}$ for some $\alpha \geq 0$ and $c_0 > 0$. Within each such *random environment* $\{x_j, E_j\}$, let us consider a *continuous-time random walk* over the points $\{x_j\}$ with energy dependent transition rates from x_j to x_k given by

$$c_{x_j,x_k}(E_j, E_k) \;=\; e^{-|x_j - x_k|}\; e^{-\beta(|E_j - E_k| + |E_j| + |E_k|)}\,, \tag{1.1}$$

where the positive parameter β is the inverse temperature. Our main result states that the random walk converges after appropriate space and time rescaling to a Brownian motion and that the associated diffusion coefficient $D(\beta)$ is bounded from below by

$$D(\beta) \;\geq\; c_1\, \beta^{-\frac{(\alpha+1)d}{\alpha+1+d}} \,\exp\left(-c_2\, \beta^{\frac{\alpha+1}{\alpha+1+d}}\right)\,, \tag{1.2}$$

where $d \geq 2$ is the dimension of space and c_1 and c_2 are some β-independent constants. The exponential factor on the r.h.s. is precisely as in Mott's law for the DC conductivity in disordered solids which is discussed below.

Based on the following heuristics due to Mott [Mot, SE], we expect that the power law in the exponential in (1.2) captures the good asymptotic behaviour of $\ln D(\beta)$ in the low temperature limit $\beta \uparrow \infty$ if $\nu([-E, E]) \sim c_0 E^{1+\alpha}$ as $E \to 0$. As β becomes larger, the rates (1.1) fluctuate widely with (x_j, x_k) because of the exponential energy factor. The low temperature limit effectively selects only jumps between points with energies in a small interval $[-E(\beta), E(\beta)]$ shrinking to zero as $\beta \to \infty$. Assuming that the diffusion coefficient is determined by those jumps with the largest rate, one can obtain directly the characteristic exponential factor on the right hand side of (1.2) by maximising these rates for a fixed temperature under the constraint that the mean density of points x_j with energies in $[-E(\beta), E(\beta)]$ is equal to $\rho\,\nu([-E(\beta), E(\beta)]) \sim c_0 \rho E(\beta)^{1+\alpha}$. As the characteristic mean distance $|x_j - x_k|$ between sites with optimal jump rates also varies heavily with the inverse temperature β, one speaks of a *variable range hopping regime*. A crucial (and physically reasonable, as discussed below) element of this argument is the independence of the energies E_j. The selection of the points $\{x_i\}$ with energies in the window $[-E(\beta), E(\beta)]$ then corresponds mathematically to a p-thinning with $p = \nu([-E(\beta), E(\beta)])$. It is then a well-known fact that an adequate rescaling of the p-thinning of a stationary point process converges in the limit $p \downarrow 0$ (corresponding to $\beta \uparrow \infty$) to a stationary Poisson point process (PPP) (*e.g.* [Kal, Theorem 16.19]). Hence one might call the stationary PPP the normal form of a model leading Mott's law, namely the exponential factor on the r.h.s. of (1.2) and we believe that proving the upper bound corresponding to (1.2) should therefore be most simple for the PPP. In dimension $d = 1$, a different behaviour of $D(\beta)$ is expected [LB] and this will not be considered here.

Our main motivation for studying the above model comes from its importance for *phonon-assisted hopping conduction* [SE] in disordered solids in which the Fermi level (set equal to 0 above) lies in a region of *strong Anderson localisation.* This means that, close to the Fermi level, the electron Hamiltonian has exponentially localised quantum eigenstates with localisation centres x_j and energy levels E_j. The DC conductivity of such materials would vanish if it were not for the lattice vibrations (phonons) at nonzero temperature. They induce transitions between the localised states, the rate of which can be calculated from first principle by means of the Fermi golden rule [MA, SE]. In the variable range hopping regime at low temperature, an adiabatic or rotating wave approximation can be used to treat quantum mechanically the electrons-phonon coupling [Spe]. Coherences between electronic eigenstates with different energies decay very rapidly under the resulting dissipative electronic dynamics and one can show that the hopping DC conductivity of the disordered solid coincides with the conductivity associated with a Markov jump process on the set of localisation centres $\{x_j\}$, hence justifying the use of a model of classical mechanics [BRSW]. Because Pauli blocking due to Fermi statistics of the electrons has to be taken into account, this leads to a rather complicated exclusion process (*e.g.* [Qua, FM]). If, however, the blocking is treated in an *effective medium approximation*, one obtains a family of independent random walks with rates which in good first approximation are given by (1.1) in the limit $\beta \uparrow \infty$ [MA, AHL]. Let us discuss the remaining aspects of the model. The stationarity of the underlying simple point process $\{x_j\}$ simply reflects that the material is homogeneous, while the independence of the energy marks is compatible with Poisson level statistics, which is a general rough indicator for the localisation regime and has been proven to hold for an Anderson model [Min]. The exponent α allows to model a possible Coulomb pseudogap in the density of states [SE].

Having in mind the Einstein relation between the conductivity and the diffusion coefficient (which can be stated as a theorem for a number of models [Spo]), the lower bound (1.2) gives a lower bound on the hopping DC conductivity. In the above materials, the DC conductivity shows experimentally Mott's law, namely a characteristic low-temperature behaviour which is well approximated by the exponential factor in the r.h.s. of (1.2) with $\alpha = 0$ or $\alpha = d-1$, as predicted by Mott [Mot] and Efros and Shklovskii [EF], respectively, based on the optimisation argument discussed above. A first convincing justification of this argument was given by Ambegoakar, Halperin and Langer [AHL], who first reduced the hopping model to a related random resistor network, in a manner similar to the work of Miller and Abrahms [MA], and then pointed out that the constant c_2 can be estimated using percolation theory [SE]. Our proof of the lower bound (1.2) is much inspired by this work. Let us also mention that the low frequency AC conductivity (response to an oscillating electric field) in disordered solids has recently been studied within a quantum-mechanical one-body approximation in [KLP]. Here the energy necessary for a jump between localised states comes from a resonance at the frequency of the external electric field rather than a phonon. It leads to another well-known formula for the conductivity which is also due to Mott.

The model described above is a *random walk in a random environment.* A main tool in this work is the early contribution of De Masi, Ferrari, Goldstein and Wick [DFGW] which is based on prior work by Kipnis and Varadhan [KV]. They construct a new process, called the *environment viewed by the particle*, which allows to translate the homogeneity of the medium into properties of the random walk. This implies weak convergence in probability to a Brownian motion whose diffusion matrix can be characterised by a variational formula. The main virtue of this variational characterisation is that it allows to bound the diffusion coefficient from below through bounds on the transition rates (1.1) and, just as in [DFGW], by the diffusion coefficient of periodic approximants (in the limit of large periods). The diffusion coefficient of these approximants can be computed as the resistance of a random resistor network. This in turn can then be bounded by invoking estimates from percolation theory. This leads to a proof of (1.2). Let us note that the optimised lower bound therefore results from a critical resistor network roughly approximating the one appearing in [AHL].

The paper is organised as follows. In Section 2 we recall some definitions and results about point processes and state some technical results needed later on. In Section 3 we first show that the continuous-time random walk in the random environment is well defined by verifying the absence of explosion phenomena caused by some possible high concentration of points x_j in finite regions. We also give the precise hypothesis on the point process and then state the main results of the paper (Theorems 1, 2 and 3). The analysis of the dynamics of the environment viewed from the particle given in Section 4 is partially inspired by [DFGW, Section 4], however, we rather investigate directly the continuous-time Markov process and prove suitable bounds on the mean square displacement (Proposition 2), thus allowing to apply the general Theorem 2.2 of [DFGW] in order to prove Theorem 1 where a variational formula for the diffusion coefficient D is given. In the remaining sections we show how to bound from below this variational formula in terms of the conductance of suitable random resistor networks, which in turn is bounded below by invoking estimates for the supercritical regime of site percolation.

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2. The random environment

In this section, we recall some properties of point processes (for more details, see [DV, FKAS, MKM, Kal, Tho]), fix our notation and give the term *random environment* a precise sense.

In the sequel, given a topological space X, $\mathcal{B}(X)$ will denote the σ-algebra of Borel subsets of X. Given a set A, $|A|$ will denote its cardinality, while, if $A \in \mathcal{B}(\mathbb{R}^d)$, $\ell(A)$ will denote its Lebesgue measure. Moreover, given a probability measure μ, we write $\mathbf{E}_\mu$ for the corresponding expectation. Finally, C_r is the open cube $C_r := \left(-\frac{r}{2}, \frac{r}{2}\right)^d$ in $\mathbb{R}^d$ with side length r.

2.1. Stationary simple marked point processes (SSMPP).

Given a bounded complete separable metric space K, consider the space $\mathcal{N} := \mathcal{N}(\mathbb{R}^d \times K)$ of all counting measures ξ on $\mathbb{R}^d \times K$, *i.e.* integer-valued measures such that $\xi(B \times K) < \infty$ for any bounded set $B \in \mathcal{B}(\mathbb{R}^d)$. One can show that $\xi \in \mathcal{N}$ if and only if $\xi = \sum_i \delta_{(x_i,k_i)}$ where δ is the Dirac measure and $\{(x_i, k_i)\}$ is a countable set of (not necessarily distinct) points in $\mathbb{R}^d \times K$ with at most finitely many points in any bounded set. Then k_i is called the *mark* at x_i. Given $\xi \in \mathcal{N}$, we write $\hat{\xi} \in \mathcal{N}(\mathbb{R}^d)$ for the counting measure on $\mathbb{R}^d$ defined by $\hat{\xi}(B) = \xi(B \times K)$ for any $B \in \mathcal{B}(\mathbb{R}^d)$. Given $x \in \mathbb{R}^d$, we write $x \in \hat{\xi}$ whenever $x \in \operatorname{supp}(\hat{\xi})$. If $\hat{\xi}(\{x\}) \leq 1$ for any $x \in \mathbb{R}^d$, we say that $\xi \in \mathcal{N}$ is *simple* and write $k_{x_i} := k_i$ for any $x_i \in \hat{\xi}$.

A metric on $\mathcal{N}$ can be defined in the following way [MKM, Section 1.15]. Fix an element $k^* \in K$. Denote by $B_r(x,k)$ and B_r the open balls of radius $r > 0$ in $\mathbb{R}^d \times K$ centred on (x,k) and on $(0,k^*)$, respectively. Let $\xi = \sum_{i\in I} \delta_{(x_i,k_i)}$ and $\xi' = \sum_{j\in J} \delta_{(x'_j,k'_j)}$ be elements of $\mathcal{N}$, where I, J are countable sets. Then ξ and ξ' are close to each other if any point (x_i, k_i) contained in B_n is close to a point (x'_j, k'_j) for arbitrary large n, up to "boundary effects". More precisely, given a positive integer n, let $d_n(\xi, \xi')$ be the infimum over all $\varepsilon > 0$ such that there is a one-to-one map f from a (possibly empty) subset D of I into a subset of J with the properties:

(i) $\mathrm{supp}(\xi) \cap B_{n-\varepsilon} \subset \{(x_i, k_i) \,:\, i \in D\}$;

(ii) $\mathrm{supp}(\xi') \cap B_{n-\varepsilon} \subset \{(x'_j, k'_j) \,:\, j \in f(D)\}$;

(iii) $(x'_{f(i)}, k'_{f(i)}) \in B_\varepsilon(x_i, k_i)$ for $i \in D$.

One can show that $d_{\mathcal{N}}(\xi, \xi') = \sum_{n=1}^{\infty} 2^{-n} d_n(\xi, \xi')$ is a bounded metric on $\mathcal{N}$ and for this metric $\mathcal{N}$ is complete and separable. Moreover, the sets $\{\xi \in \mathcal{N} \,:\, \xi(B) = j\}$, $B \in \mathcal{B}(\mathbb{R}^d \times K)$, $j \in \mathbb{N}$, generate the Borel σ-algebra $\mathcal{B}(\mathcal{N})$ and $d_{\mathcal{N}}$ generates the coarsest topology such that $\xi \in \mathcal{N} \mapsto \int \xi(dx, dk)\, f(x, k)$ is continuous for any continuous function $f \geq 0$ on $\mathbb{R}^d \times K$ with bounded support. Finally, by choosing different reference points k^* one obtains equivalent metrics.

A *marked point process* on $\mathbb{R}^d$ with marks in K is then a measurable map Φ from a probability space into $\mathcal{N}$. We denote by $\mathcal{P}$ its distribution (a probability measure on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$). We say that the process is *simple* if $\mathcal{P}$-almost all $\xi \in \mathcal{N}$ are simple.

The translations on $\mathbb{R}^d$ extend naturally to $\mathbb{R}^d \times K$ by $S_x : (y, k) \mapsto (x + y, k)$. This induces an action S of the translation group $\mathbb{R}^d$ on $\mathcal{N}$ by $(S_x \xi)(B) = \xi(S_x B)$, where $B \in \mathcal{B}(\mathbb{R}^d \times K)$ and $x \in \mathbb{R}^d$. For simple counting measures,

$$S_x \xi \;=\; \sum_{y \in \hat{\xi}} \delta_{(y-x, k_y)} \;.$$

A marked point process is said to be *stationary* if $\mathcal{P}(A) = \mathcal{P}(S_x A)$ for all $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathcal{N})$, and *(space) ergodic* if the σ-algebra of translation invariant sets is trivial, i.e. if $A \in \mathcal{B}(\mathcal{N})$ satisfies $S_x A = A$ for all $x \in \mathbb{R}^d$, then $\mathcal{P}(A) \in \{0, 1\}$. Due to [DV, Proposition 10.1.IV], if $\mathcal{P}$ is stationary and gives no weight to the trivial measure without any point (which will be assumed here), then

$$\mathcal{P}\big(\xi \in \mathcal{N} \,:\, \big|\mathrm{supp}(\hat{\xi})\big| = \infty\big) \;=\; 1\,. \tag{2.1}$$

Finally, we write ρ_κ for the κth moment of $\hat{\xi}(C_1)$:

$$\rho_\kappa \;:=\; \int_{\mathcal{N}} \mathcal{P}(d\xi)\; \hat{\xi}(C_1)^\kappa \;. \tag{2.2}$$

If the process is stationary $\rho := \rho_1$ is called the *intensity* of the process. In this case, it follows that for all $B \in \mathcal{B}(\mathbb{R}^d)$ with $\ell(B) > 0$,

$$\rho \;=\; \frac{1}{\ell(B)} \int_{\mathcal{N}} \mathcal{P}(d\xi)\; \hat{\xi}(B) \;.$$

The above definitions extend to point processes on $\mathbb{R}^d$ since they can be thought of as marked point processes with trivial mark space, *i.e.* K is reduced to one point.

Definition 1. *In this article, we call* random environment *a stationary simple marked point process (SSMPP) with marks in $K = [-1, 1]$, finite intensity $\rho > 0$ and satisfying the condition*

$$\xi \;\neq\; S_x \xi \qquad \forall\, x \in \mathbb{R}^d \setminus \{0\}\,, \qquad \mathcal{P} \;\; a.s.\;, \tag{2.3}$$

where $\mathcal{P}$ denotes the distribution of the point process. Its marks are denoted E_{x_i} and E_x instead of k_{x_i} and k_x and they are also called energy marks.

In what follows, it will always be assumed that $\mathcal{P}$ is the distribution of the random environment. In order to shorten notations, we will also write $\mathcal{N}$ and $\hat{\mathcal{N}}$ respectively for $\mathcal{N}(\mathbb{R}^d \times [-1,1])$ and $\mathcal{N}(\mathbb{R}^d)$ below.

A special role will be played by processes obtained by the procedure of randomisation, which we recall now together with the related notion of thinning (see [Kal]). Let $\hat{\Phi}$ be a stationary simple point process (SSPP) on $\mathbb{R}^d$, ν be a probability measure on $[-1,1]$ and $p \in [0,1]$. The *ν–randomisation* of $\hat{\Phi}$ is the SSMPP Φ_ν obtained by assigning to each realization $\hat{\xi} = \sum_{i\in I} \delta_{x_i}$ of $\hat{\Phi}$ the measure $\xi = \sum_{i\in I} \delta_{(x_i,E_i)}$, where $\{E_i\}_{i\in I}$ are independent random variables having the same distribution ν. Finally, the *p–thinning* $\hat{\Phi}_p$ of $\hat{\Phi}$ is the SSPP on $\mathbb{R}^d$ obtained by assigning to each realization $\sum_{i\in I} \delta_{x_i}$ of $\hat{\Phi}$ the measure $\sum_{i\in I} P_i\, \delta_{x_i}$, where $\{P_i\}_{i\in I}$ are independent Bernoulli variables with $\mathrm{Prob}(P_i = 1) = p$ and $\mathrm{Prob}(P_i = 0) = 1 - p$. Both the point processes Φ_ν and $\hat{\Phi}_p$ are examples of *stationary cluster processes*, also called *homegenous cluster fields* (see [DV, Chapter 8] and [MKM, Chapter 10]). In particular, ergodicity is conserved by ν–randomisation and p–thinning ([DV, Proposition 10.3.IX] and [MKM, Proposition 11.1.4]). Finally, note that if ν is non trivial (i.e. ν is not concentrate on a single point) then the point process Φ_ν satisfies (2.3) whenever (2.1) holds.

A *Poisson point process* (PPP) appears, as discussed in the introduction, naturally as limit distribution of thinnings. Given a measure μ on X, with X equal to $\mathbb{R}^d$ or $\mathbb{R}^d \times [-1,1]$, the PPP on X with intensity measure μ is defined by the two conditions (i) for any $B \in \mathcal{B}(X)$, $\xi(B)$ is a Poisson random variable with expectation $\mu(B)$; (ii) for any disjoint sets $B_1, \ldots, B_n \in \mathcal{B}(X)$, $\xi(B_1), \ldots, \xi(B_n)$ are independent.

Examples of random environments can be constructed as follows.

Example 1. A PPP on $\mathbb{R}^d$ is stationary if and only if its intensity measure μ is proportional to the Lebesgue measure, $\mu = \rho\, dx$. In such a case it is an ergodic process and satisfies $\rho_\kappa < \infty$ for any $\kappa > 0$. Its p–thinning is the PPP on $\mathbb{R}^d$ with intensity $p\rho$ while its ν–randomisation is the PPP on $\mathbb{R}^d \times [-1,1]$ with intensity measure $\rho\, dx \otimes \nu$, which satisfies all hypothesis of Definition 1.

Example 2. Let us associate to the uniformly distributed random variable y in the unit cube C_1 the point measure $\hat{\xi} = \sum_{z\in\mathbb{Z}^d} \delta_{z+y}$. The corresponding point process is an ergodic SSPP satisfying $\rho_\kappa = 1$ for any $\kappa > 0$. Note that if ν is non trivial, then its ν–randomisation is a random environment.

Other examples of ergodic SSMPP can be obtained by means of SSPP with short–range correlations (see [DV, Exercise 10.3.4]). Of particular relevance for solid state physics are point processes associated to random or quasiperiodic tilings [BHZ]. These processes have a uniform lower bound on the density of points. Periodic systems, like a lattice, do not form a random environment, unless combined with a randomisation procedure. The following example of a non ergodic SSMPP not satisfying (2.3) can readily be generalised to any dimension d.

Example 3. Let $d = 1$, the random variable y be uniformly distributed in the unit cube C_1 and let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed random variables with values in $[-1,1]$. To $y, Y_1, Y_2, \ldots, Y_n$ associate the marked point measure $\xi = \sum_{z\in\mathbb{Z}} \delta_{z+y,E_z}$ where $E_z := Y_j$ if $z \in j + n\mathbb{Z}$.

2.2. **The Palm distribution.** We would like now to "pick up at random" a point among $\{x_i\}$ and take it as the origin. One thus looks at the borelian subset of $\mathcal{N}$

$$\mathcal{N}_0 = \left\{ \xi \in \mathcal{N} : 0 \in \hat{\xi} \right\} .$$

Since $\mathcal{N}_0$ is closed, it defines a complete separable metric space. Note that $x \in \hat{\xi}$ if and only if $S_x\xi \in \mathcal{N}_0$. The Palm distribution $\mathcal{P}_0$ on $\mathcal{N}_0$ associated to $\mathcal{P}$ is now defined as follows. Consider the measurable map $\mathcal{G}$ from $\mathcal{N}$ into $\mathcal{N}(\mathbb{R}^d \times \mathcal{N}_0)$ given by

$$\xi \;\mapsto\; \xi^* \;=\; \sum_{x\in\hat{\xi}} \delta_{(x,S_x\xi)} \;.$$

Let $\mathcal{P}^* = \mathcal{G}_*\mathcal{P}$ be the distribution of the marked point process on $\mathbb{R}^d \times \mathcal{N}_0$ with mark space $\mathcal{N}_0$, namely $\mathcal{P}^*$ is the image under $\mathcal{G}$ of the probability measure $\mathcal{P}$ on $\mathbb{R}^d \times [-1,1]$. It is easy to show that $\mathcal{G} \circ S_x = S_x^* \circ \mathcal{G}$ for $x \in \mathbb{R}^d$ where S_x^* is the action on $\mathbb{R}^d \times \mathcal{N}_0$ of the translations given by $(y,\xi) \mapsto (y+x,\xi)$ as above. As a result, $\mathcal{P}^*$ is also stationary. Then, for any fixed $A \in \mathcal{B}(\mathcal{N}_0)$, the measure $\mu_A(B) = \int \mathcal{P}^*(d\xi^*)\,\xi^*(B\times A)$ on $\mathbb{R}^d$ is translation invariant and thus proportional to the Lebesgue measure. This implies that

$$\mathcal{C}_{\mathcal{P}}(A) \;:=\; \int_{\mathcal{N}(\mathbb{R}^d\times\mathcal{N}_0)} \mathcal{P}^*(d\xi^*)\,\xi^*(C_1\times A) \;=\; \frac{1}{N^d}\int_{\mathcal{N}(\mathbb{R}^d\times\mathcal{N}_0)} \mathcal{P}^*(d\xi^*)\,\xi^*(C_N\times A)$$

for any $N>0$ and any $A \in \mathcal{B}(\mathcal{N}_0)$. The *Palm distribution* associated to $\mathcal{P}$ is the probability measure $\mathcal{P}_0$ on $\mathcal{N}_0$ obtained from $\mathcal{C}_{\mathcal{P}}$ by normalisation, namely, $\mathcal{P}_0 = \rho^{-1}\mathcal{C}_{\mathcal{P}}$, where ρ is defined in (2.2). Thus

$$\mathcal{P}_0(A) \;:=\; \frac{1}{\rho}\frac{1}{N^d}\int_{\mathcal{N}} \mathcal{P}(d\xi)\int_{C_N} \hat{\xi}(dx)\,\chi_A(S_x\xi)\;, \tag{2.4}$$

where χ_A is the characteristic function on the Borel set $A \subset \mathcal{N}_0$. One can show [FKAS, Theorem 1.2.8] that for any nonnegative measurable function f on $\mathbb{R}^d \times \mathcal{N}_0$

$$\int_{\mathbb{R}^d} dx \int_{\mathcal{N}_0} \mathcal{P}_0(d\xi)\; f(x,\xi) \;=\; \frac{1}{\rho}\int_{\mathcal{N}} \mathcal{P}(d\xi)\int_{\mathbb{R}^d} \hat{\xi}(dx)\; f(x,S_x\xi)\;, \tag{2.5}$$

which is used in [DV] as the definition of $\mathcal{P}_0$. Similarly, there is a Palm distribution $\hat{\mathcal{P}}_0$ on $\hat{\mathcal{N}}_0 := \mathcal{N}_0(\mathbb{R}^d)$ associated to the distribution $\hat{\mathcal{P}}$ of a SSPP on $\mathbb{R}^d$.

It is known that the Palm distribution of a stationary PPP on $\mathbb{R}^d$ with distribution $\hat{\mathcal{P}}$ (Example 1 above) is the convolution $\hat{\mathcal{P}}_0 = \hat{\mathcal{P}} * \delta_{\delta_0}$ of $\hat{\mathcal{P}}$ with the Dirac measure at $\hat{\xi} = \delta_0$ (*i.e.* $\hat{\mathcal{P}}_0$ is simply obtained by adding a point at the origin). The Palm distribution of a PPP on $\mathbb{R}^d \times [-1,1]$ with intensity measure $\rho\, dx \otimes \nu$ is the convolution $\mathcal{P}_0 = \mathcal{P} * \zeta$ where ζ is the distribution of a marked point process obtained by ν–randomisation of δ_{δ_0}. The Palm distribution associated to Example 2 is $\hat{\mathcal{P}}_0 = \delta_{\sum_{x\in\mathbb{Z}^d}\delta_x}$. Its ν–randomisation is the Palm distribution of the ν–randomisation of Example 2.

