Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Large scale localization of a spatial version of Neveu's branching process

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submitted: 28. Juli 2004

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No. 951
Berlin 2004

2000 *Mathematics Subject Classification.* 60J80, 60G57.

Key words and phrases. Neveu's continuous-state branching, infinite mean branching superprocess, large scale concentration in one point, log-Laplace product formula, small epsilon asymptotics.

*) Supported by the research program "Interacting Systems of High Complexity" of the German Science Foundation
†) Supported in part by the RFBR grants 02-01-01252, 02-01-00358.

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Abstract. Recently a spatial versions of Neveu's (1992) continuous-state branching process was constructed by Fleischmann and Sturm (2004). This superprocess with infinite mean branching behaves quite differently from usual supercritical spatial branching processes. In fact, at macroscopic scales, the mass renormalized to a (random) probability measure is concentrated in a single space point which randomly fluctuates according to the underlying symmetric stable motion process.

Contents

1. Introduction and statement of results

1.1. **Motivation.** Superprocesses (spatial measure-valued branching processes) are constructed and studied usually under the assumption of finite moments, at least of order one. Recently, Fleischmann and Sturm [FS04] constructed a super–α–stable motion X in $\mathbb{R}^d$ $(0 < \alpha \leq 2$, super-Brownian if $\alpha = 2)$ with a branching mechanism of infinite mean. This process has partly strange properties compared with the ones of usual superprocesses. For instance, also in the case of a Brownian migration (i.e. if $\alpha = 2$), mass propagates instantaneously in space, that is, it is present everywhere in space at fixed times ([FS04, Proposition 15]). Or, for all α and in *all* dimensions, X_t is absolutely continuous at fixed times t (Fleischmann and Mytnik [FM04]).

If one drops the space coordinate in the model, that is, if one passes to the total mass process $t \mapsto \bar{X}_t := X_t(\mathbb{R}^d)$, one gets a continuous-state branching process with branching mechanism $u \mapsto \varrho\, u \log u$, with $\varrho > 0$ a fixed constant. This process was introduced by Neveu (1992) in the preprint [Nev92], and further studied by Bertoin and Le Gall [BLG00]. Neveu indicated that for every fixed (deterministic) initial state $\bar{X}_0 > 0$ there exists an exponentially distributed random variable V with mean $1/\bar{X}_0$, so that

$$e^{-\varrho t} \log \bar{X}_t \underset{t\uparrow\infty}{\longrightarrow} -\log V \quad \text{a.s.,} \tag{1}$$

see [FS04, Appendix] for a detailed proof. (Similar Galton-Watson results occurred earlier, for instance, in Grey [Gre77].)

Coming back to the spatial generalization X of $\bar{X}$, it was not understood so far how the total mass $\bar{X}_t$ spreads out macroscopically in space as $t \uparrow \infty$. Clearly, for supercritical super–α–stable motions of finite mean one expects that after a spatial α–rescaling the total mass normalized by its mean gets a profile described by the α–stable density function. See, for instance, Watanabe [Wat67], Fleischmann [Fle79], and Biggins [Big92] (a more detailed discussion follows after Theorem 1 below). But it was not at all clear, whether under the much stronger production of an infinite mean branching certain spatial "intermittency" effects occur. Recall that $\bar{X}_t$ has a stable distribution where its index $e^{-\varrho t}$ converges to 0 as $t \uparrow \infty$. In particular, $\bar{X}_t$ cannot be normalized by its mean. The present paper was motivated by this open problem concerning the large scale behavior of X.

1.2. **Preliminaries: notation.** Before we will describe the model in more detail, we need to introduce some notation. The class of Borel subsets of $\mathbb{R}^d$ is denoted by $\mathcal{B}$, the ring of all bounded sets in $\mathcal{B}$ by $\mathrm{b}\mathcal{B}$, and of all Lebesgue continuity sets in $\mathrm{b}\mathcal{B}$ by $\mathrm{b}\mathcal{B}_\ell$, that is, $B \in \mathrm{b}\mathcal{B}$ belongs to $\mathrm{b}\mathcal{B}_\ell$ if and only if with respect to the Lebesgue measure ℓ on $\mathbb{R}^d$ we have $\ell(\partial B) = 0$. The distance between $x \in \mathbb{R}^d$ and $B \in \mathcal{B}$ is denoted by $|x - B|$. Let 1_B stand for the indicator function of a set B, and B^{c} for the complement of B.

We denote by $\mathcal{C}_{\mathrm{l}} = \mathcal{C}_{\mathrm{l}}(\mathbb{R}^d)$ the class of continuous functions φ on $\mathbb{R}