We collect in the lemma below a number of results on the Palm distribution which will be needed in the sequel. Their proofs will be given in the Appendix.

Lemma 1. (i) *Let $k : \mathcal{N}_0 \times \mathcal{N}_0 \to \mathbb{R}$ be a measurable function such that $\int \hat{\xi}(dx)\,|k(\xi,S_x\xi)|$ and $\int \hat{\xi}(dx)\,|k(S_x\xi,\xi)|$ are in $L^1(\mathcal{N}_0,\mathcal{P}_0)$. Then*

$$\int \mathcal{P}_0(d\xi)\int \hat{\xi}(dx)\; k(\xi,S_x\xi) \;=\; \int \mathcal{P}_0(d\xi)\int \hat{\xi}(dx)\; k(S_x\xi,\xi)\;.$$

(ii) *Let $\Gamma \in \mathcal{B}(\mathcal{N})$ be such that $S_x\Gamma = \Gamma$ for all $x \in \mathbb{R}^d$. Then $\mathcal{P}(\Gamma) = 1$ if and only if $\mathcal{P}_0(\Gamma_0) = 1$ with $\Gamma_0 = \Gamma \cap \mathcal{N}_0$.*

(iii) *Let $\mathcal{P}$ be ergodic and $A, B \in \mathcal{B}(\mathcal{N}_0)$ be such that $B \subset A$, $\mathcal{P}_0(A \setminus B) = 0$ and $S_x\xi \in A$ for any $\xi \in B$ and any $x \in \hat{\xi}$. Then $\mathcal{P}_0(A) \in \{0,1\}$.*

(iv) *Let $A_j \in \mathcal{B}(\mathbb{R}^d)$ for $j = 1, \dots, n$. Then*

$$\mathbf{E}_{\mathcal{P}_0}\left(\prod_{j=1}^{n} \hat{\xi}(A_j)\right) \leq \frac{c}{\rho}\mathbf{E}_{\mathcal{P}}\big(\hat{\xi}(C_1)^{n+1}\big) + \frac{c}{\rho}\sum_{j=1}^{n}\mathbf{E}_{\mathcal{P}}\big(\hat{\xi}(\tilde{A}_j)^{n+1}\big), \tag{2.6}$$

where c is a positive constant depending on n and $\tilde{A}_j := \cup_{x \in C_1}\big(A_j + x\big)$.

Let us define the subset $\mathcal{W} \subset \mathcal{N}_0$ as

$$\mathcal{W} := \big\{\xi \in \mathcal{N}_0 \,:\, S_x\xi \neq \xi \;\forall x \in \mathbb{R}^d \setminus \{0\}\big\}. \tag{2.7}$$

Note that $S_x\xi \in \mathcal{W}$ if $\xi \in \mathcal{W}$, $x \in \hat{\xi}$. In particular, condition (2.3) and Lemma 1(ii) imply that $\mathcal{P}_0(\mathcal{W}) = 1$ (more precisely, (2.3) and $\mathcal{P}_0(\mathcal{W}) = 1$ are equivalent).

Remark 1. We remark here a very simple geometric property of point measures $\xi \in \mathcal{W}$, which will be fundamental in order to apply the methods developed in [KV] and [DFGW]:

Let us fix $\xi \in \mathcal{W}$, consider a sequence $\{x_n\}_{n \geq 0}$ of elements in $\mathrm{supp}(\hat{\xi})$ with $x_0 = 0$ and set $\xi_n := S_{x_n}\xi$. The ξ_n can be thought of as the *environment viewed from the point* x_n. Due to the definition of $\mathcal{W}$, the sequence $\{x_n\}_{n \in \mathbb{N}}$ can be recovered from $\{\xi_n\}_{n \in \mathbb{N}}$ by means of the identities

$$x_{n+1} - x_n = \Delta(\xi_n, \xi_{n+1}), \qquad n \in \mathbb{N},$$

where the function $\Delta : \mathcal{W} \times \mathcal{N}_0 \to \mathbb{R}^d$ is defined as

$$\Delta(\xi', \xi'') := \begin{cases} x & \text{if } \xi'' = S_x\xi', \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

3. Dynamics of a particle in a fixed environment

As a preamble, let us fix some notations and recall some general facts about jump Markov processes. In what follows, given a complete separable metric space Z we denote by $\mathcal{F}(Z)$ the family of bounded Borel functions on Z and, given a (not necessarily finite) interval $I \subset \mathbb{R}$, we denote by $D(I, Z)$ the space of right continuous paths $\underline{z} = (z_t)_{t \in I}$, $z_t \in Z$, having left limits. The path space $D(I, Z)$ is endowed with the Skorohod topology [Bil] which is the natural choice for the study of jump Markov processes. For a time $s \geq 0$, the time translation τ_s is defined as

$$\tau_s : D([0,\infty), Z) \to D([0,\infty), Z), \qquad (\tau_s\underline{z})_t := z_{t+s}.$$

Moreover, given $0 \leq a < b$, we denote by $R_{[a,b]}$ the function

$$R_{[a,b]} : D([0,\infty), Z) \to D([a,b], Z), \qquad (R_{[a,b]}\underline{z})_t := \lim_{\delta \downarrow 0} z_{a+b-t-\delta}.$$

$R_{[a,b]}$ is the time–reflection of $(z_t)_{t \in [a,b]}$ w.r.t. the middle point of $[a,b]$, and it can naturally be extended to paths on $[0, a+b]$.

A continuous–time Markov process with path in $D([0,\infty), Z)$ and distribution $\mathbf{p}$ is called *stationary* if

$$\mathbf{E}_{\mathbf{p}}(F) = \mathbf{E}_{\mathbf{p}}(F \circ \tau_s), \qquad \forall\, s \geq 0,$$

and for any bounded Borel function F on $D([0,\infty), Z)$. It is called *reversible* if

$$\mathbf{E}_{\mathbf{p}}(F) = \mathbf{E}_{\mathbf{p}}(F \circ R_{[a,b]}), \qquad \forall\, a, b : b > a \geq 0,$$

for any bounded Borel function F on $D([0,\infty),Z)$ such that $F(\underline{z})$ depends only on $(z_t)_{t\in[a,b]}$. Thanks to the Markov property, one can show that stationarity is equivalent to

$$\mathbf{E}_{\mathbf{p}}\big(f(z_0)\big) = \mathbf{E}_{\mathbf{p}}\big(f(z_s)\big)\ , \qquad \forall\, s\geq 0\ ,\ \ \forall\, f\in\mathcal{F}(Z), \tag{3.1}$$

while reversibility is equivalent to

$$\mathbf{E}_{\mathbf{p}}\big(f(z_0)g(z_s)\big) = \mathbf{E}_{\mathbf{p}}\big(g(z_0)f(z_s)\big)\ , \qquad \forall\, s\geq 0\ ,\ \ \forall\, f,g\in\mathcal{F}(Z)\ . \tag{3.2}$$

In particular, stationarity follows from reversibility. Finally, the Markov process is called *(time) ergodic* if $\mathbf{p}(A)\in\{0,1\}$ whenever $A\in\mathcal{B}\big(D([0,\infty))\big)$ is time-shift invariant, i.e. $A=\tau_s A$ for all $s\geq 0$.

Recall that if the Markov process is stationary then it can be extended to a Markov process with path space $D(\mathbb{R},Z)$ and the resulting distribution is univocally determined (this follows from Kolmogorov's extension theorem and the regularity of paths). Now stationarity, reversibility and ergodicity of the extended process are defined as above by means of τ_s, $s\in\mathbb{R}$, and $R_{[a,b]}$, $-\infty<a<b<\infty$. Then one can check that these properties are preserved by extension (for what concerns ergodicity, see in particular [Ros, Chapter 15, p. 96–97]). Therefore our definitions coincide with those in [DFGW].

All the above definitions and remarks can be extended in a natural way to discrete–time Markov processes (with path space $Z^{\mathbb{N}}$). Moreover, in the discrete case, stationarity and reversibility are equivalent respectively to (3.1) and (3.2) with $s=1$.

3.1. **Construction of the dynamics.** In this section, we consider a *fixed* configuration $\xi\in\mathcal{N}_0$ of the environment and define the random dynamics of a particle in this environment. To this aim, let us introduce the *symmetric* transition rates for $x,y\in\hat{\xi}$ with $x\neq y$

$$c_{x,y}(\xi)\ :=\ c_{x,y}(E_x,E_y)\ =\ \exp\big(-|x-y|-\beta(|E_x-E_y|+|E_x|+|E_y|)\big)\ , \tag{3.3}$$

where E_x,E_y are the energy marks at x,y and $\beta>0$ is the inverse temperature. It is convenient to set $c_{x,x}(\xi):=0$. Finally, we set

$$\Omega_\xi\ :=\ D([0,\infty),\mathrm{supp}(\hat{\xi}))$$

and write $(X^\xi_t)_{t\geq 0}$ for a generic element of Ω_ξ. Given $x\in\hat{\xi}$, we want to define a continuous-time random walk with paths in Ω_ξ starting at x and having the above jump rates, *i.e.* if $\mathbf{P}^\xi_x$ denotes its distribution on Ω_ξ then the set of stationary transition probabilities $p^\xi_{t-t_0}(z|y):=\mathbf{P}^\xi_x(X^\xi_t=z|X^\xi_{t_0}=y)$, $z,y\in\hat{\xi}$, $t\geq t_0\geq 0$, satisfy the following conditions for small values of t:

(C1) $p^\xi_t(y|x)=c_{x,y}(\xi)\,t+o(t)$ for $x,y\in\hat{\xi}$ and $x\neq y$;

(C2) $p^\xi_t(x|x)=1-\lambda_x(\xi)\,t+o(t)$ for $x\in\hat{\xi}$ where

$$\lambda_x(\xi)\ :=\ \sum_{y\in\hat{\xi}}c_{x,y}(\xi)\ =\ \int\xi(dy,dE)\,c_{x,y}(E_x,E)\ .$$

Note that the above formulas are meaningful for $\mathcal{P}_0$ almost all ξ if $\mathbf{E}_{\mathcal{P}_0}(\lambda_0)<\infty$. In fact, due to the bound $\lambda_z(\xi)\leq e^{4\beta}\,e^{|z|}\,\lambda_0(\xi)$, one can infer that $\lambda_z(\xi)<\infty$ for any $z\in\hat{\xi}$, $\mathcal{P}_0$ a.s. . The following lemma gives a simple criterion in order to check the condition $\mathbf{E}_{\mathcal{P}_0}(\lambda_0)<\infty$.

Lemma 2. *For any positive integer k, $\mathbf{E}_{\mathcal{P}_0}(\lambda_0^k) < \infty$ if and only if $\rho_{k+1} < \infty$.*

Proof. Note that for suitable positive constants c_1, c_2

$$c_1 \sum_{z\in\mathbb{Z}^d} \hat{\xi}(C_1+z)e^{-|z|} \;\le\; \lambda_0(\xi) \;\le c_2 \sum_{z\in\mathbb{Z}^d} \hat{\xi}(C_1+z)e^{-|z|}\,, \qquad \mathcal{P}_0\text{-a.s.}\,,$$

The criterion then follows by expanding the k-th power of the above lower and upper bounds, applying Lemma 1(iv) and using the stationarity of $\mathcal{P}$. □

Let us point out that $c_{x,y}(\xi)$ and $\lambda_x(\xi)$ are *covariant*, that is, for any $\xi \in \mathcal{N}_0$ and $y,z \in \widehat{S_x\xi}$,

$$c_{z,y}(S_x\xi) \;=\; c_{z+x,y+x}(\xi)\,, \qquad \lambda_y(S_x\xi) \;=\; \lambda_{y+x}(\xi)\,. \tag{3.4}$$

Proposition 1. *Let $\mathcal{P}$ be ergodic with $\rho_2 < \infty$ or let $\mathcal{P}$ satisfy $\rho_3 < \infty$. Then for $\mathcal{P}_0$–almost all $\xi \in \mathcal{N}_0$ and for all $x \in \hat{\xi}$, there exists a unique probability measure $\mathbf{P}_x^\xi$ on Ω_ξ of a pure jump Markov process starting at x satisfying the infinitesimal conditions (C1) and (C2).*

The construction of the dynamics follows standard references (*e.g.* [Bre, Chapter 15] and [Kal, Chapter 12]). After arriving at site $y \in \hat{\xi}$, the particle waits an exponential time with parameter $\lambda_y(\xi)$ and then jumps to another site $z \in \hat{\xi}$ with probability

$$p^\xi(z|y) \;:=\; \frac{c_{y,z}(\xi)}{\lambda_y(\xi)}\,. \tag{3.5}$$

More precisely, consider $\xi \in \mathcal{N}_0$ such that $0 < \lambda_z(\xi) < \infty$ for any $z \in \hat{\xi}$ and set $\tilde{\Omega}_\xi := \big(\operatorname{supp}(\hat{\xi})\big)^{\mathbb{N}}$. A generic path in $\tilde{\Omega}_\xi$ is denoted by $\big(\tilde{X}_n^\xi\big)_{n\ge0}$. Given $x \in \hat{\xi}$, let $\tilde{\mathbf{P}}_x^\xi$ be the distribution on $\tilde{\Omega}_\xi$ of a discrete–time random walk on $\operatorname{supp}(\hat{\xi})$ starting in x and having transition probabilities $p^\xi(z|y)$. Let $\big(\Theta,\mathbf{Q}\big)$ be another probability space where the random variables $T_{n,z}^\xi$, $z \in \hat{\xi}$, $n \in \mathbb{N}$, are independent and exponentially distributed with parameter $\lambda_z(\xi)$, namely $\mathbf{Q}\big(T_{z,n}^\xi > t\big) = \exp\big(-\lambda_z(\xi)t\big)$. On the probability space $(\tilde{\Omega}_\xi \times \Theta, \tilde{\mathbf{P}}_x^\xi \otimes \mathbf{Q})$ define the following functions:

$$\begin{aligned} &R_0^\xi \;:=\; 0\,; \quad R_n^\xi := T^\xi_{0,\tilde{X}_0^\xi} + T^\xi_{1,\tilde{X}_1^\xi} + \cdots + T^\xi_{n-1,\tilde{X}_{n-1}^\xi} \quad \text{if} \quad n \ge 1\,. \\ &n_*^\xi(t) \;:=\; n \quad \text{if} \quad R_n^\xi \le t < R_{n+1}^\xi\,, \end{aligned}$$

Note that $n_*^\xi(t)$ is well posed for any $t \ge 0$ only if $\lim_{n\uparrow\infty} R_n^\xi = \infty$. If

$$\tilde{\mathbf{P}}_x^\xi \otimes \mathbf{Q}\Big(\lim_{n\uparrow\infty} R_n^\xi = \infty\Big) \;=\; 1\,, \tag{3.6}$$

then, due to [Bre, Theorem 15.37], the random walk $(\,\tilde{X}^\xi_{n_*^\xi(t)}\,)_{t\ge0}$, defined $\tilde{\mathbf{P}}_x^\xi \otimes \mathbf{Q}$–almost everywhere, is a jump Markov process whose distribution satisfies the infinitesimal conditions (C1) and (C2). The condition $\lim_{n\uparrow\infty} R_n^\xi = \infty$ assures that no *explosion phenomenon* takes place, notably only finitely many jumps can occur in finite time intervals.

In order to prove Proposition 1, and motivated by further applications, it is convenient to consider the discrete-time Markov process $\big(S_{\tilde{X}_n^\xi}\xi\,\big)_{n\ge0}$ defined on $\big(\tilde{\Omega}_\xi, \tilde{\mathbf{P}}_0^\xi\big)$, call $\tilde{\mathbf{P}}_\xi$ its distribution on $\tilde{\Xi} := \mathcal{N}_0^{\mathbb{N}}$ and denote its generic elements by $(\xi_n)_{n\ge0}$. Such a Markov process can be thought of as the *environment viewed from the particle* performing the discrete–time random walk with distribution $\tilde{\mathbf{P}}_0^\xi$. Let us point out a few properties of the distribution $\tilde{\mathbf{P}}_\xi$. First, we remark that due to (3.4), the process $\big(\lambda_{\tilde{X}_n^\xi}(\xi)\big)_{n\in\mathbb{N}}$ defined on $(\tilde{\Omega}_\xi, \tilde{\mathbf{P}}_0^\xi)$ and the process $(\lambda_0(\xi_n))_{n\in\mathbb{N}}$ defined on $(\tilde{\Xi}, \tilde{\mathbf{P}}_\xi)$ have the same distribution. Moreover, due to Remark 1, if $\xi \in \mathcal{W}$, then the process $(\zeta_n)_{n\in\mathbb{N}}$ defined on $(\tilde{\Xi}, \tilde{\mathbf{P}}_\xi)$ as

$$\zeta_n \;=\; \begin{cases} 0 & \text{if } n = 0\,, \\ \sum_{k=0}^{n-1} \Delta(\xi_k, \xi_{k+1}) & \text{otherwise}\,, \end{cases}$$

has paths in $\tilde{\Omega}_\xi$ with distribution $\tilde{\mathbf{P}}_0^\xi$. Finally, it is convenient to consider a suitable average of the distributions $\tilde{\mathbf{P}}_\xi$. To this aim, let $\mathcal{Q}_0$ be the probability measure on $\mathcal{N}_0$ defined as

$$\mathcal{Q}_0(d\xi) \;:=\; \frac{\lambda_0(\xi)}{\mathbf{E}_{\mathcal{P}_0}(\lambda_0)}\,\mathcal{P}_0(d\xi)\;,$$

and set $\tilde{\mathbf{P}} := \int \mathcal{Q}_0(d\xi)\tilde{\mathbf{P}}_\xi$. Note that, if $\xi \in \mathcal{W}$, the transition probabilities are

$$p(\xi'|\xi) \;:=\; \tilde{\mathbf{P}}\big(\xi_{n+1} = \xi' \,|\xi_n = \xi\big) \;=\; \begin{cases} \lambda_0^{-1}(\xi) c_{0,x}(\xi) & \text{if } \xi' = S_x\xi\;, \\ 0 & \text{otherwise}\;. \end{cases}$$

Lemma 3. *The discrete-time Markov process $(\xi_n)_{n\geq 0}$ defined on $(\tilde{\Xi}, \tilde{\mathbf{P}})$ is reversible and is (time) ergodic if $\mathcal{P}$ is ergodic.*

Proof. Reversibility means that

$$\int_{\mathcal{N}_0} \mathcal{Q}_0(d\xi) \int \hat{\xi}(dx)\, p(S_x\xi|\xi)\, f(\xi)\, g(S_x\xi) \;=\; \int_{\mathcal{N}_0} \mathcal{Q}_0(d\xi) \int \hat{\xi}(dx)\, p(S_x\xi|\xi)\, g(\xi)\, f(S_x\xi)\;,$$

for any $f, g \in \mathcal{F}(\mathcal{N}_0)$. The above identity can be inferred from Lemma 1(i) by means of straightforward computation using the covariance relations (3.4) and the symmetry of the rates $c_{x,y}(\xi)$.

Due to Corollary 5 in [Ros, Chapter IV], in order to prove ergodicity it is enough to show that $\mathcal{Q}_0(A) \in \{0,1\}$ if $A \in \mathcal{B}(\mathcal{N}_0)$ has the following property: for $\mathcal{Q}_0$–almost all $\xi \in A$, $\tilde{\mathbf{P}}(\xi_1 \in A|\xi_0 = \xi) = \chi_A(\xi)$. Note that this property implies for $\mathcal{Q}_0$–almost all $\xi \in A$ that $\xi' \in A$ if $p(\xi'|\xi) > 0$. Since $\mathcal{Q}_0$ and $\mathcal{P}_0$ are absolutely continuous w.r.t. each other and since $p(S_x\xi|\xi) > 0$ for all $x \in \hat{\xi}$, the statement follows from Lemma 1(iii). $\square$

Proof of Proposition 1. The uniqueness follows from [Bre, Chapter 15]. In order to prove existence, due to the above construction, we only need to prove (3.6) for $\mathcal{P}_0$–almost all ξ and for any $x \in \hat{\xi}$. According to [Bre, Prop. 15.43], condition (3.6) is implied by the following one:

$$\tilde{\mathbf{P}}_x^\xi\Big(\sum_{n=0}^{\infty} \frac{1}{\lambda_{\tilde{X}_n^\xi}(\xi)} = \infty\Big) \;=\; 1\,. \tag{3.7}$$

Due to the identity

$$\tilde{\mathbf{P}}_0^\xi\Big(\sum_{n=1}^{\infty} \frac{1}{\lambda_{\tilde{X}_n^\xi}(\xi)} = \infty \;|\; \tilde{X}_1^\xi = x\Big) \;=\; \tilde{\mathbf{P}}_x^\xi\Big(\sum_{n=0}^{\infty} \frac{1}{\lambda_{\tilde{X}_n^\xi}(\xi)} = \infty\Big)\,, \qquad \forall\, x \in \hat{\xi}\;,$$

the proof will be completed if we can show (3.7) for $x = 0$ and $\mathcal{P}_0$–almost all ξ and, in particular, if we can show

$$\tilde{\mathbf{P}}\Big(\sum_{n=0}^{\infty}\frac{1}{\lambda_0(\xi_n)}=\infty\Big) \;=\; \int \mathcal{Q}_0(d\xi)\, \tilde{\mathbf{P}}_0^{\xi}\Big(\sum_{n=0}^{\infty}\frac{1}{\lambda_{\tilde{X}_n^{\xi}}(\xi)}=\infty\Big)=1\,,$$

or

$$\tilde{\mathbf{P}}\Big(\lim_{n\to\infty}\lambda_0(\xi_n)=\infty\Big) \;=\; \int \mathcal{Q}_0(d\xi)\, \tilde{\mathbf{P}}_0^{\xi}\Big(\lim_{n\to\infty}\lambda_{\tilde{X}_n^{\xi}}(\xi)=\infty\Big)=0\,.$$

If $\mathcal{P}$ is ergodic, then Lemma 3 implies that $\tilde{\mathbf{P}}$ is ergodic and therefore, according to ergodic theory (see [Ros, Chapter IV]),

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=0}^{N}\frac{1}{\lambda_0(\xi_n)} \;=\; \mathbf{E}_{\mathcal{Q}_0}\Big(\frac{1}{\lambda_0}\Big) \;=\; \frac{1}{\mathbf{E}_{\mathcal{P}_0}(\lambda_0)}\,, \qquad \tilde{\mathbf{P}}\text{-almost surely,}$$

thus allowing to conclude the proof.

If $\mathcal{P}$ is not necessarily ergodic and satisfies $\rho_3<\infty$, the proof can be ended as follows. Given $N>0$, by the Chebyshev inequality and the stationarity of $\tilde{\mathbf{P}}$, one obtains

$$\tilde{\mathbf{P}}\big(\lambda_0(\xi_n)\to\infty\big) \;\le\; \lim_{n\to\infty}\tilde{\mathbf{P}}\Big(\bigcap_{m\ge n}\{\lambda_0(\xi_m)\ge N\}\Big) \;\le\; \lim_{n\to\infty}\frac{1}{N}\mathbf{E}_{\tilde{\mathbf{P}}}\big(\lambda_0(\xi_0)\big)\,. \tag{3.8}$$

The assertion follows from the identity

$$\mathbf{E}_{\tilde{\mathbf{P}}}\big(\lambda_0(\xi_0)\big) \;=\; \mathbf{E}_{\mathcal{Q}_0}(\lambda_0) \;=\; \mathbf{E}_{\mathcal{P}_0}(\lambda_0^2)\mathbf{E}_{\mathcal{P}_0}(\lambda_0)^{-1}\,,$$

Lemma 2 and the arbitrariness on N. □

Remark 2. Explosions are excluded if $\sup_{x\in\hat{\xi}}\lambda_x(\xi)<\infty$ (in such a case (3.7) is always true), but this simple criterion is typically not satisfied in our case. For instance, for a PPP

$$\sup_{x\in\hat{\xi}}\lambda_x(\xi) \;\ge\; e^{-4\beta}\sup_{x\in\hat{\xi}}\sum_{y\in\hat{\xi},|y-x|\le 1}e^{-|x-y|} \;\ge\; e^{-4\beta-1}\sup_{x\in\hat{\xi}}\hat{\xi}(C_1+x) \;=\; \infty\,, \qquad \mathcal{P}_0\text{-a.s.}\,.$$

Remark 3. Some of the results of this work do not depend on the particular form of the transition rates. If the jump rates have only finite range, more care needs to be taken when studying ergodicity issues and the lower bounds on the diffusion matrix discussed below.

3.2. **Main results.** We are interested in studying the diffusion matrix D defined by

$$(a\cdot Da) \;=\; \lim_{t\to\infty}\frac{1}{t}\mathbf{E}_{\mathcal{P}_0}\Big(\mathbf{E}_{\mathbf{P}_0^{\xi}}\big((X_t^{\xi}\cdot a)^2\big)\Big)\,, \qquad a\in\mathbb{R}^d\,, \tag{3.9}$$

where $(a\cdot b)$ denotes the scalar product of the vectors a and b in $\mathbb{R}^d$. Due to Proposition 2 below, the expectation in the r.h.s. is finite if $\rho_\kappa<\infty$ for some $\kappa>5$. The following theorem will be proven in Section 4 by using the theory of Ref. [KV] and [DFGW].

Theorem 1. *Let $\mathcal{P}$ be the distribution of a random environment (in the sense of Definition 1) which is, moreover, ergodic and satisfy $\rho_{12}<\infty$. Then the limit (3.9) exists and D is given by the variational formula*

$$(a \cdot D\, a) \;=\; \inf_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx)\; c_{0,x}(\xi)\; \big(a \cdot x \,+\, \nabla_x f(\xi)\big)^2 \,, \qquad a \in \mathbb{R}^d \,, \tag{3.10}$$

with

$$\nabla_x f(\xi) \;:=\; f(S_x \xi) - f(\xi) \;. \tag{3.11}$$

Moreover, the rescaled process $\underline{Y}^{\xi,\varepsilon} = (\varepsilon X^{\xi}_{t\varepsilon^{-2}})_{t \geq 0}$ *defined on* $(\Omega_\xi, \mathbf{P}^\xi_0)$ *converges as* $\varepsilon \to 0$ *to a Brownian motion* $\underline{W}_D$ *with covariance matrix* D *in the following weak sense: for any bounded continuous function* F *on* $D([0,\infty), \mathbb{R}^d)$,

$$\mathbf{E}_{\mathbf{P}^\xi_0}\Big(F\big(\underline{Y}^{\varepsilon,\xi}\big)\Big) \;\to\; \mathbf{E}\Big(F\big(\underline{W}_D\big)\Big) \qquad \text{in } \mathcal{P}_0\text{-probability} \,.$$

We point out that this theorem does *not* necessarily imply that the motion of the particle after a proper rescaling is diffusive, since it could happen that $D = 0$. In the situation where the random environment is obtained by ν–randomisation, we shall show that the diffusion matrix is strictly positive (*i.e.* has positive eigenvalues) whenever the process has sufficiently many points or satisfies a suitable mixing condition. The following notation is needed. Given $A \subset \mathbb{R}^d$, we define $\mathcal{F}_A$ as the σ–subalgebra in $\mathcal{B}(\hat{\mathcal{N}})$ generated by the random variables $\hat{\xi}(B)$, where $B \subset A$ and $B \in \mathcal{B}(\mathbb{R}^d)$. Let us set $\mathcal{F}_r := \mathcal{F}_{\mathbb{R}^d \setminus C_r}$.

Main Hypothesis *The random environment with distribution* $\mathcal{P}$ *is the* ν*-randomisation of a SSPP with distribution* $\hat{\mathcal{P}}$ *which has a lower bound* $\rho' > 0$ *on the point density:*

$$\hat{\xi}(C_N) \;\geq\; \rho'\, \ell(C_N) \,, \qquad \forall\; N \geq N_0 \,, \quad \hat{\mathcal{P}}\text{-a.s.} \,, \tag{3.12}$$

with ρ' *and* N_0 *independent on* $\hat{\xi}$, *or satisfies the following mixing condition: there exists a function* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *with*

$$h(r) \;\leq\; \frac{c}{1 + r^{2d+16+\delta}} \,, \qquad c, \delta > 0 \,,$$

such that for $r_2 \geq r_1 > 1$

$$\Big|\hat{\mathcal{P}}(A|\mathcal{F}_{r_2}) \,-\, \hat{\mathcal{P}}(A)\Big| \;\leq\; r_1^d\, r_2^{d-1}\, h(r_2 - r_1) \,, \qquad \forall\, A \in \mathcal{F}_{C_{r_1}} \,, \quad \hat{\mathcal{P}}\text{-a.s.} \,. \tag{3.13}$$

We feel that this weak hypothesis covers nearly all cases appearing in interesting examples. The uniform lower bound (3.12) holds in the case of random and quasiperiodic tilings and, more generally, the so-called Delone sets [BHZ]. It holds true in particular for the SSPP of Example 2. The type of mixing condition (3.13) is inspired by decorrelation estimates holding for Gibbs measures of spin systems in a high temperature phase [Mar]. Note that, due to the stationarity of $\hat{\mathcal{P}}$, the mixing condition (3.13) imply that $\hat{\mathcal{P}}$ is a mixing, and in particular ergodic, point process (see [DV, Chapter 10]). It is trivially satisfied for the PPP as well as point processes with finite range correlations, that is for point processes for which there exists some $r > 0$ such that $\mathcal{F}_A$ and $\mathcal{F}_B$ are independent whenever $\mathrm{dist}(A, B) \geq r$.

Theorem 2. *Let* $d \geq 2$, $\rho_{12} < \infty$ *and let* $\mathcal{P}$ *satisfy the Main Hypothesis. Then the diffusion matrix* D *is strictly positive.*

Theorem 2 also covers the case of a trivial ν–randomisation, *i.e.* ν is supported by a single point. Under a further assumption on the randomisation measure ν which is physically reasonable as discussed in the introduction, one obtains the following quantitative lower bound.

Theorem 3. (Mott law as lower bound) *Let $d \geq 2$ and let $\mathcal{P}$ be as in Theorem 2 with a randomisation measure ν satisfying*

$$\nu([-E, E]) \;\geq\; c_0 \, E^{1+\alpha} \;, \tag{3.14}$$

for some positive constants α, c_0. Then

$$D \;\geq\; c_1 \, \beta^{-\frac{d(\alpha+1)}{\alpha+1+d}} \, \exp\Big(-c_2 \, \beta^{\frac{\alpha+1}{\alpha+1+d}} \Big) \; \mathbf{1}_d \;, \tag{3.15}$$

where $\mathbf{1}_d$ is the $d \times d$ identity matrix and c_1 and c_2 are some positive constants.

The proofs of Theorems 2 and 3 will be given in Section 7.2.

Remark 4. Given an invertible linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ we define the map S_T on $\mathbb{R}^d \times K$ by $S_T(x,k) = (Tx, k)$. This induces a map S_T on $\mathcal{N}$ by $(S_T\xi)(B) = \xi(S_T B)$ where $B \in \mathcal{B}(\mathbb{R}^d \times K)$. Then, as in [DFGW, p. 824], one can prove that if $\mathcal{P}$ is reflection invariant (*i.e.* $\mathcal{P} \circ S_{R_i} = \mathcal{P}$ for all reflections R_i with $R_i x = (x^{(1)}, \dots, -x^{(i)}, \dots, x^{(d)})$), then the diffusion matrix D appearing in Theorem 1 is diagonal, while if $\mathcal{P}$ is isotropic (here in the sense that $\mathcal{P} \circ S_R = \mathcal{P}$ for all rotations R by $\pi/2$ in a coordinate plane), then D is multiple of the identity.

In order to simplify the exposition, in the rest of this article we assume $\mathcal{P}$ to be isotropic. As the reader can check, all the proofs below can be adapted to the general case considered in Theorems 1, 2 and 3.

3.3. **L^p–property of X_t^ξ at fixed times.** In order to apply the results of [DFGW], one needs that $\mathbf{E}_{\mathcal{P}_0} \mathbf{E}_{\mathbf{P}_0^\xi}(|X_t^\xi|) < \infty$. The following statement is more general.

Proposition 2. *Let $\mathcal{P}$ satisfy $\rho_\kappa < \infty$ for some integer $\kappa > 3$. Then, given $t > 0$ and $0 < \gamma < \kappa - 3$,*

$$\mathbf{E}_{\mathcal{P}_0} \mathbf{E}_{\mathbf{P}_0^\xi} \Big(|X_t^\xi|^\gamma \Big) \;<\; \infty \;.$$

Proof. Due to the construction of the dynamics given in Section 3.1

$$\mathbf{E}_{\mathcal{P}_0} \mathbf{E}_{\mathbf{P}_0^\xi} \Big(|X_t^\xi|^\gamma \Big) \;=\; \mathbf{E}_{\mathcal{P}_0} \, \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}} \big(|\tilde{X}^\xi_{n_*^\xi(t)}|^\gamma \big) \,.$$

Let $p, q > 1$ be such that $1/p + 1/q = 1$. Due to the Hölder inequality,

$$\begin{aligned} \mathbf{E}_{\mathcal{P}_0} \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}} \big(|\tilde{X}^\xi_{n_*^\xi(t)}|^\gamma \big) \;&=\; \sum_{n=1}^{\infty} \mathbf{E}_{\mathcal{P}_0} \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}} \Big(|\tilde{X}_n^\xi|^\gamma \, \chi\big(n_*^\xi(t) \geq 1\big) \chi\big(n_*^\xi(t) = n\big) \Big) \\ &\leq \sum_{n=1}^{\infty} \Big(\mathbf{E}_{\mathcal{P}_0} \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}} \Big(|\tilde{X}_n^\xi|^{\gamma\, q} \chi\big(n_*^\xi(t) \geq 1\big) \Big) \Big)^{\frac{1}{q}} \Big(\mathbf{E}_{\mathcal{P}_0} \big(\, \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q} \; (n_*^\xi(t) = n) \big) \Big)^{\frac{1}{p}} . \end{aligned}$$

Clearly, $n_*^\xi(t) \geq 1$ means $T^\xi_{0, \tilde{X}_0^\xi} \leq t$. It then follows from the estimate $1 - e^{-u} \leq u$, $u \geq 0$, that

$$\mathbf{E}_{\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}} \big(|\tilde{X}_n^\xi|^{\gamma\, q} \, \chi\big(n_*^\xi(t) \geq 1\big) \big) \;=\; \big(1 - e^{-\lambda_0(\xi) t} \big) \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi} \big(|\tilde{X}_n^\xi|^{\gamma\, q} \big) \;\leq\; \lambda_0(\xi) t \; \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi} \big(|\tilde{X}_n^\xi|^{\gamma\, q} \big) \,. \tag{3.16}$$

We then obtain

$$\mathbf{E}_{\mathcal{P}_0} \, \mathbf{E}_{\mathbf{P}_0^\xi} \big(|X_t^\xi|^\gamma \big) \;\leq\; C \sum_{n=1}^{\infty} \Big(\int \mathcal{Q}_0(d\xi) \, \mathbf{E}_{\tilde{\mathbf{P}}_0^\xi} \Big(|\tilde{X}_n^\xi|^{\gamma\, q} \Big) \Big)^{1/q} \Big(\int \mathcal{P}_0(d\xi) \, \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q} \Big(n_*^\xi(t) = n \Big) \Big)^{1/p} , \tag{3.17}$$

with $C = [t\,\mathbf{E}_{\mathcal{P}_0}(\lambda_0)]^{1/q}$. We claim that there is a (time-dependent) constant $C' > 0$ such that

$$\int \mathcal{Q}_0(d\xi)\,\mathbf{E}_{\tilde{\mathbf{P}}_0^\xi}\big(|\tilde{X}_n^\xi|^{\gamma q}\big) \;\leq\; C'\, n^{\gamma q}\;. \tag{3.18}$$

To show this, let us note first that, given $\tilde{X}_0^\xi = 0$, by another application of the Hölder inequality,

$$|\tilde{X}_n^\xi|^{\gamma q} \;=\; \left|\sum_{m=0}^{n-1}\big(\tilde{X}_{m+1}^\xi - \tilde{X}_m^\xi\big)\right|^{\gamma q} \;\leq\; n^{\gamma q - 1}\sum_{m=0}^{n-1}|\tilde{X}_{m+1}^\xi - \tilde{X}_m^\xi|^{\gamma q}\;,$$

where it has been assumed that $\gamma q > 1$. One can derive from the stationarity of $\tilde{\mathbf{P}}$ and Remark 1 that

$$\int \mathcal{Q}_0(d\xi)\,\mathbf{E}_{\tilde{\mathbf{P}}_0^\xi}\big(|\tilde{X}_{n+1}^\xi - \tilde{X}_n^\xi|^{\gamma q}\big) \;=\; \int \mathcal{Q}_0(d\xi)\,\mathbf{E}_{\tilde{\mathbf{P}}_0^\xi}\big(|\tilde{X}_1^\xi|^{\gamma q}\big) \;:=\; C'\;.$$

for any $n \in \mathbb{N}$. One concludes the proof of (3.18) by checking that C' is finite. Actually, by (3.5), $\mathbf{E}_{\mathcal{P}_0}(\lambda_0)\, C'$ is equal to

$$\int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx)\, c_{0,x}(\xi)|x|^{\gamma q} \;\leq\; c\int \mathcal{P}_0(d\xi)\int \hat{\xi}(dx)\, e^{-\frac{|x|}{2}}\;,$$

for a suitable constant c. The r.h.s. can be bounded by means of Lemma 1(iv) and the same argument leading to Lemma 2.

In view of (3.17) and (3.18), the proposition will be proved if we can show that the expectation $\mathbf{E}_{\mathcal{P}_0}(\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}(n_*^\xi(t) = n))$ converges to zero more rapidly than $n^{-(\gamma+1)p}$ as $n \to \infty$. Let us fix $0 < \alpha < 1$. We will show that, if $l > 0$ is such that $\mathbf{E}_{\mathcal{P}_0}(\lambda_0^{l+1}) < \infty$, then

$$\mathbf{E}_{\mathcal{P}_0}\big(\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}\big(n_*^\xi(t) = n\big)\big) \;=\; \mathcal{O}(n^{-\alpha l})\;. \tag{3.19}$$

To this end, let us first make a general observation. Let $\lambda > 0$ and let $T_1, \ldots, T_k$ be independent exponential variables on some probability space (Ω, μ), with parameters $\lambda_1, \ldots, \lambda_k \leq \lambda$. Define the random variables $T_j' := (\lambda_j/\lambda)T_j$, $j = 1, \ldots, k$. These are independent identically distributed exponential variables with parameter λ. As $T_j' \leq T_j$, this shows that

$$\mu\big(T_1 + \cdots + T_k \leq t\big) \;\leq\; \mu\big(T_1' + \cdots + T_k' \leq t\big) \;=\; e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^{j+k}}{(j+k)!} \;\leq\; \frac{(\lambda t)^k}{k!}\;. \tag{3.20}$$

In order to proceed, we set for all $\xi \in \mathcal{N}_0$

$$B_n^\xi \;:=\; \big\{x \in \hat{\xi} \,:\, \lambda_x(\xi) \leq n^\alpha\big\}\;,$$

as well as

$$A_n^\xi \;:=\; \Big\{(\tilde{X}_k^\xi)_{k\geq 0} \in \tilde{\Omega}_\xi \,:\, \exists\, J \subset I_n\,,\, |J| > \frac{n}{2}\,,\, \tilde{X}_j^\xi \in B_n^\xi \;\; \forall\, j \in J\Big\}$$

where $I_n := \{0, \ldots, n-1\}$ and $|J|$ is the cardinality of J. We write

$$\tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}\big(n_*^\xi(t) = n\big) \;=\; g_n(\xi) \;+\; h_n(\xi)\;,$$

with

$$g_n(\xi) \;:=\; \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}\Big(\big\{n_*^\xi(t) = n\big\} \cap A_n^\xi\Big)\;, \qquad h_n(\xi) \;:=\; \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}\Big(\big\{n_*^\xi(t) = n\big\} \cap (A_n^\xi)^c\Big)\;.$$

We first estimate g_n. Obviously $\{n_*^\xi(t)=n\}$ is contained in $\{\sum_{j\in J} T^\xi_{j,\tilde X^\xi_j}\le t\}$. As a result,

$$\begin{aligned}
g_n(\xi) \;&\le\; \sum_{J\subset I_n, |J|>n/2}\;\sum_{x_0,\dots,x_{n-1}\in\hat\xi} \chi\big(x_j\in B_n^\xi\ \forall\, j\in J\big)\,\chi\big(x_i\notin B_n^\xi\ \forall\, i\in I_n\setminus J\big)\\
&\qquad\qquad \tilde{\mathbf P}_0^\xi\big(\tilde X_0^\xi = x_0,\dots,\tilde X_{n-1}^\xi = x_{n-1}\big)\;\mathbf Q\Big(\sum_{j\in J} T^\xi_{j,x_j}\le t\Big)\\
&\le\; \max_{k=[n/2]+1,\dots,n-1}\left\{\frac{(n^\alpha\, t)^k}{k!}\right\}\,.
\end{aligned}$$

Thanks to the Stirling formula $k!\sim k^k e^{-k}\sqrt{2\pi k}$ as $k\to\infty$, the last expression can be bounded by a constant times $(2\,e\,t)^{n/2}\, n^{-n(1-\alpha)/2}$ and is thus exponentially small. We now turn to $\mathbf E_{\mathcal P_0}(h_n)$, $n\ge 1$. Clearly,

$$\tilde{\mathbf P}_0^\xi\Big(\big(A_n^\xi\big)^c\Big)\;\le\;\frac{2}{n}\,\mathbf E_{\tilde{\mathbf P}_0^\xi}\Big(\chi\big(\tilde X_0^\xi\notin B_n^\xi\big)+\ldots+\chi\big(\tilde X_{n-1}^\xi\notin B_n^\xi\big)\Big)\;=\;\frac{2}{n}\sum_{m=0}^{n-1}\mathbf E_{\tilde{\mathbf P}_\xi}\big(\lambda_0(\xi_m)>n^\alpha\big)\,.$$

By Lemma 3 and invoking Chebyshev's inequality, one obtains for any $l>0$

$$\begin{aligned}
\mathbf E_{\mathcal P_0}(h_n)\;&\le\;\int \mathcal P_0(d\xi)\,\tilde{\mathbf P}_0^\xi\otimes\mathbf Q\Big(\big\{n_*^\xi(t)\ge 1\big\}\cap\big(A_n^\xi\big)^c\Big)\;\le\; t\int\mathcal P_0(d\xi)\,\lambda_0(\xi)\tilde{\mathbf P}_0^\xi\Big(\big(A_n^\xi\big)^c\Big)\\
&\le\;\frac{2t}{n}\sum_{m=0}^{n-1}\int\mathcal P_0(d\xi)\,\lambda_0(\xi)\,\mathbf E_{\tilde{\mathbf P}_\xi}\big(\lambda_0(\xi_m)>n^\alpha\big)\;=\;2t\,\mathbf E_{\mathcal P_0}\big(\lambda_0\,\chi(\lambda_0>n^\alpha)\big)\\
&\le\;\frac{2t}{n^{\alpha l}}\,\mathbf E_{\mathcal P_0}\big(\lambda_0^{l+1}\big)\,,
\end{aligned}$$

where the second inequality follows from the same argument leading to (3.16) and the equality follows from the stationarity of $\tilde{\mathbf P}$. This proves (3.19). We may now choose $p=\alpha^{-1}>1$ arbitrarily close to 1 so that $\gamma q>1$ and such that one may take for l the smallest integer strictly greater than $\gamma+1$. For such a choice the sum (3.17) converges. We can now invoke Lemma 2 to get the result. □

4. Dynamics of the environment viewed from the particle

We check in this section that our model satisfies all hypothesis of Theorem 2.2 in [DFGW], which then directly implies Theorem 1. Throughout, we assume to have a *random environment in the sense of Definition 1 fulfilling the assumptions of Proposition 1.*

4.1. Construction of the process.

Due to (2.3), for $\mathcal P_0$–almost all ξ, the random walk on $\mathrm{supp}(\hat\xi)$ of a particle starting at the origin and having transition rates $c_{y,z}(\xi)$ given in (3.3) is well defined and its distribution on the path space $\Omega_\xi = D([0,\infty),\mathrm{supp}(\hat\xi))$ has been denoted by $\mathbf P_0^\xi$. The *environment viewed from the particle* is the continuous–time jump Markov process $\big(S_{X_t^\xi}\xi\big)_{t\ge 0}$ defined on $(\Omega_\xi,\mathbf P_0^\xi)$. Let $\mathbf P_\xi$ be its distribution on the path space $\Xi := D([0,\infty),\mathcal N_0)$. A generic element of Ξ will be denoted by $\underline\xi=(\xi_t)_{t\ge 0}$. Let us set $\mathbf P := \int\mathcal P_0(d\xi)\mathbf P_\xi$. Then the *environment process* is $(\xi_t)_{t\ge 0}$ defined on $(\Xi,\mathbf P)$ with distribution $\mathbf P$. It is a continuous–time jump Markov process such that

$$\mathbf P(\xi_{s+t}=\xi'\,|\,\xi_s=\xi)\;=\;\mathbf P_\xi(\xi_t=\xi')\;=:\;p_t(\xi'|\xi)\qquad\forall\, s,t\ge 0\,.$$

Then, if $\xi\in\mathcal W$,

$$p_t(\xi'|\xi) \;=\; \begin{cases} p_t^{\xi}(x|0) & \text{if } \xi' = S_x\xi \text{ for some } x \in \hat{\xi}\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{4.1}$$

Given $t > 0$, we further introduce the function $n_*(t)$ on the path space Ξ defined by

$$n_*(t)\,(\,\underline{\xi}\,) \;=\; \text{number of jumps of } \underline{\xi} \text{ in the time interval }\ .$$

Proposition 3. *Let $\mathcal{P}$ be ergodic with $\rho_2 < \infty$ or let $\mathcal{P}$ satisfy $\rho_3 < \infty$. Then the process $(\xi_t)_{t\geq 0}$ defined on $(\Xi, \mathbf{P})$ is reversible, i.e.*

$$\mathbf{E}_{\mathbf{P}}\big(f(\xi_0)g(\xi_t)\big) \;=\; \mathbf{E}_{\mathbf{P}}\big(g(\xi_0)f(\xi_t)\big) \qquad \forall\, f, g \in \mathcal{F}(\mathcal{N}_0)\ ,\ \ \forall\, t > 0\ , \tag{4.2}$$

and is (time) ergodic if $\mathcal{P}$ is ergodic.

Proof. We first verify the symmetric property

$$p_t(\xi'|\xi) \;=\; p_t(\xi|\xi')\ . \tag{4.3}$$

Actually, thanks to the construction of the dynamics given in Section 3, one can show that for any positive integer n and any $\xi = \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(n-1)}, \xi^{(n)} = \xi' \in \mathcal{N}_0$,

$$\mathbf{P}_{\xi}\big(\,n_*(t) = n, \xi_{R_1} = \xi^{(1)}, \dots, \xi_{R_n} = \xi^{(n)}\big) \;=\; \mathbf{P}_{\xi'}\big(\,n_*(t) = n, \xi_{R_1} = \xi^{(n-1)}, \dots, \xi_{R_n} = \xi^{(0)}\big)\ .$$

where, given $\underline{\xi} \in \Xi$, $R_1(\underline{\xi}) < R_2(\underline{\xi}) < \dots$ denote the jump times of the path $\underline{\xi}$. Next, given $f, g \in \mathcal{F}(\mathcal{N}_0)$ one gets by applying Lemma 1(i) and using (4.3) that

$$\int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx)\, p_t(S_x\xi|\xi)\ f(\xi)g(S_x\xi) \;=\; \int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx)\, p_t(S_x\xi|\xi)\ f(S_x\xi)g(\xi)\ , \tag{4.4}$$

which is equivalent to (4.2). Hence $\mathbf{P}$ is reversible.

Due to Corollary 5 in [Ros, Chapter IV], in order to prove ergodicity it is enough to show that $\mathcal{P}_0(A) \in \{0,1\}$ if $A \in \mathcal{B}(\mathcal{N}_0)$ has the following property: $\mathbf{P}_{\xi}(\xi_t \in A) = \chi_A(\xi)$ for $\mathcal{P}_0$–almost all ξ. Given such a set A, then there exists a Borel subset $B \subset A$ such that $\mathcal{P}_0(A \setminus B) = 0$ and $\mathbf{P}_{\xi}(\xi_t \in A) = 1$ for any $\xi \in B$. Fix $\xi \in B$ and $x \in \hat{\xi}$, then $\mathbf{P}_{\xi}(\xi_t = S_x\xi, \xi_t \in A) = \mathbf{P}_{\xi}(\xi_t = S_x\xi) > 0$ (the last bound follows from the positivity of the jump rates). Hence $S_x\xi \in A$. Lemma 1(iii) implies that $\mathcal{P}_0(A) \in \{0,1\}$, thus allowing to conclude the proof. □

4.2. **Markov semigroup.** Let $\mathcal{P}$ fulfil the assumption of Proposition 3. Then,

$$(\mathcal{T}_t f)(\xi) \;:=\; \mathbf{E}_{\mathbf{P}_{\xi}}\big(f(\xi_t)\big) \;=\; \int \hat{\xi}(dx)\ p_t(S_x\xi, \xi)\, f(S_x\xi)\ , \qquad \mathcal{P}_0 \text{ a.s.} \tag{4.5}$$

defines a strongly continuous contraction semigroup on $L^2(\mathcal{N}_0, \mathcal{P}_0)$ (Markov semigroup). Actually, (i) $\mathcal{T}_t : L^2(\mathcal{N}_0, \mathcal{P}_0) \to L^2(\mathcal{N}_0, \mathcal{P}_0)$ is self-adjoint by (4.2) and is a contraction by the Cauchy-Schwarz inequality and the stationarity of $\mathbf{P}$; (ii) $\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_s$ follows from the Markov nature of the process; (iii) the continuity follows from the following argument: first observe that it is enough to prove the continuity of $\mathcal{T}_t f$ at $t = 0$ for $f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)$, which is obtained from the dominated convergence theorem and the estimate $|(\mathcal{T}_t f - f)(\xi)| \leq 2\|f\|_\infty(1 - p_t^{\xi}(0|0))$.

Let us denote by $\mathcal{L}$ the generator of the Markov semigroup $(\mathcal{T}_t)_{t\geq 0}$ and by $D(\mathcal{L}) \subset L^2(\mathcal{N}_0, \mathcal{P}_0)$ its domain.

Proposition 4. *Let $\mathcal{P}$ satisfy $\rho_4 < \infty$. Then $\mathcal{L}$ is nonpositive and self–adjoint with core $L^\infty(\mathcal{N}_0, \mathcal{P}_0)$. For any $f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)$, one has*

$$(\mathcal{L}f)(\xi) \;=\; \int \hat{\xi}(dx)\, c_{0,x}(\xi)\, \nabla_x f(\xi)\,, \qquad \textit{for } \mathcal{P}_0\textit{-a.e. } \xi\,, \tag{4.6}$$

where $\nabla_x f$ is defined in (3.11), and, moreover,

$$\langle f, (-\mathcal{L})f\rangle_{\mathcal{P}_0} \;=\; \frac{1}{2}\int \mathcal{P}_0(d\xi) \int \hat{\xi}(dx)\, c_{0,x}(\xi)\, \left(\nabla_x f(\xi)\right)^2\,. \tag{4.7}$$

Proof. We use the abbreviation L^p for $L^p = L^p(\mathcal{N}_0, \mathcal{P}_0)$, $p = 2$ or ∞. For any $f \in L^\infty$, denote by Λf the function defined by the r.h.s. of (4.6). Due to Lemma 2, $\mathbf{E}_{\mathcal{P}_0}(\lambda_0^2) < \infty$ and in particular

$$\int \mathcal{P}_0(d\xi) \left|(\Lambda f)(\xi)\right|^2 \;\leq\; 4\,\|f\|_\infty^2\, \mathbf{E}_{\mathcal{P}_0}(\lambda_0^2) < \infty\,,$$

thus implying that $\Lambda : L^\infty \to L^2$ is a well-defined operator. We claim that

$$L^2 - \lim_{t\downarrow 0} \frac{\mathcal{T}_t f - f}{t} \;=\; \Lambda f\,, \qquad \forall\, f \in L^\infty\,. \tag{4.8}$$

Note that (4.8) implies that $L^\infty \subset D(\mathcal{L})$ and $\mathcal{L}f = \Lambda f$ for all $f \in L^\infty$. As $\mathcal{T}_t$ is self-adjoint and is a contraction, this also implies that Λ is a symmetric nonpositive operator and, thanks to [RS, Vol.2, Theorem X.1], essentially self–adjoint. Since $\mathcal{L}$ is closed [RS, Vol.2, Chapter X.8] and extends Λ, we deduce that $\mathcal{L}$ is the closure of Λ, thus implying that $\mathcal{L}$ is self-adjoint and L^∞ is a core for $\mathcal{L}$. Finally, using (4.4) in the limit $t \to 0$, by straightforward computations (4.7) can be derived from (4.6).

Let us now prove (4.8). We assume $\xi \in \mathcal{W}$ and we set, for $\xi' \neq \xi$,

$$p_{t,1}(\xi'|\xi) \;:=\; \mathbf{P}_\xi(\xi_t = \xi', n_*(t) = 1) \;=\; p_t(\xi'|\xi) - \mathbf{P}_\xi(\xi_t = \xi', n_*(t) \geq 2)\,.$$

Thanks to the construction of the dynamics described in Section 3.1 and due to the estimate $1 - e^{-u} \leq u$, $u \geq 0$, one has for any $x \in \hat{\xi}$ and $x \neq 0$

$$p_{t,1}(S_x\xi|\xi) \;\leq\; \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}(\tilde{X}_1^\xi = x,\, T_{0,0}^\xi \leq t) \;=\; p^\xi(x|0)(1 - e^{-\lambda_0(\xi)t}) \;\leq\; c_{0,x}(\xi)\, t\,. \tag{4.9}$$

Let $f \in L^\infty$. In view also of (4.5) and $\int \hat{\xi}(dx)\, p_t(S_x\xi, \xi) = 1$,

$$\begin{aligned}\Big|\, (\mathcal{T}_t f - f - t\,\Lambda f)(\xi)\Big| \;&=\; \Big|\int \hat{\xi}(dx) \big(f(S_x\xi) - f(\xi)\big)\big(p_t(S_x\xi|\xi) - c_{0,x}(\xi)\, t\big)\Big| \\ &\leq\; 2\,\|f\|_\infty \left(\int_{\{x\neq 0\}} \hat{\xi}(dx)\big(p_t(S_x\xi|\xi) - p_{t,1}(S_x\xi|\xi)\big) + \int_{\{x\neq 0\}} \hat{\xi}(dx)\big(-p_{t,1}(S_x\xi|\xi) + c_{0,x}(\xi)\, t\big)\right).\end{aligned}$$

The first integral in second line can be bounded by $\mathbf{P}_\xi(n_*(t) \geq 2)$. The second integral is equal to

$$-\,\mathbf{P}_\xi(\, n_*(t) = 1\,) + \lambda_0(\xi)\, t \;=\; -1 + e^{-\lambda_0(\xi)t} + \lambda_0(\xi)\, t + \mathbf{P}_\xi(n_*(t) \geq 2)\,.$$

By collecting the above estimates, we get

$$\frac{1}{t^2}\, \mathbf{E}_{\mathcal{P}_0}\left[\left(\mathcal{T}_t f - f - t\Lambda f\right)^2\right] \leq \frac{16\,\|f\|_\infty^2}{t^2}\, \mathbf{E}_{\mathcal{P}_0}\Big(\mathbf{P}_\xi^2(n_*(t) \geq 2)\Big) + \frac{4\|f\|_\infty^2}{t^2}\, \mathbf{E}_{\mathcal{P}_0}\Big(\left(-1 + e^{-\lambda_0 t} + \lambda_0\, t\right)^2\Big)\,. \tag{4.10}$$

By using the estimate $(e^{-u}-1+u)^2 \leq u^3/2$ for $u \geq 0$ and the finiteness of $\mathbf{E}_{\mathcal{P}_0}(\lambda_0^3)$, it is easy to check that the second term in the r.h.s. tends to zero as $t \to 0$. In order to bound the first term, we observe that

$$\mathbf{P}_\xi\big(n_*(t) \geq 2\big) \leq \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}(T_{0,0}^\xi \leq t,\, T_{1,\tilde{X}_1^\xi}^\xi \leq t) \;=\; \sum_{x\in\hat{\xi}} \tilde{\mathbf{P}}_0^\xi \otimes \mathbf{Q}\big(\tilde{X}_1^\xi = x,\, T_{0,0}^\xi \leq t,\, T_{1,x}^\xi \leq t\big)$$

$$= \big(1-e^{-\lambda_0(\xi)t}\big)\int \hat{\xi}(dx)\, p(S_x\xi|\xi)\big(1-e^{-\lambda_0(S_x\xi)t}\big)\ .$$

Due to the estimate $1-e^{-u} \leq u$, this implies the bound

$$\mathbf{P}_\xi(n_*(t) \geq 2) \;\leq\; t^2\,\lambda_0(\xi)\int \hat{\xi}(dx)\, p(S_x\xi|\xi)\lambda_0(S_x\xi) \;=\; t^2\,\lambda_0(\xi)\,\mathbf{E}_{\tilde{\mathbf{P}}_\xi}\big(\lambda_0(\xi_1)\big)\ . \tag{4.11}$$

Due also to the estimate $1-e^{-u} \leq 1$, it is also true that

$$\mathbf{P}_\xi(n_*(t) \geq 2) \;\leq\; t\,\mathbf{E}_{\tilde{\mathbf{P}}_\xi}\big(\lambda_0(\xi_1)\big)\ . \tag{4.12}$$

By multiplying the last two inequalities, and using the stationarity of $\tilde{\mathbf{P}}$, one obtains

$$\begin{aligned}\frac{1}{t^2}\mathbf{E}_{\mathcal{P}_0}\big(\mathbf{P}_\xi^2(n_*(t) \geq 2)\big) \;&\leq\; t\,\mathbf{E}_{\mathcal{P}_0}\Big(\lambda_0(\xi)\big[\mathbf{E}_{\tilde{\mathbf{P}}_\xi}\big(\lambda_0(\xi_1)\big)\big]^2\Big)\\ &\leq\; t\,\mathbf{E}_{\mathcal{P}_0}\Big(\lambda_0(\xi)\mathbf{E}_{\tilde{\mathbf{P}}_\xi}\big(\lambda_0^2(\xi_1)\big)\Big) \;=\; t\,\mathbf{E}_{\mathcal{P}_0}\big(\lambda_0^3\big)\ ,\end{aligned}$$

thus implying that the first term on the r.h.s. of (4.10) goes to 0 as $t \to 0$. □

4.3. **The $\mathcal{H}_{-1}$–norm.** Let us recall some general results concerning self–adjoint operators which will be useful in the proof of Theorem 1 and in Section 5. Let (Ω,μ) be a probability space and denote by $\langle\,.\,,\,.\,\rangle_\mu$ and by $\|.\|_\mu$ the scalar product and the norm on $\mathcal{H} = L^2(\Omega,\mu)$. Let $\mathcal{L} : D(\mathcal{L}) \to \mathcal{H}$ be a nonpositive self–adjoint operator with (dense) domain $D(\mathcal{L}) \subset \mathcal{H}$ and assume $\mathcal{C} \subset D(\mathcal{L})$ is a core of $\mathcal{L}$. The space $\mathcal{H}_1$ is the completion of $D(|\mathcal{L}|^{1/2}) \cap (\mathrm{Ker}(\mathcal{L}))^\perp$ under the norm

$$\|f\|_1 \;:=\; \big\||\mathcal{L}|^{\frac{1}{2}} f\big\|_\mu, \qquad f \in D(|\mathcal{L}|^{1/2})$$

while the dual $\mathcal{H}_{-1}$ of $\mathcal{H}_1$ under $\langle\,.\,,\,.\,\rangle_\mu$ can be identified with the completion of $D(|\mathcal{L}|^{-1/2}) = \mathrm{Ran}(|\mathcal{L}|^{1/2})$ under the $\|.\|_{-1}$-norm defined as

$$\|\varphi\|_{-1} \;:=\; \big\||\mathcal{L}|^{-\frac{1}{2}}\varphi\big\|_\mu, \quad \varphi \in D(|\mathcal{L}|^{-1/2})\ .$$

Given $\varphi \in \mathcal{H}\cap\mathcal{H}_{-1}$, the dual norm $\|\varphi\|_{-1}$ admits several useful characterisations:

$$\|\varphi\|_{-1}^2 \;=\; \sup_{f\in\mathcal{H}_1\cap\mathcal{H}} \frac{|\langle\varphi,f\rangle_\mu|^2}{\|f\|_1^2} \;=\; \sup_{f\in\mathcal{C}\cap(\mathrm{Ker}(\mathcal{L}))^\perp} \frac{|\langle\varphi,f\rangle_\mu|^2}{\|f\|_1^2}\ , \tag{4.13}$$

where the last identity results from the fact that $\mathcal{C}$ is a core for $\mathcal{L}$. Moreover, the useful identity

$$\|\varphi\|_{-1}^2 \;=\; \sup_{f\in\mathcal{C}}\Big(2\,\langle\varphi,f\rangle_\mu - \langle f,(-\mathcal{L})f\rangle_\mu\Big) \tag{4.14}$$

is easily obtained by using the inhomogeneity in f of the expression in the r.h.s. of (4.13) and observing that $\varphi \in (\mathrm{Ker}(\mathcal{L}))^\perp$. Finally, it follows from spectral calculus that

$$\|\varphi\|_{-1}^2 \;=\; \int_0^\infty dt\,\langle\varphi, e^{t\mathcal{L}}\varphi\rangle_\mu\ . \tag{4.15}$$

In what follows, we extend the definition of $\|\cdot\|_{-1}$ to the whole space $\mathcal{H}$ by setting $\|\varphi\|_{-1} := \infty$ whenever $\varphi \in \mathcal{H}$ and $\varphi \notin \mathcal{H}_{-1}$. Thanks to this choice, identities (4.13), (4.14) and (4.15) are true for all $\varphi \in \mathcal{H}$.

4.4. **Position as anti-symmetric random variable.** For any time $t \geq 0$, let us introduce the random variable $X_t : \Xi \to \mathbb{R}^d$ defined by

$$X_t(\underline{\xi}) \; = \; \sum_{s \in [0,t]} \Delta_s(\underline{\xi}) \; , \tag{4.16}$$

where

$$\Delta_s(\underline{\xi}) \; = \; \begin{cases} x & \text{if } \xi_s \; = \; S_x \xi_{s^-} \; , \\ 0 & \text{otherwise} \; , \end{cases}$$

and the sum runs over all jump times s for which $\Delta_s(\underline{\xi}) \neq 0$. Note that the family $\{X_{[s,t]} := X_t - X_s : t > s \geq 0\}$ of random variables satisfies the following properties (see [DFGW]):

(P1) $X_{[s,t]}$ depends only on $\{\xi_u\}_{u \in [s,t]}$,

(P2) additivity: $X_{[s,t]} + X_{[t,v]} = X_{[s,v]}$ for any $s, t, v,\ 0 \leq s < t < v$,

(P3) time–covariance: $X_{[s,t]} \circ \tau_r \; = \; X_{[s+r,t+r]}$ for any $r, s, t \geq 0,\ s < t$,

(P4) $(X_t)_{t \geq 0}$ has paths in $D([0,\infty), \mathbb{R}^d)$,

(P5) anti–symmetry:

$$X_{[s,t]} \circ R_{[s,t]} \; = \; -X_{[s,t]} \; , \qquad \text{for any } s, t,\ 0 \leq s < t, \qquad \mathbf{P} \text{ a.s. } .$$

The crucial link to the dynamics of a particle in a fixed environment is now the following: due to Remark 1, for any $\xi \in \mathcal{W}$, the distribution of the random process $(X_t)_{t \geq 0}$ defined on $(\Xi, \mathbf{P}_\xi)$ is equal to the distribution $\mathbf{P}_0^\xi$ of the continuous–time random walk on $\hat{\xi}$ (naturally embedded in $\mathbb{R}^d$) starting at the origin. Recalling that $\mathbf{P} = \int \mathcal{P}_0(d\xi)\, \mathbf{P}_\xi$, this implies

$$\mathbf{E}_{\mathcal{P}_0}\left(\mathbf{E}_{\mathbf{P}_0^\xi}\left((X_t^\xi \cdot a)^2 \right) \right) \; = \; \mathbf{E}_{\mathbf{P}}\left((X_t \cdot a)^2 \right) \; , \tag{4.17}$$

which gives a way to calculate the diffusion matrix D from the reversible distribution $\mathbf{P}$ on Ξ.

4.5. **Proof of Theorem 1.** The lemma below shows that the *mean forward velocity* and the *infinitesimal mean square displacement* defined as in [DFGW] are well-defined in the present context.

Lemma 4. *Let $\mathcal{P}$ satisfy $\rho_{12} < \infty$.*

(i) *Let φ be the $\mathbb{R}^d$-valued function on $\mathcal{N}_0$ defined by*

$$\varphi(\xi) \; = \; \int \hat{\xi}(dx)\; c_{0,x}(\xi)\; x \; . \tag{4.18}$$

Then $\varphi \in L^2(\mathcal{N}_0, \mathcal{P}_0)$ and φ is equal to the mean forward velocity *given by the convergent L^2-strong limit*

$$\varphi(\xi) \; = \; L^2\!-\!\lim_{t \downarrow 0} \frac{1}{t}\, \mathbf{E}_{\mathbf{P}_\xi}(X_t) \; . \tag{4.19}$$

(ii) *Let us introduce the function ψ on $\mathcal{N}_0$ with values in the real symmetric $d \times d$ matrices by*

$$(a \cdot \psi(\xi) a) \; = \; \int \hat{\xi}(dx)\; c_{0,x}(\xi)\; (a \cdot x)^2 \; . \tag{4.20}$$

Then $\psi \in L^2(\mathcal{N}_0, \mathcal{P}_0)$ *and* ψ *is equal to the* infinitesimal mean square displacement *given by the convergent* L^2*-strong limit*

$$(a \cdot \psi(\xi) a) \;=\; L^2\!-\!\lim_{t \downarrow 0} \, \frac{1}{t} \, \mathbf{E}_{\mathbf{P}_\xi} \left((a \cdot X_t)^2 \right) \;. \tag{4.21}$$

Proof. (i) One has

$$\begin{aligned}\frac{1}{t^2} \int \mathcal{P}_0(d\xi) \left| \mathbf{E}_{\mathbf{P}_\xi}(X_t) - t\,\varphi(\xi) \right|^2 \;\leq\; & \frac{2}{t^2} \int \mathcal{P}_0(d\xi) \left| \mathbf{E}_{\mathbf{P}_\xi}\left(X_t \, \chi\big(n_*(t) = 1\big)\right) - t\,\varphi(\xi) \right|^2 \\ & + \frac{2}{t^2} \int \mathcal{P}_0(d\xi) \left| \mathbf{E}_{\mathbf{P}_\xi}\left(X_t \, \chi\big(n_*(t) \geq 2\big)\right) \right|^2 \;. \end{aligned} \tag{4.22}$$

We first show that the first term on the r.h.s. vanishes as $t \to 0$. Using the same notation as in the proof of Proposition 4 and invoking (4.9),

$$\mathbf{E}_{\mathcal{P}_0}\left(\left| \mathbf{E}_{\mathbf{P}_\xi}\left(X_t \, \chi\big(n_*(t) = 1\big)\right) - t\,\varphi(\xi) \right|^2 \right) \;=\; \mathbf{E}_{\mathcal{P}_0}\left(\left| \int \hat{\xi}(dx) \big(p_{t,1}(S_x \xi | \xi) - t\, c_{0,x}(\xi) \big) \, x \right|^2 \right) ,$$

is bounded according to the Cauchy-Schwarz inequality by

$$\mathbf{E}_{\mathcal{P}_0}\left(\int \hat{\xi}(dx) \big(-p_{t,1}(S_x \xi | \xi) + t\, c_{0,x}(\xi) \big) \; \int \hat{\xi}(dy) \big(-p_{t,1}(S_y \xi | \xi) + t\, c_{0,y}(\xi) \big) |y|^2 \right) . \tag{4.23}$$

Let us denote by $I_1(\xi)$ and $I_2(\xi)$ the (non negative) integrals over $\hat{\xi}(dx)$ and $\hat{\xi}(dy)$ respectively. Using the identities of the proof of Proposition 4, the inequality $0 \leq -1 + e^{-u} + u \leq u^2$, $u \geq 0$, and (4.11), we deduce

$$I_1(\xi) \;=\; -1 + e^{-t\lambda_0(\xi)} + t\lambda_0(\xi) + \mathbf{P}_\xi(n_*(t) \geq 2) \;\leq\; t^2 \lambda_0(\xi)^2 + t^2 \lambda_0(\xi) \, \mathbf{E}_{\tilde{\mathbf{P}}_\xi}(\lambda_0(\xi_1)) \;.$$

Moreover, $I_2(\xi) \leq t \int \hat{\xi}(dy) \, c_{0,y}(\xi) |y|^2$. Hence (4.23) is bounded by

$$t^3 \left(\mathbf{E}_{\mathcal{P}_0}\left(\lambda_0^2(\xi) \int \hat{\xi}(dx) \, c_{0,x}(\xi) x^2 \right) + \mathbf{E}_{\mathcal{P}_0}\left(\lambda_0(\xi) \mathbf{E}_{\tilde{\mathbf{P}}_\xi}(\lambda_0(\xi_1)) \int \hat{\xi}(dx) \, c_{0,x}(\xi) x^2 \right) \right) .$$

As long as $\rho_5 < \infty$, the first expression can be bounded by applying Lemma 1(iv) (see the argument leading to Lemma 2). A short calculation shows that the second expression is equal to

$$\int \mathcal{P}_0(d\xi) \; \int \hat{\xi}(dx) \, c_{0,x}(\xi) \; \int \hat{\xi}(dz) \, c_{x,z}(S_x \xi) \; \int \hat{\xi}(dy) \, c_{0,y}(\xi) \, y^2 \;,$$

and is therefore bounded even if $\rho_4 < \infty$ (again by means of Lemma 1(iv)). Resuming the results obtained so far, one gets

$$\frac{1}{t^2} \int \mathcal{P}_0(d\xi) \; \left| \mathbf{E}_{\mathbf{P}_\xi}\left(X_t \, \chi\big(n_*(t) = 1\big)\right) - t\,\varphi(\xi) \right|^2 \;=\; \mathcal{O}(t) \;. \tag{4.24}$$

We now turn to the second term in (4.22). By Proposition 2, $\mathbf{E}_{\mathcal{P}_0}(\mathbf{E}_{\mathbf{P}_\xi}(|X_t|^\gamma)) < \infty$ as long as $0 < \gamma < \kappa - 3$ whenever $\rho_\kappa < \infty$ for κ integer. By applying twice the Hölder inequality, if $\gamma > 2$,

$$\mathbf{E}_{\mathcal{P}_0}\left(\left| \mathbf{E}_{\mathbf{P}_\xi}\left(X_t \, \chi\big(n_*(t) \geq 2\big)\right) \right|^2 \right) \;\leq\; \left(\mathbf{E}_{\mathcal{P}_0}\left(\mathbf{E}_{\mathbf{P}_\xi}\left(|X_t|^\gamma \right) \right) \right)^{\frac{2}{\gamma}} \left(\mathbf{E}_{\mathcal{P}_0}\left(\mathbf{P}_\xi\big(n_*(t) \geq 2\big)^{\frac{2\gamma - 2}{\gamma - 2}} \right) \right)^{1 - \frac{2}{\gamma}} \;.$$

Let us take (4.12) to the power $\gamma/(\gamma - 2)$, multiply the result by (4.11). This yields

$$\mathbf{E}_{\mathcal{P}_0}\left(\mathbf{P}_\xi\big(n_*(t) \geq 2\big)^{\frac{2\gamma - 2}{\gamma - 2}} \right) \;\leq\; t^{\frac{3\gamma - 4}{\gamma - 2}} \, \mathbf{E}_{\mathcal{P}_0}\left(\lambda_0(\xi) \, \mathbf{E}_{\tilde{\mathbf{P}}_\xi}\left(\lambda_0^{\frac{2\gamma - 2}{\gamma - 2}}(\xi_1) \right) \right) \;=\; t^{\frac{3\gamma - 4}{\gamma - 2}} \, \mathbf{E}_{\mathcal{P}_0}\left(\lambda_0^{\frac{3\gamma - 4}{\gamma - 2}} \right) \;.$$

Hence, by Lemma 2, if $\rho_\kappa < \infty$ is satisfied for integer $\kappa > (4\gamma - 6)/(\gamma - 2)$, there is a finite constant $C > 0$ such that

$$\mathbf{E}_{\mathcal{P}_0}\left(\left| \mathbf{E}_{\mathbf{P}_\xi}\left(X_t \, \chi\big(n_*(t) \geq 2\big)\right) \right|^2 \right) \leq C \, t^{\frac{3\gamma - 4}{\gamma}} \;. \tag{4.25}$$

One concludes the proof by choosing $\gamma > 4$ and by combining (4.22), (4.24) and (4.25), as long as $\kappa > 7$.

(ii) One follows the same strategy. The first term in the equation corresponding to (4.22) can be dealt with in exactly the same way. In the argument for the second term, $|X_t|$ is replaced by $|X_t|^2$ so that one needs $2\gamma < \kappa - 3$, hence $\kappa > 11$. $\square$

Proof of Theorem 1. All statements in this theorem follow from [DFGW, Theorem 2.2] and from (4.17). Actually, we have already checked all the relevant hypothesis, namely (i) the random variables $X_{[s,t]}$, $0 \le s < t$, satisfy the properties (P1) to(P5) described in Section 4.4 and are in $L^1(\mathcal{N}_0, \mathcal{P}_0)$ (see Proposition 2); (ii) the Markov process $(\xi_t)_{t\ge 0}$ with distribution $\mathbf{P}$ is reversible and ergodic (see Proposition 3); (iii) the mean forward velocity exists (see Lemma 4(i)); (iv) the martingale $X_t - \int_0^t ds\, \varphi(\xi_s)$ is in $L^2(\Xi, \mathbf{P})$. The last point is a consequence of Proposition 2 assuring that $X_t \in L^2(\Xi, \mathbf{P})$ and the fact that $\int_0^t ds\, \varphi(\xi_s) \in L^2(\Xi, \mathbf{P})$, which can be proved by means of the Cauchy–Schwarz inequality, the stationarity of $\mathbf{P}$ and the property $\varphi \in L^2(\mathcal{N}, \mathcal{P}_0)$. One can deduce from [DFGW, Theorem 2.2] that $\varphi \cdot a \in \mathcal{H}_{-1}$ and (see [DFGW, Remark 4, p. 802] together with Lemma 4(ii)) that the rescaled process $(\varepsilon X_{t\varepsilon^2})_{t\ge 0}$ on $(\Xi, \mathbf{P})$ converges as $\varepsilon \to 0$ (in the sense specified in Theorem 1) to the Brownian motion $\underline{W}_D$ with covariance matrix D given by

$$(a \cdot Da) \;=\; \mathbf{E}_{\mathcal{P}_0}\big(\, (a \cdot \psi a)\,\big) \;-\; 2 \int_0^\infty dt\; \langle \varphi \cdot a\,,\, e^{t\mathcal{L}}\, \varphi \cdot a \rangle_{\mathcal{P}_0} \;.$$

Invoking (4.14) and (4.15), this gives

$$\int_0^\infty dt\, \langle \varphi \cdot a\,,\, e^{t\mathcal{L}}\, \varphi \cdot a \rangle_{\mathcal{P}_0} \;=\; \sup_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \Big(2\, \langle \varphi \cdot a\,,\, f \rangle_{\mathcal{P}_0} \;-\; \langle f\,,\, (-\mathcal{L}) f \rangle_{\mathcal{P}_0} \Big) \,.$$

Using (4.7), (4.18), (4.20) and Lemma 1(i), a short calculation yields (3.10). $\square$

5. Bound by cut-off on the transition rates

The variational formula (3.10) is particularly suited in order to derive bounds on the diffusion matrix D. For example, due to the monotonicity of the jump rates $c_{x,y}(\xi)$ in the inverse temperature β, one deduces that the diffusion matrix is non-increasing function of β. The aim of this section is to obtain more quantitative bounds. Throughout this section, we suppose that the *random environment satisfies the assumptions of Theorem 1*, but we *do not* suppose that $\mathcal{P}$ is a ν-randomisation of a SSPP.

Given an energy $0 \le E_c \le 1$, we define the map $\Phi_c : \mathcal{N} \to \hat{\mathcal{N}} := \mathcal{N}(\mathbb{R}^d)$ as follows:

$$\big(\Phi_c(\xi)\big)(A) \;:=\; \xi(A \times [-E_c, E_c])\,, \qquad A \in \mathcal{B}(\mathbb{R}^d)\;. \tag{5.1}$$

Note that $\hat{\mathcal{P}}^c := \mathcal{P} \circ \Phi_c^{-1}$ is the distribution of a point process on $\mathbb{R}^d$ with finite intensity $\rho_c := \mathbf{E}_{\hat{\mathcal{P}}^c}(\,\hat{\xi}(C_1)\,) \le \rho$, and in general

$$\mathbf{E}_{\hat{\mathcal{P}}^c}(\,\hat{\xi}(C_1)^\kappa\,) \;\le\; \rho_\kappa\,, \qquad \forall\, \kappa > 0\,. \tag{5.2}$$

In what follows, we assume that $\rho_c > 0$. It can readily be checked that $\hat{\mathcal{P}}^c$ is an ergodic SSPP on $\mathbb{R}^d$. We write $\hat{\mathcal{P}}^c_0$ for the Palm distribution associated to $\hat{\mathcal{P}}^c$. Finally, note that in the case where $\mathcal{P}$ is obtained by ν–randomisation of a SSPP ζ on $\mathbb{R}^d$, the distribution $\hat{\mathcal{P}}^c$ is the δ_c–thinning of ζ, with $\delta_c := \nu([-E_c, E_c])$. The relation between the Palm distribution $\mathcal{P}_0$ and $\hat{\mathcal{P}}^c_0$ is described in the following lemma.

Lemma 5. *For any Borel set $A \in \mathcal{B}(\hat{\mathcal{N}}_0)$,*

$$\hat{\mathcal{P}}^c_0(A) \;=\; \frac{\rho}{\rho_c}\, \mathcal{P}_0(\,|E_0| \le E_c,\; \Phi_c(\xi) \in A\,)\;.$$

Proof. Comparing the identities

$$\hat{\mathcal{P}}_0^c(A) \;=\; \frac{1}{\rho_c}\int_{\hat{\mathcal{N}}}\hat{\mathcal{P}}^c(d\hat{\xi})\int_{C_1}\hat{\xi}(dx)\chi_A(S_x\hat{\xi})$$

and

$$\begin{aligned}\mathcal{P}_0(\,|E_0|\le E_c,\;\Phi_c(\xi)\in A\,) \;&=\; \frac{1}{\rho}\int_{\mathcal{N}}\mathcal{P}(d\xi)\int_{C_1}\hat{\xi}(dx)\,\chi(|E_x|\le E_c)\,\chi_A(\Phi_c(S_x\xi))\\ &=\; \frac{1}{\rho}\int_{\mathcal{N}}\mathcal{P}(d\xi)\int_{C_1}\big(\Phi_c(\xi)\big)(dx)\,\chi_A(\Phi_c(S_x\xi))\,,\end{aligned} \tag{5.3}$$

following from (2.4), proves the assertion. □

Proposition 5. *Fix a distance $r_c>0$, an energy $0\le E_c\le 1$ and let $\hat{\mathcal{P}}_0^c$ be as above. Moreover, define*

$$\varphi_c(\hat{\xi}) \;:=\; \int\hat{\xi}(dx)\;\hat{c}_{0,x}\;x\;,\qquad (a\cdot\psi_c(\hat{\xi})a)\;:=\;\int\hat{\xi}(dx)\;\hat{c}_{0,x}\;(a\cdot x)^2\;, \tag{5.4}$$

as functions on $\hat{\mathcal{N}}_0$ where $\hat{c}_{0,x}:=\chi(|x|\le r_c)$. Then the diffusion matrix D for the process $(X_t^{\xi})_{t\ge 0}$ in Theorem 1 admits the following lower bound

$$D\;\ge\;\frac{\rho_c}{\rho}\,e^{-r_c-4\beta E_c}\,D_c(r_c,E_c)\;,$$

where

$$(a\cdot D_c(r_c,E_c)\,a)\;:=\;\mathbf{E}_{\hat{\mathcal{P}}_0^c}\Big((a\cdot\psi_c a)\Big)\;-\;2\int_0^\infty dt\;\langle\varphi_c\cdot a\,,\,e^{t\mathcal{L}_c}\,\varphi_c\cdot a\rangle_{\hat{\mathcal{P}}_0^c}\;, \tag{5.5}$$

and $\mathcal{L}_c$ is the unique self–adjoint operator on $L^2(\hat{\mathcal{N}}_0,\hat{\mathcal{P}}_0^c)$ such that

$$(\mathcal{L}_c f)(\hat{\xi})\;=\;\int\hat{\xi}(dx)\;\hat{c}_{0,x}\nabla_x f(\hat{\xi})\;,\qquad\forall\,f\in L^\infty(\hat{\mathcal{N}}_0,\hat{\mathcal{P}}_0^c)\,. \tag{5.6}$$

One can prove by the same arguments used in the proof of Proposition 4 that the above operator $\mathcal{L}_c$ is well-defined and self–adjoint. We will write $\hat{\mathbf{P}}^c$ for the probability measure on the path space $\hat{\Xi}:=D(\,[0,\infty),\hat{\mathcal{N}}_0\,)$ associated to the Markov process with generator $\mathcal{L}_c$ and initial distribution $\hat{\mathcal{P}}_0^c$. If the initial distribution is $\delta_{\hat{\xi}}$, with $\hat{\xi}\in\hat{\mathcal{N}}_0$, then we will write $\hat{\mathbf{P}}_{\hat{\xi}}^c$. One can prove that these Markov processes are well defined (in particular, $\hat{\mathbf{P}}_{\hat{\xi}}^c$ is well defined for $\hat{\mathcal{P}}_0^c$–almost all $\hat{\xi}$) and exhibit a realization as jump processes by means of the same arguments used in Section 3 (note that, for a suitable positive constant c, $\int\hat{\xi}(dx)\hat{c}_{0,x}\le c\,\lambda_0(\xi)$ for any $\xi\in\mathcal{N}_0$, thus allowing to exclude explosion phenomena from the results of Section 3.1). Finally, given $\underline{\hat{\xi}}\in\hat{\Xi}$, $X_t(\underline{\hat{\xi}})$ is defined as in (4.16).

Proof. Note that

$$c_{0,x}(\xi)\;\ge\;e^{-r_c-4\beta E_c}\;\tilde{c}_{0,x}(\xi)\;,$$

where

$$\tilde{c}_{x,y}(\xi)\;:=\;\chi\big(|E_x|\le E_c,\;|E_y|\le E_c,\;|x-y|\le r_c\big)\;.$$

Then (3.10) implies

$$(a\cdot Da)\;\ge\;e^{-r_c-4\beta E_c}\;g(a)$$

where

$$g(a)\;:=\;\inf_{f\in L^\infty(\mathcal{N}_0,\mathcal{P}_0)}\int\mathcal{P}_0(d\xi)\int\hat{\xi}(dx)\;\tilde{c}_{0,x}(\xi)\;\Big(a\cdot x\;+\;\nabla_x f(\xi)\Big)^2\;\ge 0\;.$$

By the same arguments used in the proof of Proposition 4 one can show that there is a unique self–adjoint operator $\tilde{\mathcal{L}}$ on $L^2(\mathcal{N}_0,\mathcal{P}_0)$ such that

$$(\tilde{\mathcal{L}}f)(\xi)\;:=\;\int\hat{\xi}(dx)\;\tilde{c}_{0,x}(\xi)\;\nabla_x f(\xi)\;,\qquad\forall\,f\in L^\infty(\mathcal{N}_0,\mathcal{P}_0).$$

Moreover, $L^\infty(\mathcal{N}_0, \mathcal{P}_0)$ is a core of $\tilde{\mathcal{L}}$ and

$$\langle f, (-\tilde{\mathcal{L}})f\rangle_{\mathcal{P}_0} \;=\; \frac{1}{2}\int \mathcal{P}_0(d\xi)\int \hat{\xi}(dx)\; \tilde{c}_{0,x}(\xi)\;\left(\nabla_x f(\xi)\right)^2\;, \qquad \forall\, f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)\;. \tag{5.7}$$

Next let us introduce the functions

$$\tilde{\varphi}(\xi) \;=\; \int \hat{\xi}(dx)\; \tilde{c}_{0,x}(\xi)\; x\;, \qquad (a\cdot\tilde{\psi}(\xi)a) \;=\; \int \hat{\xi}(dx)\; \tilde{c}_{0,x}(\xi)\; (a\cdot x)^2\;.$$

Then, we obtain by means of straightforward computations and the identities (4.14), (4.15) and (5.7) that

$$\begin{aligned} g(a) \;&=\; \mathbf{E}_{\mathcal{P}_0}\Big(\,(a\cdot\tilde{\psi}a)\,\Big) - 2 \sup_{f\in L^\infty(\mathcal{N}_0,\mathcal{P}_0)} \Big(2\,\langle\tilde{\varphi}\cdot a, f\rangle_{\mathcal{P}_0} - \langle f, (-\tilde{\mathcal{L}})f\rangle_{\mathcal{P}_0}\Big) \\ &=\; \mathbf{E}_{\mathcal{P}_0}\Big(\,(a\cdot\tilde{\psi}a)\,\Big) \;-\; 2\int_0^\infty dt\;\Big\langle \tilde{\varphi}\cdot a\,,\, e^{t\tilde{\mathcal{L}}}\,\tilde{\varphi}\cdot a\Big\rangle_{\mathcal{P}_0}\,. \end{aligned}$$

At this point, in order to get (5.5), it is enough to show that

$$\mathbf{E}_{\mathcal{P}_0}\Big(\,(a\cdot\tilde{\psi}a)\,\Big) = \frac{\rho_c}{\rho}\mathbf{E}_{\hat{\mathcal{P}}_0^c}\Big(\,(a\cdot\psi_c a)\,\Big)\;,$$

and

$$\Big\langle \tilde{\varphi}\cdot a\,,\, e^{t\tilde{\mathcal{L}}}\,\tilde{\varphi}\cdot a\Big\rangle_{\mathcal{P}_0} = \frac{\rho_c}{\rho}\,\langle \varphi_c\cdot a\,,\, e^{t\mathcal{L}_c}\,\varphi_c\cdot a\rangle_{\hat{\mathcal{P}}_0^c}\;.$$

This can be derived from Lemma 5 and the identities

$$\begin{aligned} \tilde{\psi} \;&=\; \chi(|E_0|\leq E_c)\,\psi_c\circ\Phi_c\;, \\ \tilde{\varphi} \;&=\; \chi(|E_0|\leq E_c)\,\varphi_c\circ\Phi_c\;, \\ \tilde{\mathcal{L}}(f\circ\Phi_c) \;&=\; \chi(|E_0|\leq E_c)\,(\mathcal{L}_c f)\circ\Phi_c\;, \end{aligned}$$

where Φ_c has been introduced in (5.1). □

6. Periodic approximants and resistor networks

In this section, we compare $D_c(r_c, E_c)$ to the diffusion coefficient of adequately defined periodic approximants, which then in turn can be calculated as the conductance of a random resistor network. There have been numerous works on periodic approximants; a recent one containing further references is [Owh]. Still, we merely suppose that the *random environment satisfies the assumptions of Theorem 1 as well as the isotropy property of Remark 4.*

6.1. **Random walk on a periodized medium.** Let us choose a given direction in $\mathbb{R}^d$, say, the direction parallel to the axis of the first coordinate. Given a fixed configuration $\hat{\xi}\in\hat{\mathcal{N}}$ and $N > r_c$, we define the following subsets of $\mathbb{R}^d$

$$\begin{aligned} Q_N^{\hat{\xi}} \;&:=\; \mathrm{supp}(\hat{\xi})\cap C_{2N}\;, \qquad \Gamma_N^{\pm} \;:=\; \mathbb{Z}^d\cap\{x : x^{(1)} = \pm N,\; |x^{(j)}| < N \text{ for } j = 2,\dots,d\}\;, \\ \overline{\mathcal{V}}_N^{\hat{\xi}} \;&:=\; Q_N^{\hat{\xi}}\cup\Gamma_N^+\cup\Gamma_N^-\;, \qquad B_N^{\hat{\xi}\pm} \;:=\; Q_N^{\hat{\xi}}\cap B_N^{\pm}\;, \end{aligned}$$

where as before $C_{2N} = (-N,N)^d$ as well as $B_N^- = \{x\in C_{2N} \,:\, x^{(1)}\in(-N,-N+r_c]\}$ and $B_N^+ = \{x\in C_{2N} \,:\, x^{(1)}\in[N-r_c,N)\}$.

Next let us introduce a graph $(\overline{\mathcal{V}}_N^{\hat{\xi}}, \overline{\mathcal{E}}_N^{\hat{\xi}})$ with set of vertices $\overline{\mathcal{V}}_N^{\hat{\xi}}$ and set of edges $\overline{\mathcal{E}}_N^{\hat{\xi}}$. Two vertices $x,y\in Q_N^{\hat{\xi}}$ are connected by a non-oriented edge $\{x,y\}\in\overline{\mathcal{E}}_N^{\hat{\xi}}$ if and only if $|x-y|\leq r_c$;

moreover, all vertices $x \in B_N^{\hat{\xi}+}$ (respectively $x \in B_N^{\hat{\xi}-}$) are connected to all $y \in \Gamma_N^+$ (respectively $y \in \Gamma_N^-$) by an edge $\{x,y\} \in \overline{\mathcal{E}}_N^{\hat{\xi}}$ and the points of $\Gamma_N^{\pm}$ are not connected between themselves.

We now define another graph $(\mathcal{V}_N^{\hat{\xi}}, \mathcal{E}_N^{\hat{\xi}})$ obtained from $(\overline{\mathcal{V}}_N^{\hat{\xi}}, \overline{\mathcal{E}}_N^{\hat{\xi}})$ by identifying the vertices

$$x_- \;=\; (-N, x^{(2)}, \ldots, x^{(d)}) \qquad \text{and} \qquad x_+ \;=\; (N, x^{(2)}, \ldots, x^{(d)}) \,.$$

Let us write $\pi : \overline{\mathcal{V}}_N^{\hat{\xi}} \to \mathcal{V}_N^{\hat{\xi}}$ for the identification map on the sets of vertices. Hence $\pi(\Gamma_N^-) = \pi(\Gamma_N^+)$ and π restricted to $Q_N^{\hat{\xi}}$ is the identity map. The set $\mathcal{V}_N^{\hat{\xi}} = \pi(\overline{\mathcal{V}}_N^{\hat{\xi}})$ represents the medium periodized along the first coordinate.

Now a continuous–time random walk with state space $\mathcal{V}_N^{\hat{\xi}}$ and infinitesimal generator $\mathcal{L}_N^{\hat{\xi}}$ is given by

$$\big(\mathcal{L}_N^{\hat{\xi}} f\big)(x) \;=\; \sum_{y \in \mathcal{V}_N^{\hat{\xi}} \,:\, \{x,y\} \in \mathcal{E}_N^{\hat{\xi}}} c(\{x,y\}) \big(f(y) - f(x) \big) \,, \qquad \forall\, x \in \mathcal{V}_N^{\hat{\xi}} \,,$$

where the bond-dependent transition rates $c(\{x,y\})$ are defined for any $\{x,y\} \in \mathcal{E}_N^{\hat{\xi}}$ by

$$c(\{x,y\}) \;=\; \begin{cases} 1 & \text{if } x, y \in Q_N^{\hat{\xi}} \,, \\ \frac{1}{|\Gamma_N^-|} & \text{if } x \in \pi(\Gamma_N^-) \text{ or } y \in \pi(\Gamma_N^-) \,. \end{cases} \tag{6.1}$$

Clearly the generator $\mathcal{L}_N^{\hat{\xi}}$ is symmetric w.r.t. the uniform distribution $m_N^{\hat{\xi}}$ on $\mathcal{V}_N^{\hat{\xi}}$ given by

$$m_N^{\hat{\xi}} \;=\; \frac{1}{|\mathcal{V}_N^{\hat{\xi}}|} \sum_{x \in \mathcal{V}_N^{\hat{\xi}}} \delta_x \,.$$

Hence the Markov process with generator $\mathcal{L}_N^{\hat{\xi}}$ and initial distribution $m_N^{\hat{\xi}}$ is reversible. Note that it is not ergodic, however, if there are more than 1 cluster (equivalence class of edges). In the latter case, the ergodic measures are the uniform distributions on a given cluster and this is sufficient for the present purposes.

We write $\mathbf{P}_N^{\hat{\xi}}$ (respectively $\mathbf{P}_{N,x}^{\hat{\xi}}$) for the probability on the path space $\Omega_N^{\hat{\xi}} = D\big([0,\infty), \mathcal{V}_N^{\hat{\xi}}\big)$ associated to the random walk with initial distribution $m_N^{\hat{\xi}}$ (respectively δ_x) and generator $\mathcal{L}_N^{\hat{\xi}}$.

Let us introduce an antisymmetric function $d_1(x,y)$ on $\mathcal{V}_N^{\hat{\xi}}$ such that

$$d_1(x,y) \;=\; \begin{cases} y^{(1)} - x^{(1)} & \text{if } x, y \in Q_N^{\hat{\xi}} \,, \\ y^{(1)} + N & \text{if } y \in Q_N^{\hat{\xi}},\ y^{(1)} < 0,\ x \in \pi(\Gamma_N^-) \,, \\ y^{(1)} - N & \text{if } y \in Q_N^{\hat{\xi}},\ y^{(1)} > 0,\ x \in \pi(\Gamma_N^-) \,. \end{cases}$$

Finally, given $t \geq 0$, we define the random variable

$$X_{N,t}^{(1)\hat{\xi}}(\underline{\omega}) \;=\; \sum_{s \in [0,t] \,:\, \omega_s \neq \omega_{s-}} d_1(\omega_{s-}, \omega_s) \,,$$

where $(\omega_s)_{s \geq 0} \in \Omega_N^{\hat{\xi}}$. It is the sum of position increments along the first coordinate axis for all jumps occurring in the time interval $[0,t]$. Clearly, $X_{N,t}^{(1)\hat{\xi}}$ gives rise to a time-covariant and anti-symmetric family so that, as in Section 4.5, [DFGW, Theorem 2.2] can be used in order to deduce the following result.

Proposition 6. *Given $N \in \mathbb{N}$, $N > r_c$, and $\hat{\xi} \in \hat{\mathcal{N}}$*

$$\lim_{t\uparrow\infty} \frac{1}{t} \, \mathbf{E}_{\mathbf{P}_N^{\hat{\xi}}} \big((X_{N,t}^{(1)\hat{\xi}})^2 \big) \; = \; D_N^{\hat{\xi}} \, ,$$

where the diffusion coefficient $D_N^{\hat{\xi}}$ is finite and given by

$$D_N^{\hat{\xi}} \; = \; m_N^{\hat{\xi}}(\psi_N^{\hat{\xi}}) \; - \; 2 \int_0^\infty dt \, \langle \varphi_N^{\hat{\xi}} \, , \, e^{t\mathcal{L}_N^{\hat{\xi}}} \, \varphi_N^{\hat{\xi}} \rangle_{m_N^{\hat{\xi}}} \, , \tag{6.2}$$

with $\psi_N^{\hat{\xi}}$, $\varphi_N^{\hat{\xi}}$ (scalar) functions on $\mathcal{V}_N^{\hat{\xi}}$ defined as

$$\psi_N^{\hat{\xi}}(x) \; = \sum_{y \,:\, \{y,x\} \in \mathcal{E}_N^{\hat{\xi}}} c(\{x,y\}) \, d_1(x,y)^2 \, , \qquad \varphi_N^{\hat{\xi}}(x) \; = \sum_{y \,:\, \{y,x\} \in \mathcal{E}_N^{\hat{\xi}}} c(\{x,y\}) \, d_1(x,y) \, . \tag{6.3}$$

6.2. **Link to periodized medium.** Here we show that the diffusion matrix (5.5) can be bounded below in terms of the average of the diffusion coefficient associated to the periodized random media. Our proof follows the arguments of [DFGW, Prop. 4.13], but additional technical problems are related to the randomness of geometry (absence of any lattice structure) and possible (albeit integrable) singularities of the mean forward velocity and infinitesimal mean square displacement.

Proposition 7. *Suppose that for $1 \le p \le 8$*

$$\lim_{N\uparrow\infty} \frac{\rho_c \, \ell(C_{2N})}{\hat{\xi}(C_{2N}) + a_{2N}} \; = \; 1 \, , \qquad \text{in} \quad L^p(\hat{\mathcal{N}}, \hat{\mathcal{P}}^c) \, , \tag{6.4}$$

where $\rho_c := \mathbf{E}_{\hat{\mathcal{P}}^c}(\hat{\xi}(C_1))$ and $a_{2N} := |\Gamma_N^{\pm}| = (2N-1)^{d-1}$. Then, for any $t > 0$,

$$\lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c} \Big(m_N^{\hat{\xi}}(\psi_N^{\hat{\xi}}) \Big) \; = \; \mathbf{E}_{\hat{\mathcal{P}}_0^c}(\psi_c^{(11)}) \, , \tag{6.5}$$

$$\lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c} \left(\langle \varphi_N^{\hat{\xi}} \, , \, e^{t\mathcal{L}_N^{\hat{\xi}}} \, \varphi_N^{\hat{\xi}} \rangle_{m_N^{\hat{\xi}}} \right) \; = \; \left\langle \varphi_c^{(1)} \, , \, e^{t\mathcal{L}_c} \, \varphi_c^{(1)} \right\rangle_{\hat{\mathcal{P}}_0^c} \, , \tag{6.6}$$

where $\psi_c^{(11)}$ and $\varphi_c^{(1)}$ are the first diagonal matrix element of the matrix ψ_c and the first component of the vector φ_c introduced in (5.4), respectively.

Since D_c is given by (5.5) and is a multiple of the identity (*cf.* Remark 4), the identities (6.5) and (6.6) combined with Fatou's Lemma immediately imply:

Corollary 1. *Under the same hypothesis as above,*

$$D_c(r_c, E_c) \; \ge \; \left(\limsup_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c}\big(D_N^{\hat{\xi}}\big) \right) \mathbf{1}_d \, , \tag{6.7}$$

where $\mathbf{1}_d$ is the $d \times d$ identity matrix.

Before giving the proof, let us comment on its assumptions. Due to (5.2), $\rho_p < \infty$ implies $\mathbf{E}_{\hat{\mathcal{P}}^c}(\, \hat{\xi}(C_1)^p \,) < \infty$ for any $p > 0$. As $\hat{\mathcal{P}}^c$ is ergodic, this implies the following ergodic theorem, an extension of [DV, Theorem 10.2]. We recall that a convex averaging sequence of sets $\{A_n\}$ in $\mathbb{R}^d$ is a sequence of convex sets such that $A_n \subset A_{n+1}$ and A_n contains a ball of radius r_n with $r_n \to \infty$ as $n \to \infty$.

Lemma 6. *Suppose that $\rho_p < \infty$, $p \geq 1$. Then, given a convex averaging sequence of Borel sets $\{A_n\}$ in $\mathbb{R}^d$,*

$$\frac{\hat{\xi}(A_n)}{\rho_c\,\ell(A_n)} \to 1 \qquad \textit{in} \quad L^p(\hat{\mathcal{N}},\hat{\mathcal{P}}^c)\,,$$

and

$$\frac{\hat{\xi}(A_n)}{\rho_c\,\ell(A_n)} \to 1 \qquad \hat{\mathcal{P}}^c\text{-}a.s.\,.$$

We will also need a bound on $\mathbf{E}_{\hat{\mathcal{P}}^c}((\hat{\xi}(A_n)/\ell(A_n))^p)$, uniformly in n, for a sequence of sets that does not satisfy the assumptions of Lemma 6. To this aim we note that, given a Borel set $B \subset \mathbb{R}^d$ which is a union of k non-overlapping cubes of side 1, one has

$$\mathbf{E}_{\hat{\mathcal{P}}^c}\big(\big(\frac{\hat{\xi}(B)}{k}\big)^p\big) \;\leq\; \mathbf{E}_{\hat{\mathcal{P}}^c}\big(\hat{\xi}(\overline{C}_1)^p\big) \;\leq\; \rho_p\,, \qquad \forall\, p \geq 1\,, \tag{6.8}$$

where $\overline{C}_1$ is the closed cube $[-\frac{1}{2},\frac{1}{2}]^d$. This follows from the stationarity of $\hat{\mathcal{P}}^c$ and the convexity of the function $f(x) = x^p$, $x \geq 0$.

Proof of Proposition 7. Without loss of generality, we assume $r_c = 1$. A key observation in order to prove (6.5) and (6.6) is the following identity, valid for any nonnegative measurable function h defined on $\hat{\mathcal{N}}_0$. It follows easily from (2.5):

$$\mathbf{E}_{\hat{\mathcal{P}}^c}\left(\int_B \hat{\xi}(dx)h(S_x\hat{\xi})\right) \;=\; \rho_c\,\ell(B)\,\mathbf{E}_{\hat{\mathcal{P}}^c_0}(h), \qquad \forall\; B \in \mathcal{B}(\mathbb{R}^d). \tag{6.9}$$

From this identity we can deduce for any $h \in L^2(\mathcal{N}_0,\hat{\mathcal{P}}^c_0)$ that

$$\lim_{N\uparrow\infty}\mathbf{E}_{\hat{\mathcal{P}}^c}\left(\frac{1}{\hat{\xi}(C_{2N})+a_{2N}}\int_{C_{2N-2}}\hat{\xi}(dx)h(S_x\hat{\xi})\right) \;=\; \mathbf{E}_{\hat{\mathcal{P}}^c_0}(h)\,. \tag{6.10}$$

In fact, due to (6.9), it is enough to show that

$$\mathbf{E}_{\hat{\mathcal{P}}^c}\left(\Big(\frac{1}{\hat{\xi}(C_{2N})+a_{2N}} - \frac{1}{\rho_c\ell(C_{2N-2})}\Big)\int_{C_{2N-2}}\hat{\xi}(dx)h(S_x\hat{\xi})\right) \downarrow 0\,, \qquad \text{as } N \uparrow \infty\,. \tag{6.11}$$

By applying twice the Cauchy-Schwarz inequality and by invoking (6.9), we obtain

$$\begin{aligned}
&\Big(\text{l.h.s. of (6.11)}\Big)^2 \\
&\leq\; \mathbf{E}_{\hat{\mathcal{P}}^c}\left(\Big(\frac{\rho_c\,\ell(C_{2N-2})}{\hat{\xi}(C_{2N})+a_{2N}}-1\Big)^2\frac{\hat{\xi}(C_{2N-2})}{\rho_c^2\,\ell(C_{2N-2})^2}\right)\mathbf{E}_{\hat{\mathcal{P}}^c}\left(\frac{1}{\hat{\xi}(C_{2N-2})}\Big(\int_{C_{2N-2}}\hat{\xi}(dx)h(S_x\hat{\xi})\Big)^2\right) \\
&\leq\; \mathbf{E}_{\hat{\mathcal{P}}^c}\left(\Big(\frac{\rho_c\,\ell(C_{2N-2})}{\hat{\xi}(C_{2N})+a_{2N}}-1\Big)^2\frac{\hat{\xi}(C_{2N-2})}{\rho_c\,\ell(C_{2N-2})}\right)\mathbf{E}_{\hat{\mathcal{P}}^c_0}(h^2)\,.
\end{aligned}$$

At this point, (6.11) follows by applying the Cauchy-Schwarz inequality to the first expectation above and then applying (6.8) and the limit (6.4) for $p = 4$.

Let now $h_N^{\hat{\xi}}$ be a function on $\mathcal{V}_N^{\hat{\xi}}$ such that for some constant $c > 0$ independent of N

$$|h_N^{\hat{\xi}}(x)| \;\leq\; c\begin{cases}\hat{\xi}(\overline{B_1(x)}) & \text{if } x \in Q_N^{\hat{\xi}}, \\ \frac{|B_N^{\hat{\xi}}|}{a_{2N}} & \text{otherwise}\,,\end{cases}$$

where $B_N^{\hat\xi} = B_N^{\hat\xi -} \cup B_N^{\hat\xi +}$ and $\overline{B_1(x)}$ is the closed unit ball centered in x. Note that $\psi_N^{\hat\xi}$ and $\varphi_N^{\hat\xi}$ satisfy this inequality. We claim that the mean boundary contribution vanishes in the limit:

$$\lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c}\Big(\frac{1}{\hat\xi(C_{2N}) + a_{2N}} \sum_{x\in \mathcal{V}_N^{\hat\xi}\setminus Q_{N-1}^{\hat\xi}} |h_N^{\hat\xi}(x)|^p \Big) \;=\; 0\,, \qquad \text{for } 1\le p\le 4. \tag{6.12}$$

In fact, the sum in (6.12) can be bounded by

$$c^p\, a_{2N} \frac{|B_N^{\hat\xi}|^p}{a_{2N}^p} \;+\; c^p \sum_{x\in Q_N^{\hat\xi}\setminus Q_{N-1}^{\hat\xi}} \hat\xi(\overline{B_1(x)})^p\,. \tag{6.13}$$

By the Cauchy-Schwarz inequality

$$\mathbf{E}_{\hat{\mathcal{P}}^c}\Big(\frac{a_{2N}}{\hat\xi(C_{2N}) + a_{2N}} \frac{|B_N^{\hat\xi}|^p}{a_{2N}^p}\Big) \;\le\; \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big(\frac{a_{2N}^2}{(\hat\xi(C_{2N}) + a_{2N})^2}\Big)\, \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big(\frac{|B_N^{\hat\xi}|^{2p}}{a_{2N}^{2p}}\Big)\,.$$

The first factor on the r.h.s. is negligible as $N\uparrow\infty$ because of the limit (6.4) for $p=2$, while the second factor is bounded, uniformly in N, because of (6.8). For the second summand in (6.13), we use twice the Cauchy-Schwarz inequality and invoke (6.9) to deduce

$$\begin{aligned}
&\mathbf{E}_{\hat{\mathcal{P}}^c}\Big(\frac{1}{\hat\xi(C_{2N}) + a_{2N}} \sum_{x\in Q_N^{\hat\xi}\setminus Q_{N-1}^{\hat\xi}} \hat\xi(\overline{B_1(x)})^p\Big) \\
&\qquad \le \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big(\frac{\hat\xi(C_{2N}\setminus C_{2N-2})}{(\hat\xi(C_{2N}) + a_{2N})^2}\Big) \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big(\frac{1}{\hat\xi(C_{2N}\setminus C_{2N-2})} \Big(\sum_{x\in Q_N^{\hat\xi}\setminus Q_{N-1}^{\hat\xi}} \hat\xi(\overline{B_1(x)})^p\Big)^2\Big) \\
&\qquad \le \Big(\rho_c\, \ell(C_{2N}\setminus C_{2N-2}) \mathbf{E}_{\hat{\mathcal{P}}^c}\Big(\frac{\hat\xi(C_{2N}\setminus C_{2N-2})}{(\hat\xi(C_{2N}) + a_{2N})^2}\Big)\Big)^{\frac12} \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big(\hat\xi(\overline{B_1(0)})^{2p}\Big)\,.
\end{aligned}$$

The last factor is bounded by hypothesis, the first one converges to 0 as $N\uparrow\infty$ because of Lemma 6 and (6.4).

In order to prove (6.5) observe that $\psi_c^{(11)}(S_x\hat\xi) = \psi_N^{\hat\xi}(x)$ if $x\in Q_{N-1}^{\hat\xi}$. Therefore we can write

$$m_N^{\hat\xi}(\psi_N^{\hat\xi}) \;=\; \frac{1}{\hat\xi(C_{2N}) + a_{2N}} \int_{C_{2N-2}} \hat\xi(dx)\psi_c^{(11)}(S_x\hat\xi) \;+\; \frac{1}{\hat\xi(C_{2N}) + a_{2N}} \sum_{x\in\mathcal{V}_N^{\hat\xi}\setminus C_{2N-2}} \psi_N^{\hat\xi}(x)\,.$$

Now (6.5) follows easily from (6.10) and (6.12) with $h_N^{\hat\xi} := \psi_N^{\hat\xi}$. Note that by the same arguments one can prove

$$\lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c}\Big\{ m_N^{\hat\xi}\big[\, |\varphi_N^{\hat\xi}(x)|^p\,\big]\Big\} \;=\; \mathbf{E}_{\hat{\mathcal{P}}_0^c}(|\varphi_c^{(1)}|^p) \;<\; \infty\,, \qquad 1\le p\le 4\,, \tag{6.14}$$

which will be useful below.

In order to prove (6.6), we fix $0<\alpha<1$ and set $M = 2N - 2[N^\alpha]$, where $[N^\alpha]$ denotes the integer part of N^α. Moreover, we define the hitting times

$$\tau_N^{\hat\xi}(\underline{\omega}) \;=\; \inf\{s\ge 0\,:\, \omega_s\notin C_{2N-2}\}\;, \qquad \underline{\omega} \;=\; (\omega_s)_{s\ge 0}\in \Omega_N^{\hat\xi} = D([0,\infty),\mathcal{V}_N^{\hat\xi})\,. \tag{6.15}$$

Recall the definitions of the distribution $\hat{\mathbf{P}}^c_{\hat\xi}$, $\mathbf{P}^{\hat\xi}_{N,x}$ and $\mathbf{P}^{\hat\xi}_N$ given in Sections 5 and 6.1. Thanks to the identity $(e^{t\mathcal{L}_N^{\hat\xi}}\varphi_N^{\hat\xi})(x) = \mathbf{E}_{\mathbf{P}^{\hat\xi}_{N,x}}\big(\varphi_N^{\hat\xi}(\omega_t)\big)$, we can write

$$\mathbf{E}_{\hat{\mathcal{P}}^c}\Big(\langle\varphi_N^{\hat{\xi}}\,,\,e^{t\mathcal{L}_N^{\hat{\xi}}}\,\varphi_N^{\hat{\xi}}\rangle_{m_N^{\hat{\xi}}}\Big)\;=\;\mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{1,N}^{\hat{\xi}}+A_{2,N}^{\hat{\xi}}+A_{3,N}^{\hat{\xi}}\big)\;,$$

where

$$\begin{aligned}
A_{1,N}^{\hat{\xi}} &= m_N^{\hat{\xi}}\Big(\chi(x\notin C_M)\,\varphi_N^{\hat{\xi}}(x)\,\mathbf{E}_{\mathbf{P}_{N,x}^{\hat{\xi}}}\big(\varphi_N^{\hat{\xi}}(\omega_t)\big)\Big)\;,\\
A_{2,N}^{\hat{\xi}} &= m_N^{\hat{\xi}}\Big(\chi(x\in C_M)\,\varphi_N^{\hat{\xi}}(x)\,\mathbf{E}_{\mathbf{P}_{N,x}^{\hat{\xi}}}\big(\chi(\tau_N^{\hat{\xi}}\le t)\,\varphi_N^{\hat{\xi}}(\omega_t)\big)\Big)\;,\\
A_{3,N}^{\hat{\xi}} &= m_N^{\hat{\xi}}\Big(\chi(x\in C_M)\,\varphi_N^{\hat{\xi}}(x)\,\mathbf{E}_{\mathbf{P}_{N,x}^{\hat{\xi}}}\big(\chi(\tau_N^{\hat{\xi}}> t)\,\varphi_N^{\hat{\xi}}(\omega_t)\big)\Big)\;.
\end{aligned}$$

Then (6.6) follows from

$$\lim_{N\uparrow\infty}\mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{1,N}^{\hat{\xi}}\big)\;=\;0\;,\qquad\lim_{N\uparrow\infty}\mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{2,N}^{\hat{\xi}}\big)\;=\;0\;,\qquad\lim_{N\uparrow\infty}\mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{3,N}^{\hat{\xi}}\big)\;=\;\langle\varphi_c^{(1)}\,,\,e^{t\mathcal{L}_c}\,\varphi_c^{(1)}\rangle_{\hat{\mathcal{P}}_0^c}\;.\tag{6.16}$$

Let us first prove the first limit in (6.16). By several applications of Cauchy-Schwarz inequality and due to the identity $\mathbf{P}_N^{\hat{\xi}}=\int m_N^{\hat{\xi}}(dx)\mathbf{P}_{N,x}^{\hat{\xi}}$ we get

$$\begin{aligned}
\big|\mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{1,N}^{\hat{\xi}}\big)\big| &\le \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big(\mathcal{V}_N^{\hat{\xi}}\setminus C_M\big)\Big\}\,\mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big[\,\varphi_N^{\hat{\xi}}(x)^2\mathbf{E}_{\mathbf{P}_{N,x}^{\hat{\xi}}}\big(\varphi_N^{\hat{\xi}}(\omega_t)^2\big)\,\big]\Big\}\\
&\le \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big(\mathcal{V}_N^{\hat{\xi}}\setminus C_M\big)\Big\}\,\mathbf{E}^{\frac14}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big[\,\varphi_N^{\hat{\xi}}(x)^4\,\big]\Big\}\,\mathbf{E}^{\frac14}_{\hat{\mathcal{P}}^c}\Big\{\mathbf{E}_{\mathbf{P}_N^{\hat{\xi}}}\big(\varphi_N^{\hat{\xi}}(\omega_t)^4\big)\,\big]\Big\}\\
&= \mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big(\mathcal{V}_N^{\hat{\xi}}\setminus C_M\big)\Big\}\,\mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big[\,\varphi_N^{\hat{\xi}}(x)^4\,\big]\Big\}\;,
\end{aligned}$$

where the last identity follows from the stationarity of $\mathcal{L}_N^{\hat{\xi}}$ w.r.t. $m_N^{\hat{\xi}}$. Due to the dominated convergence theorem, the first expectation on the r.h.s. goes to 0, while the second expectation is bounded due to (6.14).

In order to prove the second limit in (6.16), we apply twice the Cauchy-Schwarz inequality in order to obtain the bound $\mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{2,N}^{\hat{\xi}}\big)$ by

$$\mathbf{E}^{\frac12}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big[\,\varphi_N^{\hat{\xi}}(x)^2\,\big]\Big\}\mathbf{E}^{\frac14}_{\hat{\mathcal{P}}^c}\Big\{\mathbf{E}_{\mathbf{P}_N^{\hat{\xi}}}\big[\,\varphi_N^{\hat{\xi}}(\omega_t)^4\,\big]\Big\}\mathbf{E}^{\frac14}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\big[\chi(x\in C_M)\,\mathbf{P}_{N,x}^{\hat{\xi}}(\tau_N^{\hat{\xi}}\le t)\big]\Big\}\;.\tag{6.17}$$

Again, because of stationarity and (6.14), the first two factors on the r.h.s. are bounded while the last one converges to 0 due to Lemma 7 below.

Finally we prove the last limit in (6.16). To this aim, given $\underline{\hat{\xi}}\in\hat{\Xi}=D([0,\infty),\hat{\mathcal{N}}_0)$ and $x\in C_M$, we set

$$\tau_{N,x}(\underline{\hat{\xi}})=\inf\Big\{s\ge 0\,:\,x+X_s(\underline{\hat{\xi}})\notin C_{2N-2}\Big\}\;,\tag{6.18}$$

where $X_s(\underline{\hat{\xi}})$ is defined as in (4.16). Note that for $x\in C_M\cap\operatorname{supp}(\hat{\xi})$,

$$\varphi_N^{\hat{\xi}}(x)\;=\;\varphi_c^{(1)}(S_x\hat{\xi})\;,\qquad\mathbf{E}_{\mathbf{P}_{N,x}^{\hat{\xi}}}\big(\chi(\tau_N^{\hat{\xi}}>t)\,\varphi_N^{\hat{\xi}}(\omega_t)\big)\;=\;\mathbf{E}_{\hat{\mathbf{P}}^c_{S_x\hat{\xi}}}\big(\chi(\tau_{N,x}>t)\,\varphi_c^{(1)}(\hat{\xi}_t)\big)\;.$$

Therefore

$$\mathbf{E}_{\hat{\mathcal{P}}^c}\big(\,A_{3,N}^{\hat{\xi}}\,\big)\;=\;\mathbf{E}_{\hat{\mathcal{P}}^c}\Big\{m_N^{\hat{\xi}}\Big[\chi(x\in C_M)\,\varphi_c^{(1)}(S_x\hat{\xi})\,\mathbf{E}_{\hat{\mathbf{P}}^c_{S_x\hat{\xi}}}\big(\,\chi(\tau_{N,x}>t)\,\varphi_c^{(1)}(\hat{\xi}_t)\,\big)\Big]\Big\}\;.$$

On the other hand, by applying the Cauchy-Schwarz inequality as in (6.17) and due to Lemma 7, we obtain

$$\lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c}\Big\{ m_N^{\hat{\xi}}\Big[\chi(x\in C_M)\,|\varphi_c^{(1)}(S_x\hat{\xi})|\,\mathbf{E}_{\hat{\mathbf{P}}^c_{S_x\hat{\xi}}}\big(\chi(\tau_{N,x}\le t)\,|\varphi_c^{(1)}(\hat{\xi}_t)|\,\big)\Big]\Big\} \;=\; 0\,.$$

The last two identities imply

$$\lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c}\big(A_{3,N}^{\hat{\xi}}\big) \;=\; \lim_{N\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}^c}\Big\{ m_N^{\hat{\xi}}\Big[\chi(x\in C_M)\,\varphi_c^{(1)}(S_x\hat{\xi})\,\mathbf{E}_{\hat{\mathbf{P}}^c_{S_x\hat{\xi}}}\big(\varphi_c^{(1)}(\hat{\xi}_t)\big)\Big]\Big\}\,. \tag{6.19}$$

Observe now that (6.10) remains valid if the integral is performed on C_M in place of C_{2N-2} (the arguments used in the proof there work also in this case) and the function $h(\hat{\xi})$ is defined as

$$h(\hat{\xi}) \;=\; \varphi_c^{(1)}(\hat{\xi})\,\mathbf{E}_{\hat{\mathbf{P}}^c_{\hat{\xi}}}\big(\varphi_c^{(1)}(\hat{\xi}_t)\big) \;=\; \varphi_c^{(1)}(\hat{\xi})\big(e^{t\mathcal{L}_c}\varphi_c^{(1)}\big)(\hat{\xi})\,.$$

Note that $h \in L^2(\hat{\mathcal{N}}_0, \hat{\mathcal{P}}_0^c)$. Therefore we can conclude that the r.h.s. of (6.19) is equal to $\langle \varphi_c^{(1)}\,,\, e^{t\mathcal{L}_c}\,\varphi_c^{(1)}\rangle_{\hat{\mathcal{P}}_0^c}$. □

Lemma 7. *Let $\tau_N^{\hat{\xi}}$ and $\tau_{N,x}$ be defined as in* (6.15) *and* (6.18), *and let $M = 2N - 2[N^\alpha]$. Then*

$$\lim_{N\uparrow\infty} \;\mathbf{E}_{\hat{\mathcal{P}}^c}\Big\{ m_N^{\hat{\xi}}\Big[\chi(x\in C_M)\,\mathbf{P}^{\hat{\xi}}_{N,x}\big(\tau_N^{\hat{\xi}}\le t\,\big)\Big]\Big\} \;=\; 0\,, \tag{6.20}$$

$$\lim_{N\uparrow\infty} \;\mathbf{E}_{\hat{\mathcal{P}}^c}\Big\{ m_N^{\hat{\xi}}\Big[\chi(x\in C_M)\;\hat{\mathbf{P}}^c_{S_x\hat{\xi}}\big(\tau_{N,x}\le t\,\big)\Big]\Big\} \;=\; 0\,. \tag{6.21}$$

Proof. One can check by a coupling argument that the two expectations in (6.20) and (6.21) coincide: for each $N \in \mathbb{N}_+$, $\hat{\xi}\in\hat{\mathcal{N}}$ and $x \in C_M \cap \mathrm{supp}(\hat{\xi})$, one can define a probability measure μ on $\Omega_N^{\hat{\xi}} \times \hat{\Xi}$ such that

$$\mu(A\times\hat{\Xi}) \;=\; \mathbf{P}^{\hat{\xi}}_{N,x}(A)\,, \qquad \mu(\Omega_N^{\hat{\xi}}\times B) \;=\; \hat{\mathbf{P}}^c_{S_x\hat{\xi}}(B)\,, \qquad \forall\, A\in\mathcal{B}(\Omega_N^{\hat{\xi}})\,,\quad \forall\, B\in\mathcal{B}(\hat{\Xi})\,,$$

and such that, μ almost surely, $\tau_N^{\hat{\xi}}(\underline{\omega}) = \tau_{N,x}(\underline{\hat{\xi}})$ and $\omega_s = x + X_s(\underline{\hat{\xi}})$ for any $0\le s<\tau_N^{\hat{\xi}}$. Such a coupling μ implies $\hat{\mathbf{P}}^c_{S_x\hat{\xi}}\big(\tau_{N,x}\le t\,\big) = \mathbf{P}^{\hat{\xi}}_{N,x}\big(\tau_N^{\hat{\xi}}\le t\big)$. Thus we need to prove only (6.20). Moreover, without loss of generality, we assume $r_c = 1$.

To this aim let us cover $C_{2N-2}\setminus C_M$ by disjoint cubes $C_{1,i}$ of side 1, $i\in I$, so that $C_{2N-2}\setminus C_M = \cup_{i\in I} C_{1,i}$ (the boundaries of these cubes are suitably chosen for them to be disjoint). Finally, given a positive integer n, we set

$$I_*^n \;=\; \{(l_1,\dots,l_n)\in I^n \;:\; l_j\neq l_k \;\;\text{if}\; j\neq k\}\,.$$

It is simple to verify that, given $\hat{\xi}\in\hat{\mathcal{N}}$ and $x\in C_M\cap\mathrm{supp}(\hat{\xi})$, $\tau_N^{\hat{\xi}}(\underline{\omega})<\infty$ for $\mathbf{P}^{\hat{\xi}}_{N,x}$ almost all $\underline{\omega}\in\Omega_N^{\hat{\xi}}$. For such paths $\underline{\omega}$ we we define $k=k(\underline{\omega})$ as the number of different cubes $C_{1,i}$, $i\in I$, visited by the particle in the time interval $[0,\tau_N^{\hat{\xi}}(\underline{\omega})\,)$ and moreover we define by induction $(i_1,\dots,i_k)\in I_k^*$, $(x_1,\dots,x_k)\in\big(C_{2N-2}\setminus C_M\big)^k$ with $x_j\in C_{1,i_j}$ $\forall j\,:\,1\le j\le k$, and $(t_1,\dots,t_k)$ as follows: Let x_1 be the first point reached in $C_{2N-2}\setminus C_M$ and t_1 be the time spent in x_1 before jumping away. The index i_1 is characterised by the requirement that $x_1\in C_{1,i_1}$. Suppose now that $i_1,\dots,i_j$, $x_1,\dots,x_j$ and $t_1,\dots,t_j$ have been defined and that $j<k$. Then x_{j+1} is the first point in $C_{2N-2}\setminus\big(C_M\cup C_{1,i_1}\cup\dots\cup C_{1,i_j}\big)$ visited during the time interval $[0,\tau_N^{\hat{\xi}}(\underline{\omega})\,)$ and t_{j+1} is the time spent at x_{j+1} during such a first visit. Moreover, i_{j+1} is such that $x_{j+1}\in C_{1,i_{j+1}}$.

Now let $T_i^{\hat\xi}$, $i \in I$ and $\hat\xi \in \hat{\mathcal N}$, be a family of independent exponential random variables (all independent from the above random objects) and such that $T_i^{\hat\xi}$ has parameter $\hat\xi\big(\widetilde C_{1,i}\big)$, where

$$\widetilde C_{1,i} \;=\; \{y \in \mathbb R^d \,:\, \mathrm{dist}(y, C_{1,i}) \le 1\}.$$

Since, given $\hat\xi$, k and $(x_1,\dots,x_k)$, t_j $(1 \le j \le k)$ are independent exponential variables and t_j has parameter non larger than $\hat\xi\big(\widetilde C_{1,i_j}\big)$ and since $k \ge k_{\min} := [N^\alpha]-1$, we obtain

$$\begin{aligned}
&\mathbf E_{\hat{\mathcal P}^c}\Big\{ m_N^{\hat\xi}\Big[\chi(x\in C_M)\,\mathbf P^{\hat\xi}_{N,x}\big(\tau_N^{\hat\xi}\le t\big)\Big]\Big\}\\
&\quad = \sum_{n=k_{\min}}^{|I|}\sum_{\underline l\in I_*^n}\sum_{\underline y\in\prod_{j=1}^n C_{1,l_j}} \mathbf E_{\hat{\mathcal P}^c}\Big\{ m_N^{\hat\xi}\Big[\chi(x\in C_M)\,\mathbf P^{\hat\xi}_{N,x}\big(\tau_N^{\hat\xi}\le t,\, k=n, x_1=y_1,\dots,x_n=y_n\big)\Big]\Big\}\\
&\quad \le \sum_{n=k_{\min}}^{|I|}\sum_{\underline l\in I_*^n} \mathbf E_{\hat{\mathcal P}^c}\Big\{ m_N^{\hat\xi}\Big[\chi(x\in C_M)\,\mathbf P^{\hat\xi}_{N,x}\big(k=n, i_1=l_1,\dots,i_n=l_n\,\big)\Big]\cdot \qquad (6.22)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Prob }\big(T^{\hat\xi}_{l_1}+\cdots+T^{\hat\xi}_{l_n}\le t\big)\Big\}.
\end{aligned}$$

where the last inequality follows from the bound

$$\mathbf P^{\hat\xi}_{N,x}\big(\tau_N^{\hat\xi}\le t\,|\,k=n, x_1=y_1,\dots,x_n=y_n\big) \le \text{ Prob }\big(T^{\hat\xi}_{l_1}+\cdots+T^{\hat\xi}_{l_n}\le t\big)\,.$$

Let us define $m := \mathbf E_{\hat{\mathcal P}^c}\big(\hat\xi(\widetilde C_1)\big)$, where $\widetilde C_1 = \{y\in\mathbb R^d : \mathrm{dist}(y, C_1)\le 1\}$. Given $\kappa>0$ and $\underline l\in I_*^n$ as above, we define $\mathcal A=\mathcal A(\kappa,\underline l)$ as follows

$$\mathcal A \;=\; \Big\{\hat\xi\in\hat{\mathcal N} \,:\, \big|\{j \,:\, 1\le j\le n \text{ and } \hat\xi(\widetilde C_{1,l_j})>\kappa\, m\}\big| > \frac n2\Big\}.$$

Then, by the Chebyshev inequality and the stationarity of $\hat{\mathcal P}^c$,

$$\hat{\mathcal P}^c(\mathcal A) \;\le\; \frac 2n\,\mathbf E_{\hat{\mathcal P}^c}\Big(\big|\{j \,:\, 1\le j\le n \text{ and } \hat\xi(\widetilde C_{1,l_j})>\kappa\, m\}\big|\Big) \;\le\; 2\hat{\mathcal P}^c\big(\hat\xi(\widetilde C_1)>\kappa\, m\big) \;\to\; 0\,,$$

as $\kappa\to\infty$. Note that the complement $\mathcal A^c$ of $\mathcal A$ can be written as

$$\mathcal A^c \;=\; \Big\{\hat\xi\in\hat{\mathcal N} \,:\, \big|\{j \,:\, 1\le j\le n \text{ and } \hat\xi(\widetilde C_{1,l_j})\le\kappa\, m\}\big| \ge \big[\frac n2\big]_*\Big\}\,,$$

where $[n/2]_*$ is defined as $n/2$ for n even and as $[n/2]+1$ for n odd. If $\hat\xi\in\mathcal A^c$ then at least $\big[\frac n2\big]_*$ of the exponential variables $T^{\hat\xi}_{l_1},\dots,T^{\hat\xi}_{l_n}$ have parameter non larger than $\kappa\, m$. Then, by a coupling argument (*e.g.* the proof of Proposition 2), we get for all $\hat\xi\in\mathcal A^c$

$$\text{Prob}\big(T^{\hat\xi}_{l_1}+\cdots+T^{\hat\xi}_{l_n}\le t\big) \;\le\; e^{-\kappa\, m t}\sum_{r=[n/2]_*}^{\infty}\frac{(\kappa\, m\, t)^r}{r!} \;=:\; \phi(\kappa, n)\,.$$

Due to the above estimates and since $n\ge k_{\min}=[N^\alpha]-1$, we get

$$\text{r.h.s. of (6.22)} \;\;\le\; 2\,\hat{\mathcal P}^c\big(\hat\xi(\widetilde C_1)>\kappa\, m\big) \;+\; \phi(\kappa, N^\alpha)\,.$$

The lemma follows by taking first the limit $N\uparrow\infty$ and then the limit $\kappa\uparrow\infty$. $\square$

6.3. **Random resistor networks.** We conclude this section by pointing out that the diffusion coefficient $D_N^{\hat{\xi}}$ of the periodized medium can be expressed in terms of the effective conductance of the graph $(\overline{\mathcal{V}}_N^{\hat{\xi}}, \overline{\mathcal{E}}_N^{\hat{\xi}})$ when assigning suitable bond conductances. More precisely, consider the electrical network given by the graph $(\overline{\mathcal{V}}_N^{\hat{\xi}}, \overline{\mathcal{E}}_N^{\hat{\xi}})$ where the bond $\{x,y\} \in \overline{\mathcal{E}}_N^{\hat{\xi}}$ has conductivity $c(\{\pi(x),\pi(y)\})$ with $c(\{\cdot,\cdot\})$ defined in (6.1). Then, the effective conductance $G_N^{\hat{\xi}}$ of this network is defined as the current flowing from Γ_N^- to Γ_N^+ when a unit potential difference between Γ_N^- to Γ_N^+ is imposed. It can be calculated from Ohm's law and the Kirchoff rule as follows. Let the electrical potential $V(x)$ vanish on the left border Γ_N^-, be equal to 1 on the right border Γ_N^+, and satisfy:

$$\sum_{y\,:\,\{y,x\}\in\overline{\mathcal{E}}_N^{\hat{\xi}}} c(\{\pi(x),\pi(y)\})\,\big(V(y)-V(x)\big) \;=\; 0 \qquad \text{for any} \qquad x \in Q_N^{\hat{\xi}}\;.$$

Then the effective conductance is given by the current flowing through the surfaces $\{x \in [-N,N]^d : x^{(1)} = \pm N\}$:

$$G_N^{\hat{\xi}} \;=\; \sum_{x\in B_N^{\hat{\xi}-}} V(x) \;=\; \sum_{x\in B_N^{\hat{\xi}+}} \big(1-V(x)\big)\,. \tag{6.23}$$

By a well-known analogy it is linked to the diffusion coefficient $D_N^{\hat{\xi}}$ (see *e.g.* [DFGW, Proposition 4.15] for a similar proof):

Proposition 8. *One has*

$$D_N^{\hat{\xi}} \;=\; \frac{8\,N^2}{|\overline{\mathcal{V}}_N^{\hat{\xi}}|}\,G_N^{\hat{\xi}}\;. \tag{6.24}$$

7. Estimates for SSMPP obtained by randomisation

The purpose of this section is to prove Theorems 2 and 3. The random environment with distribution $\mathcal{P}$ is the ν-randomisation of a SSPP with distribution $\hat{\mathcal{P}}$ and density ρ. Furthermore is $\hat{\mathcal{P}}^c$ the δ_c-thinning of $\hat{\mathcal{P}}$ with $\delta_c = \nu([-E_c,E_c])$. Thus $\rho_c = \delta_c\rho$. We also assume that the *Main Hypothesis holds, that $\rho_{12} < \infty$ and that the dimension d is larger than* 1.

7.1. **Point density estimates.** Here we show how the ergodic properties of Lemma 6 combined with the Main Hypothesis imply (6.4).

Proposition 9. *Suppose that $\rho_8 < \infty$ and that the Main Hypothesis holds. Then for $1 \le p \le 8$*

$$\lim_{N\uparrow\infty} \frac{\rho_c\,\ell(C_N)}{\hat{\xi}(C_N)+a_N} \;=\; 1\,, \qquad \textit{in} \quad L^p(\hat{\mathcal{N}},\hat{\mathcal{P}}^c)\,, \tag{7.1}$$

where $a_N = (N-1)^{d-1}$.

We will first prove the following criterion.

Lemma 8. *Property (7.1) holds if one has, for some $0 < \rho' < \rho$,*

$$\lim_{N\uparrow\infty} N^p\;\hat{\mathcal{P}}\left(\hat{\xi}(C_N) \le \rho'\,N^d\right) \;=\; 0\;. \tag{7.2}$$

Proof. We first check that (7.2) implies that, for some $0 < \rho'' < \rho'\delta_c$,

$$\lim_{N\uparrow\infty} N^p\, \hat{\mathcal{P}}^c\left(\hat{\xi}(C_N) \le \rho''\, N^d\right) \;=\; 0\,. \tag{7.3}$$

If $\delta_c = 1$, this is clearly true so let us suppose that $0 < \delta_c < 1$. Set $\tilde{\delta}_c = 1 - \delta_c$. If $\mathcal{C}_j^k$ denotes the binomial coefficient, we have

$$\begin{aligned}
\hat{\mathcal{P}}^c\left(\hat{\xi}(C_N) \le \rho'' N^d\right) \;&=\; \sum_{k=0}^{[\rho'' N^d]} \hat{\mathcal{P}}(\hat{\xi}(C_N) = k) \;+\; \sum_{k=[\rho'' N^d]+1}^{\infty} \hat{\mathcal{P}}(\hat{\xi}(C_N) = k) \sum_{j=k-[\rho'' N^d]}^{k} \mathcal{C}_j^k\, \tilde{\delta}_c^j \delta_c^{k-j} \\
&\le\; \sum_{k=0}^{[\rho' N^d]} \hat{\mathcal{P}}(\hat{\xi}(C_N) = k) \;+\; \sup_{k>[\rho' N^d]} \sum_{j=k-[\rho'' N^d]}^{k} \mathcal{C}_j^k\, \tilde{\delta}_c^j \delta_c^{k-j} \\
&\le\; \hat{\mathcal{P}}\left(\hat{\xi}(C_N) \le \rho' N^d\right) \;+\; \exp(-c[\rho' N^d](\delta_c - \rho''/\rho')^2)\;,
\end{aligned}$$

where the last inequality, given $\rho'' < \delta_c\rho'$, follows from a standard large deviation type estimate for Bernoulli variables with some $c > 0$. Multiplying by N^p, (7.2) thus implies (7.3).

Now set $A_N = \{\hat{\xi} \,:\, \hat{\xi}(C_N) \le \rho'' N^d\}$. Then, for some $c' > 0$ independent of N,

$$f_N(\hat{\xi}) \;:=\; \left|\frac{\rho_c\,\ell(C_N)}{\hat{\xi}(C_N) + a_N} - 1\right|^p \;\le\; c'\,\rho_c^p\, N^p\, \chi_{A_N}(\hat{\xi}) \;+\; f_N(\hat{\xi})\,\chi_{A_N^c}(\hat{\xi})\;.$$

Integrating w.r.t. $\hat{\mathcal{P}}^c$, the first term vanishes in the limit $N \uparrow \infty$ because of (7.3). For the second, let us first note that Lemma 6 implies that $\lim_{N\uparrow 0} f_N \chi_{A_N^c} = 0$ holds $\hat{\mathcal{P}}^c$-a.s.. Furthermore, $|f_N \chi_{A_N^c}| \le c'' < \infty$ uniformly in N so that the dominated convergence theorem assures that $\lim_{N\uparrow 0} \mathbf{E}_{\hat{\mathcal{P}}^c}(f_N \chi_{A_N^c}) = 0$. □

Proof of Proposition 9. Due to Lemma 8 we only need to show that (7.2) is satisfied for some $\rho' < \rho$. This is trivially true if $\hat{\mathcal{P}}$ has a uniform lower bound on the point density. Hence let us consider the other case where (3.13) holds. This implies

$$\left|\mathbf{E}_{\hat{\mathcal{P}}}(f \,|\, \mathcal{F}_{r_2}) - \mathbf{E}_{\hat{\mathcal{P}}}(f)\right| \;\le\; \|f\|_\infty\, r_1^d r_2^{d-1}\, h(r_2 - r_1)\;, \qquad \hat{\mathcal{P}}\text{-a.s.}\;, \tag{7.4}$$

where f is a bounded $\mathcal{F}_{C_{r_1}}$–measurable function.

Let $C_N = \cup_{i\in I_N} C_1^i$ be a disjoint union of cubes of side 1 such that C_1^i is centered at $i \in \mathbb{R}^d$. Hence $|I_N| = N^d$. Given $M > 0$, set $\tilde{Y}_i(\hat{\xi}) = \min\{\hat{\xi}(C_1^i), \frac{M}{2}\}$ and $Y_i = \tilde{Y}_i - \mathbf{E}_{\hat{\mathcal{P}}}(\tilde{Y}_i)$. Note that Y_i is centred, $\mathcal{F}_{C_1^i}$–measurable and $\|Y_i\|_\infty \le M$ for any $i \in I$. We choose M large enough so that $\rho'' := \mathbf{E}_{\hat{\mathcal{P}}}(\tilde{Y}_i) > \rho'$ which is possible because $\rho' < \rho$ and $\lim_{M\uparrow\infty} \mathbf{E}_{\hat{\mathcal{P}}}(\tilde{Y}_i) = \rho$. Now

$$\left\{\hat{\xi}(C_N) \le \rho' N^d\right\} \;\subset\; \left\{\sum_{i\in I_N} \tilde{Y}_i(\hat{\xi}) \le \rho' N^d\right\} \;\subset\; \left\{\Big|\sum_{i\in I_N} Y_i(\hat{\xi})\Big| \ge (\rho'' - \rho') N^d\right\}\;.$$

Hence it is sufficient to show that, for $a > 0$,

$$\lim_{N\uparrow\infty} N^p\, \hat{\mathcal{P}}\left(\Big|\sum_{i\in I_N} Y_i\Big| \ge aN^d \right) \;=\; 0\;. \tag{7.5}$$

By Chebyshev inequality, one has for any even $q\in\mathbb{N}$:

$$\hat{\mathcal{P}}\left(\Big|\sum_{i\in I_N} Y_i\Big| \ge aN^d \right) \;\le\; \frac{1}{a^q\,N^{dq}} \sum_{i_1,\dots,i_q\in I_N} \mathbf{E}_{\hat{\mathcal{P}}}\left(Y_{i_1}\cdots Y_{i_q}\right)\;. \tag{7.6}$$

Let us introduce the notation $\underline{i}=(i_1,\dots,i_q)$ as well as the norm $\|x\|=\max\{|x^{(k)}|\,:\,1\le k\le d\}$, $x\in\mathbb{R}^d$. Moreover, we introduce the integer $r_j(\underline{i})=\min_{k\neq j}\|i_j-i_k\|$. Since the expectation inside the sum on the r.h.s. of (7.6) is invariant under cyclic permutation of the indices $i_1,\dots,i_q$, one may assume that $\underline{i}$ is such that $r_1(\underline{i})\ge r_j(\underline{i})$ for $j=2,\dots,q$. If $r_1(\underline{i})=0$, then use the bound

$$\mathbf{E}_{\hat{\mathcal{P}}}(Y_{i_1}\cdots Y_{i_q}) \;\le\; M^q\;.$$

If $r_1(\underline{i})>0$, let $A(\underline{i})$ be the complement of the cube centred at i_1 with side of length $2r_1(\underline{i})-1$. Then Y_{i_j} is $\mathcal{F}_{A(\underline{i})}$-measurable for any $j=2,\dots,q$. Using conditional expectation, (7.4) and the fact that Y_{i_1} is centred imply that

$$\mathbf{E}_{\hat{\mathcal{P}}}(Y_{i_1}\cdots Y_{i_q}) \;\le\; M^{q-1}\,\mathbf{E}_{\hat{\mathcal{P}}}\left(\left|\mathbf{E}_{\hat{\mathcal{P}}}(Y_{i_1}|\mathcal{F}_{A(\underline{i})})\right|\right) \;\le\; M^q\,h(2r_1(\underline{i})-2)\,(2r_1(\underline{i})-1)^{d-1}\;.$$

It remains to estimate the number $\eta(r)$ of configurations $\underline{i}\in I_N$ such that $r_1(\underline{i})=r$. If $r_1(\underline{i})=0$, then each point i_j appears at least twice. Setting $s=\frac{q}{2}$, the number of these configurations is bounded above by c_1N^{ds} for some c_1 depending on q through a combinatorial factor. In order to deal with arbitrary r, let us regroup the points $i_1,\dots,i_q$ (treated as distinguishable) into clusters according to the relation $x\sim y$ iff $\|x-y\|\le r$. We denote these clusters by $B_1,\dots,B_K$ and set $n_k=|B_k|$. Hence $n_1+\ldots+n_K=q$ and $n_k\ge 2$ (since $r_1(\underline{i})=r$) so that $K\le s$. Moreover, since $\|x-y\|\le n_kr$ for any $x,y\in B_k$, the number of the possible configurations of the cluster B_k under the constraint $|B_k|=n_k$ can by bounded by $c_2N^dr^{d(n_k-1)}$ for a suitable constant c_2 depending on q. Hence we get:

$$\eta(r)\;\le\; c_2\sum_{n_1,\dots,n_K} N^{dK}\prod_{k=1}^K r^{d(n_k-1)} \;\le\; c_2\,N^{ds}\sum_{n_1,\dots,n_K} r^{d(q-K)} \;\le\; c_3\,N^{ds}\,r^{d(q-1)}\;,$$

where c_3 contains a q-dependent combinatorial factor counting the number of possible cluster configurations. Combining the above,

$$\sum_{\underline{i}\in I_N}\mathbf{E}_{\hat{\mathcal{P}}}(Y_{i_1}\cdots Y_{i_q}) \;\le\; N^{ds}\,M^q\left(c_1\;+\;c_4\sum_{r\in\mathbb{N}_+} h(2r-2)r^{dq-1}\right)\;. \tag{7.7}$$

In order to derive (7.5) from (7.6) and (7.7), we need $dq>2p$ for $p\le 8$, hence q needs to be the smallest even integer larger than $16/d$. In particular, $dq\le 2d+16$. For such q, the sum on the r.h.s. is indeed bounded by the Main Hypothesis. □

7.2. **Percolation and domination.** Due to Proposition 9, we may apply the results of Section 6 so that combining with Proposition 5

$$D \;\geq\; \nu([-E_c,E_c])\; e^{-r_c-4\beta E_c}\;\limsup_{N\to\infty}\;\mathbf{E}_{\hat{\mathcal{P}}^c}\left(\frac{8\,N^2}{|\overline{\mathcal{V}}^{\hat{\xi}}_N|}\;G^{\hat{\xi}}_N\right)\;. \tag{7.8}$$

In order to bound the conductance $G^{\hat{\xi}}_N$ for $N \gg r_c$ from below, we will discretize the space $\mathbb{R}^d$ using cubes of appropriate size and spacing. Given $r_2 \geq r_1 > 0$, let us then consider the following functions on $\hat{\mathcal{N}}$:

$$\sigma_j(\hat{\xi}) \;:=\; \chi\big(\hat{\xi}(C_{r_1}+r_2\,j)>0\,\big)\;, \qquad j\in\mathbb{Z}^d\,. \tag{7.9}$$

They form a random field $\Sigma=(\sigma_j)_{j\in\mathbb{Z}^d}$ on the probability space $(\hat{\mathcal{N}},\hat{\mathcal{P}}^c)$. If $\hat{\mathcal{P}}$ is a PPP, the σ_j are independent random variables. For a process with finite range correlations, this independence can also be assured by an adequate choice of r_1 and r_2, but in general the σ_j are correlated. The side length r_1 and spacing r_2 are going to be chosen of order $\mathcal{O}(r_c)$ in such a way that all points of neighbouring cubes have an euclidian distance less than r_c and they are thus connected by an edge of the graph $(\overline{\mathcal{V}}^{\hat{\xi}}_N,\overline{\mathcal{E}}^{\hat{\xi}}_N)$.

Next note that the σ_j take values in $\{0,1\}$. We shall consider the associated site percolation problem with bonds between nearest neighbours only [Gri]. For this purpose, we shall compare Σ with a random field $Z^p=(z^p_j)_{j\in\mathbb{Z}^d}$ of independent and identically distributed random variables with $\mathrm{Prob}(z^p_j=1)=p$ and $\mathrm{Prob}(z^p_j=0)=1-p$. In this independent case, it is well-known that there is a critical probability $p_c(d)\in(0,1)$ such that, if $p>p_c(d)$, there is almost surely a unique infinite cluster, while for $p<p_c(d)$ there is almost surely none [Gri]. We will need somewhat finer estimates for the super-critical regime. A left-right crossing (LR-crossing) with length $k-1$ of C_{2N} of a configuration $(z^p_j)_{j\in\mathbb{Z}^d}$ is a sequence of distinct points $y_1,\ldots,y_k$ in $C_{2N}\cap\mathbb{Z}^d$ such that $|y_i-y_{i+1}|=1$ for $1\leq i<k$, $z^p_{y_i}=1$ for $1\leq i\leq k$, $y^{(1)}_1=-N$, $y^{(1)}_k=N$, $-N<y^{(1)}_i<N$ for $1<i<k$ and finally $y^{(s)}_i=y^{(s)}_j$ for any $s\geq 3$ and for $1\leq i<j\leq k$. Two crossings are called disjoint if all the involved y_j's are distinct. In the same way, one defines disjoint LR-crossings for $(\sigma_j)_{j\in\mathbb{Z}^d}$. Note that this definition of LR-crossings for $d\geq 3$ uses LR-crossings in $2d$-slices only. For the random field Z^p, the techniques of [Gri, Section 2.6 and 11.3] transposed to site percolation imply that, if $p>p_c(2)$, there are positive constants $a=a(p)$ and $b=b(p)$ such that for all $N\in\mathbb{N}_+$

$$\mathrm{Prob}\big(Z^p \text{ has less than } bN^{d-1} \text{ disjoint LR–crossings in } C_{2N}\big)\;\leq\; N^{d-2}e^{-a\,N}\,. \tag{7.10}$$

In order to transpose this result on Z^p to one for Σ, we will use the concept of stochastic dominance [Gri, Section 7.4]. One writes $\Sigma\geq_{\mathrm{st}} Z^p$ whenever

$$\mathbf{E}_{\mathcal{P}^c}(f(\Sigma))\;\geq\;\mathbf{E}_{\mathrm{Prob}}(f(Z^p))\;, \tag{7.11}$$

for any bounded, increasing, measurable function $f:\mathbb{Z}^d\to\mathbb{R}$ (recall that a function is increasing if $f((z_j)_{j\in\mathbb{Z}^d})\geq f((z'_j)_{j\in\mathbb{Z}^d})$ whenever $z_j\geq z'_j$ for all $j\in\mathbb{Z}^d$). As the event on the l.h.s. of (7.10) is decreasing, $\Sigma\geq_{\mathrm{st}} Z^p$ with $p>p_c(2)$ implies that for all $N\in\mathbb{N}_+$

$$\hat{\mathcal{P}}^c\big((\sigma_j)_{j\in\mathbb{Z}^d} \text{ has less than } bN^{d-1} \text{ disjoint LR–crossings in } C_{2N}\big)\;\leq\; N^{d-2}e^{-a\,N}\,. \tag{7.12}$$

Moreover, let us call the configurations $\hat{\xi}$ in the set on the l.h.s. *N-bad*, those in the complementary set *N-good*. For every N–good configuration $\hat{\xi}$, let us fix a set of at least bN^{d-1} disjoint LR–crossings in C_{2N} for $\big(\sigma_j(\hat{\xi})\big)_{j\in\mathbb{Z}^d}$ and denote it $\mathcal{C}_N(\hat{\xi})$. Given an LR–crossing γ in C_{2N}, we write $L(\gamma)$ for its length. In particular, due to Jensen inequality, for any N–good configuration $\hat{\xi}$

$$\sum_{\gamma \in \mathcal{C}_N(\hat{\xi})} \frac{1}{L(\gamma)} \;\geq\; \frac{|\mathcal{C}_N(\hat{\xi})|^2}{\sum_{\gamma \in \mathcal{C}_N(\hat{\xi})} L(\gamma)} \;\geq\; \frac{b^2\, N^{d-2}}{2^d}\,. \tag{7.13}$$

This will allow us to prove a lower bound on (7.8). Hence we need the following criterion for domination.

Lemma 9. $\Sigma \geq_{\rm st} Z^p$ *holds with* $r_1 = r$, $r_2 = 2r$ *if* $\hat{\mathcal{P}}$ *and* $r > 0$ *satisfy either of the following two cases:*

(i) $E_c = 1$ *and*

$$\hat{\mathcal{P}}\big(\,\hat{\xi}(C_r) > 0\,|\,\mathcal{F}_{2r}\,\big) \;\geq\; p\,, \qquad \hat{\mathcal{P}}\text{–}a.s.\,. \tag{7.14}$$

(ii) *There exists* $\rho' > 0$ *such that*

$$r^d\,\nu([-E_c, E_c]) \;\geq\; -\frac{\ln(p/2)}{\rho'}\,, \tag{7.15}$$

and

$$\hat{\mathcal{P}}\Big(\,\hat{\xi}(C_r) < \rho' r^d\,\big|\mathcal{F}_{2\,r}\Big) \;\leq\; 1 \,-\, \frac{3\,p}{2}\,, \qquad \hat{\mathcal{P}}\text{–}a.s.\,\,. \tag{7.16}$$

Proof. The proof is based on the following criterion [Gri, Section 7.4]: if for any finite subset J of $\mathbb{Z}^d$, $i \in \mathbb{Z}^d \setminus J$ and $z_j \in \{0,1\}$ for $j \in J$ satisfying

$$\hat{\mathcal{P}}^c(\sigma_j = z_j \quad \forall j \in J) > 0\,,$$

one has

$$\hat{\mathcal{P}}^c(\sigma_i = 1\,|\,\sigma_j = z_j \;\;\forall\, j \in J) \;\geq\; p\,, \tag{7.17}$$

then $\Sigma \geq_{\rm st} Z^p$.

Hence let J, i, z_j be as above and set $\tilde{\delta}_c := 1 - \delta_c$ and

$$J_0 \;:=\; \{j \in J \;:\; z_j = 0\}\,, \qquad J_1 \;:=\; \{j \in J \;:\; z_j = 1\}\,.$$

Moreover, given $\underline{k} \in \mathbb{N}^{J_0}$ and $\underline{s} \in \mathbb{N}_+^{J_1}$, let

$$W(\underline{k}, \underline{s}) \;:=\; \big\{\,\hat{\xi} \in \hat{\mathcal{N}} \;:\; \hat{\xi}(C_r + 2rj) = k_j \quad \forall j \in J_0,\;\; \hat{\xi}(C_r + 2rj) = s_j \quad \forall j \in J_1\,\big\}\,.$$

Then

$$\begin{aligned}
&\hat{\mathcal{P}}^c(\sigma_i = 0\,|\,\sigma_j = z_j \quad \forall j \in J) \\
&\quad= \frac{\sum_{\underline{k} \in \mathbb{N}^{J_0}} \sum_{\underline{s} \in \mathbb{N}_+^{J_1}} \sum_{n \in \mathbb{N}} \hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) = n\,,\, W(\underline{k}, \underline{s})\big)\, \tilde{\delta}_c^{n} \prod_{j \in J_0} \tilde{\delta}_c^{k_j} \prod_{j \in J_1} (1 - \tilde{\delta}_c^{s_j})}{\sum_{\underline{k} \in \mathbb{N}^{J_0}} \sum_{\underline{s} \in \mathbb{N}_+^{J_1}} \hat{\mathcal{P}}\big(W(\underline{k}, \underline{s})\big) \prod_{j \in J_0} \tilde{\delta}_c^{k_j} \prod_{j \in J_1} (1 - \tilde{\delta}_c^{s_j})}\,.
\end{aligned}$$

Within this, we can, moreover, replace

$$\hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) = n\,,\, W(\underline{k}, \underline{s})\big) \;=\; \hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) = n\,|\, W(\underline{k}, \underline{s})\big)\; \hat{\mathcal{P}}\big(W(\underline{k}, \underline{s})\big)\,.$$

Finally, note that $W(\underline{k},\underline{s}) \in \mathcal{F}_A$ where $A = \mathbb{R}^d \setminus \big(C_{2r} + 2ri\big)$.

(i) If $E_c = 1$, then $\delta_c = 1$ and

$$\sum_{n\in\mathbb{N}} \hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) = n \,|\, W(\underline{k},\underline{s})\big)\tilde{\delta}_c^n \;=\; \hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) = 0 \,|\, W(\underline{k},\underline{s})\big) \;\leq\; 1 - p\;,$$

where the inequality follows from (7.14) and the stationarity of $\hat{\mathcal{P}}$. Due to the above computations, this inequality implies (7.17).

(ii) As $\tilde{\delta}_c \leq e^{-\delta_c}$, we obtain the following bound

$$\sum_{n\in\mathbb{N}} \hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) = n \,|\, W(\underline{k},\underline{s})\big)\tilde{\delta}_c^n \;\leq\; \hat{\mathcal{P}}\big(\hat{\xi}(C_r + 2ri) < \rho' r^d \,|\, W(\underline{k},\underline{s})\big) \;+\; e^{-\delta_c \rho' r^d}\;.$$

Due the stationarity of $\hat{\mathcal{P}}$, (7.15) and (7.16) imply (7.17). □

Proof of Theorem 2. We fix $p > p_c(2)$ and verify that (7.14) holds if r is large enough. This is trivial for a process with a uniform lower bound (3.12) on the point density. For a mixing point process satisfying (3.13), one has

$$\hat{\mathcal{P}}\Big(\hat{\xi}(C_r) < \rho' r^d \,\big|\mathcal{F}_{2r}\Big) \;\leq\; \hat{\mathcal{P}}\Big(\hat{\xi}(C_r) < \rho' r^d\Big) \;+\; r^d\,(2r)^{d-1}\,h(r)\;, \qquad \hat{\mathcal{P}} - \text{a.s.}.$$

Due to the hypothesis on h, the second term converges to 0 in the limit $r \uparrow \infty$. If $\rho' < \rho$, the first one can be bounded by the Chebychev inequality:

$$\hat{\mathcal{P}}(\hat{\xi}(C_r) \leq \rho'\, r^d) \;\leq\; \mathbf{E}_{\hat{\mathcal{P}}}\left(\left|\frac{\hat{\xi}(C_r)}{\ell(C_r)} - \rho\right| > \rho - \rho'\right) \;\leq\; \frac{1}{\rho - \rho'}\int \hat{\mathcal{P}}(d\xi)\,\left|\frac{\hat{\xi}(C_r)}{\ell(C_r)} - \rho\right|\;.$$

By Lemma 6, the expression on the r.h.s. can be made arbitrarily small by choosing r sufficiently large, thus implying that (7.14) is satisfied for r sufficiently large. In conclusion, due to Lemma 9, (7.12) holds for r large enough. We fix such a value r satisfying (7.12) and call it r_p.

Consider the variables $(\sigma_j)_{j\in\mathbb{Z}^d}$ defined for $r_1 = r_p, r_2 = 2r_p$ and choose $r_c = (d+8)^{\frac{1}{2}} r_p$. This assures that, if neighbouring sites j and j' in $\mathbb{Z}^d$ have $\sigma_j(\hat{\xi}) = \sigma_{j'}(\hat{\xi}) = 1$, then $C_{r_p} + 2jr_p$ and $C_{r_p} + 2j'r_p$ contain each a point and these points are separated by a distance less than r_c. Two neighbouring sites j and j' in $\mathbb{Z}^d$ such that $\sigma_j(\hat{\xi}) = \sigma_{j'}(\hat{\xi}) = 1$ define a bond of the site percolation problem. To such a bond one can associate (at least) two points $x \in \operatorname{supp}\hat{\xi} \cap (C_{r_p} + 2jr_p)$ and $y \in \operatorname{supp}\hat{\xi} \cap (C_{r_p} + 2j'r_p)$ separated by a distance less than r_c. Given N integer, we define

$$\hat{N} := \max\{n \in \mathbb{N} \,:\, C_{r_p} + 2r_p j \subset C_{2[r_p N]},\;\; \forall j \in C_{2n} \cap \mathbb{Z}^d\}\;.$$

Note that $\hat{N} = \mathcal{O}(N)$. If $j, j' \in C_{2\hat{N}} \cap \mathbb{Z}^d$, then the above associated points x and y are linked by an edge of the graph $(\overline{\mathcal{V}}^{\hat{\xi}}_{[r_pN]}, \overline{\mathcal{E}}^{\hat{\xi}}_{[r_pN]})$ defined in section 6.1. Each LR-crossing of $C_{2\hat{N}}$ for the site percolation problem gives in a natural way a connected path of edges of the graph $(\overline{\mathcal{V}}^{\hat{\xi}}_{[r_pN]}, \overline{\mathcal{E}}^{\hat{\xi}}_{[r_pN]})$ which connects the boundary faces $\Gamma^{\pm}_N$.

For a $\hat{N}$–good configuration $\hat{\xi}$, we now bound the conductance $G^{\hat{\xi}}_{[r_pN]}$ from below. For this purpose, let us consider the random resistor network with vertices $Q^{\hat{\xi}}_{[r_pN]} \cup \{\hat{\Gamma}^+_N, \hat{\Gamma}^-_N\}$ where unit conductances are put on all edges in $\overline{\mathcal{E}}^{\hat{\xi}}_{[r_pN]}$ with vertices in $Q^{\hat{\xi}}_{[r_pN]}$ as well as between the two added boundary points $\hat{\Gamma}_N^{\pm}$ and all points of $B_{[r_pN]}^{\hat{\xi}\pm}$. This new network is obtained from the one of Section 6.1 upon placing superconducting wires between all vertices of $\Gamma_{[r_pN]}^{+}$ and $\Gamma_{[r_pN]}^{-}$ so that they can be identified with a single point $\hat{\Gamma}_N^{+}$ and $\hat{\Gamma}_N^{-}$. The conductance $g_N^{\hat{\xi}}$ of this new network (defined as the current flowing from $\hat{\Gamma}_N^{-}$ to $\hat{\Gamma}_N^{+}$ when a unit potential difference is imposed between these two points) is precisely equal to $G_{[r_pN]}^{\hat{\xi}}$ because all points of $\Gamma_{[r_pN]}^{\pm}$ have the same potential (0 or 1 respectively) and each has links to all points of $B_{[r_pN]}^{\hat{\xi}\pm}$ with equal conductances summing up to 1.

In order to bound $g_N^{\hat{\xi}}$ from below, we now invoke Rayleigh's monotonicity law which states that eliminating links (*i.e.* conductances) from the network always lowers its conductance. For a given $\hat{N}$-good configuration $\hat{\xi}$, we cut all links but those belonging to the family of disjoint paths associated to $\mathcal{C}_{\hat{N}}(\hat{\xi})$. Each of these paths γ connecting $\hat{\Gamma}_N^{+}$ and $\hat{\Gamma}_N^{-}$ has a conductance bounded below by $1/L(\gamma)$. As all the paths of $\mathcal{C}_{\hat{N}}(\hat{\xi})$ are disjoint and they are connecting $\hat{\Gamma}_N^{+}$ and $\hat{\Gamma}_N^{-}$ in parallel, $g_N^{\hat{\xi}}$ is the sum of the conductances of all paths and it follows from (7.13) that $g_N^{\hat{\xi}} \geq c(b)N^{d-2}$ for some positive constant $c(b)$ depending on b. We therefore deduce that

$$\mathbf{E}_{\hat{\mathcal{P}}}\left(\frac{[r_pN]^2}{|\overline{\mathcal{V}}_{[r_pN]}^{\hat{\xi}}|}G_{[r_pN]}^{\hat{\xi}}\right) \;\geq\; c(b)\,\mathbf{E}_{\hat{\mathcal{P}}}\left(\frac{[r_pN]^2}{|\overline{\mathcal{V}}_{[r_pN]}^{\hat{\xi}}|}\,N^{d-2}\chi(\hat{\xi}\text{ is }\;\hat{N}\text{–good })\right).$$

Due to (7.12) and Lemma 6 the r.h.s. converges to a positive value. □

Proof of Theorem 3. This is a variation of the above proof, using the second criterion in Lemma 9. In what follows, we fix $p > p_c(2)$ and $\rho' < \rho$. Then, given E_c, we choose r_c such that (7.15) is satisfied, *i.e.* $r_c = c(E_c^{\alpha+1})^{-1/d}$ for some constant c. As we will have $r_c \uparrow \infty$ in the limit of low temperature, the condition (7.16) also holds by the same argument as in the proof of Theorem 2. Following further the argument of the last proof,

$$D \;\geq\; C\;\nu([-E_c,E_c])\;e^{-r_c-4\beta E_c} \;\geq\; C'\,E_c^{1+\alpha}\,\exp(-cE_c^{-\frac{\alpha+1}{d}}-4\beta E_c)\,,$$

where C and C' are positive constants. Optimising the exponent leads to $E_c = c'\,\beta^{-\frac{d}{\alpha+1+d}}$ which completes the proof. □

Appendix A. Proof of Lemma 1

Note that the statements (ii) and (iii) of Lemma 1 are proved in [FKAS, Corollary 1.2.11 and Theorem 1.3.9] in dimension $d = 1$. The proof below is valid for any dimension d.

Proof of Lemma 1. (i) Let $h(\xi,\xi') := k(\xi,\xi') - k(\xi',\xi)$. By the definition (2.4) of the Palm distribution $\mathcal{P}_0$, $\forall N > 0$, $\forall A \in \mathcal{B}(\mathbb{R}^d)$ and for any non negative measurable function f

$$\int \mathcal{P}_0(d\xi)\int_A \hat{\xi}(dx)f(\xi,S_x\xi) = \frac{1}{\rho N^d}\int \mathcal{P}(d\xi)\int_{C_N}\hat{\xi}(dy)\int_{S_{-y}A}\hat{\xi}(dx)f(S_y\xi,S_x\xi)\,. \tag{A.1}$$

The antisymmetry of $h(\xi,\xi')$ and the identity above imply

$$\int \mathcal{P}_0(d\xi)\int_{\mathbb{R}^d}\hat{\xi}(dx)\,h(\xi,S_x\xi) = \frac{1}{\rho N^d}\int \mathcal{P}(d\xi)\int_{C_N}\hat{\xi}(dy)\int_{\mathbb{R}^d\setminus C_N}\hat{\xi}(dx)\,h(S_y\xi,S_x\xi)\,. \tag{A.2}$$

Let us split the last integral into two integrals over $\mathbb{R}^d \setminus C_{N+\sqrt{N}}$ and over $C_{N+\sqrt{N}} \setminus C_N$. Using (A.1) again,

$$\frac{1}{\rho N^d}\left|\int \mathcal{P}(d\xi)\int_{C_N}\hat{\xi}(dy)\int_{\mathbb{R}^d\setminus C_{N+\sqrt{N}}}\hat{\xi}(dx)h(S_y\xi,S_x\xi)\right|$$
$$\leq \int \mathcal{P}_0(d\xi)\int_{\mathbb{R}^d\setminus C_{\sqrt{N}}}\hat{\xi}(dx)\big(|k(\xi,S_x\xi)|+|k(S_x\xi,\xi)|\big)\ ,$$

which converges to zero as $N\to\infty$ by the dominated convergence theorem. The same holds for

$$\frac{1}{\rho N^d}\left|\int \mathcal{P}(d\xi)\int_{C_N}\hat{\xi}(dy)\int_{C_{N+\sqrt{N}}\setminus C_N}\hat{\xi}(dx)\,h(S_y\xi,S_x\xi)\right|\ ,$$

since, due to (A.1), it can be bounded by

$$\frac{1}{\rho N^d}\int \mathcal{P}(d\xi)\int_{C_{N+\sqrt{N}}\setminus C_N}\hat{\xi}(dx)\int_{\mathbb{R}^d}\hat{\xi}(dy)\big(|k(S_y\xi,S_x\xi)|+|k(S_x\xi,S_y\xi)|\big)$$
$$=\frac{(N+\sqrt{N})^d-N^d}{N^d}\int \mathcal{P}_0(d\xi)\int_{\mathbb{R}^d}\hat{\xi}(dy)\big(|k(S_y\xi,\xi)|+|k(\xi,S_y\xi)|\big)\ .$$

Letting $N\to\infty$ in (A.2) leads to the result.

(ii) Since $\Gamma\in\mathcal{B}(\mathcal{N})$ is translation invariant,

$$\chi_{\Gamma_0}(S_x\xi)=\chi_\Gamma(\xi)\ ,\qquad \forall\,\xi\in\mathcal{N},\ \forall\,x\in\hat{\xi}\ .$$

The above remark together with (2.4) gives

$$\mathcal{P}_0(\Gamma_0)\ =\ \frac{1}{\rho}\int\mathcal{P}(d\xi)\int_{C_1}\hat{\xi}(dx)\chi_{\Gamma_0}(S_x\xi)\ =\ \frac{1}{\rho}\int_\Gamma\mathcal{P}(d\xi)\,\hat{\xi}(C_1)\ .$$

Comparing with (2.2), this yields $\mathcal{P}_0(\Gamma_0)=1$ if $\mathcal{P}(\Gamma)=1$. Reciprocally, always due to (2.2), if $\mathcal{P}_0(\Gamma_0)=1$, one gets $\hat{\xi}(C_1)=0$ for $\mathcal{P}$–almost all $\xi\in\mathcal{N}\setminus\Gamma$, and by translation invariance $\xi=0$ for $\mathcal{P}$–almost all $\xi\in\mathcal{N}\setminus\Gamma$, thus implying that $\mathcal{P}(\Gamma)=1$.

(iii) Let us suppose that $\mathcal{P}_0(A)=\mathcal{P}_0(B)>0$. and set $\Gamma:=\bigcup_{x\in\mathbb{R}^d}S_xB$. This is a translation-invariant Borel subset of $\mathcal{N}$ (see Lemma 10) and $B\subset\Gamma\cap\mathcal{N}_0\subset A$. In particular, $\mathcal{P}(\Gamma)\in\{0,1\}$ by the ergodicity of $\mathcal{P}$. Since

$$\chi_B(S_y\xi)\leq\chi_\Gamma(\xi)\ ,\qquad \forall\,\xi\in\mathcal{N},\ \forall\,y\in\mathbb{R}^d\ ,$$

if follows from (2.4) that

$$\mathcal{P}_0(B)\ =\ \frac{1}{\rho}\int_\mathcal{N}\mathcal{P}(d\xi)\int_{C_1}\hat{\xi}(dy)\,\chi_B(S_y\xi)\ \leq\ \frac{1}{\rho}\int_\Gamma\mathcal{P}(\xi)\hat{\xi}\,(C_1)\ .$$

Therefore, $\mathcal{P}(\Gamma)=0$ would imply that $\mathcal{P}_0(B)=0$, in contradiction with our assumption. Thus $\mathcal{P}(\Gamma)=1$. But $\Gamma\cap\mathcal{N}_0\subset A$, therefore the statement follows from (ii).

(iv) The thesis follows by observing that (2.4) implies

$$\mathbf{E}_{\mathcal{P}_0}\Big(\prod_{j=1}^k\hat{\xi}(A_j)\Big)\ =\ \frac{1}{\rho}\int_\mathcal{N}\mathcal{P}(d\xi)\int_{C_1}\hat{\xi}(dx)\prod_{j=1}^k\hat{\xi}(A_j+x)\ \leq\ \frac{1}{\rho}\int_\mathcal{N}\mathcal{P}(d\xi)\hat{\xi}(C_1)\prod_{j=1}^k\hat{\xi}(\tilde{A}_j)$$

and by applying the estimate $a_1\cdots a_{k+1}\leq c(k+1)\,(a_1^{k+1}+\cdots+a_{k+1}^{k+1})$, $a_1,\ldots,a_{k+1}\geq 0$. □

Lemma 10. *Let $A\in\mathcal{B}(\mathcal{N}_0)$. Then $\bigcup_{x\in\mathbb{R}^d}S_xA\in\mathcal{B}(\mathcal{N})$.*

Proof. Let us introduce the following lexicographic ordering on $\mathbb{R}^d$: $x\prec y$ if and only if either $|x|<|y|$ or $|x|=|y|$ and there is k, $1\leq k\leq d$, such that $x^{(k)}<y^{(k)}$ and $x^{(l)}=y^{(l)}$ for $l<k$

(here $x^{(k)}$ is the k-th component of the vector x). Given $\hat{\xi} \in \hat{\mathcal{N}}$, one can then order the support of $\hat{\xi}$ according to $\prec$:

$$\operatorname{supp}(\hat{\xi}) \;=\; \begin{cases} \{y_1(\hat{\xi}), y_2(\hat{\xi}), \dots, y_N(\hat{\xi})\} & \text{if } N := \hat{\xi}(\mathbb{R}^d) < \infty \ , \\ \{y_j(\hat{\xi})\}_{j\in\mathbb{N}_+} & \text{otherwise} \ , \end{cases}$$

where $y_j \prec y_k$ whenever $j < k$. For any $n \in \mathbb{N}$, let $x_n : \hat{\mathcal{N}} \to \mathbb{R}^d$ then be defined as

$$x_n(\hat{\xi}) \;=\; \begin{cases} y_n(\hat{\xi}) & \text{if } n \le \hat{\xi}(\mathbb{R}^d) \ , \\ y_N(\hat{\xi}) & \text{if } n > N := \hat{\xi}(\mathbb{R}^d) \ . \end{cases}$$

Using an adequate family of finite disjoint covers of $\mathbb{R}^d$ and the fact that $\hat{\xi} \in \hat{\mathcal{N}} \mapsto \hat{\xi}(B)$ is a Borel function for every Borel set $B \subset \mathbb{R}^d$, one can verify that x_n is a Borel function for each n. Moreover, $\operatorname{supp}(\hat{\xi}) = \{x_n(\hat{\xi}) \,:\, n \in \mathbb{N}\}$ for all $\hat{\xi} \in \hat{\mathcal{N}}$.

Due to the definition of the Borel sets in $\mathcal{N}$ and $\hat{\mathcal{N}}$, the map $\pi : \mathcal{N} \to \hat{\mathcal{N}}$ given by $\pi(\xi) = \hat{\xi}$ is Borel, and by [MKM, Section 6.1] the function $F : \mathbb{R}^d \times \mathcal{N} \to \mathcal{N}$ given by $F(x,\xi) = S_x\xi$ is even continuous. Hence we conclude that

$$H_n : \mathcal{N} \to \mathcal{N}_0 \ , \qquad H_n(\xi) \;:=\; F\big(x_n(\hat{\xi}), \xi\big) \;=\; S_{x_n(\hat{\xi})}\xi \ ,$$

is a Borel function. Its restriction $\hat{H}_n : \mathcal{N}_0 \to \mathcal{N}_0$ is then also a Borel function. Now given a Borel subset A of $\mathcal{N}_0$, we conclude that $\Phi(A) := \bigcup_{n=1}^{\infty} \hat{H}_n^{-1}(A)$ is a Borel subset in $\mathcal{N}_0$. One can check that

$$\Phi(A) \;=\; \{\xi \,:\, \xi \;=\; S_x\eta \ \text{ for some } \eta \in A \text{ and } x \in \hat{\eta}\,\}.$$

Since $\mathcal{N}_0$ is a Borel subset of $\mathcal{N}$, it follows that $\Phi(A)$ is a Borel subset of $\mathcal{N}$ as is $H_1^{-1}\big(\Phi(A)\big)$ since H_1 is a Borel function. The identity

$$H_1^{-1}\big(\Phi(A)\big) \;=\; \bigcup_{x\in\mathbb{R}^d} S_x A \ ,$$

now completes the proof. □

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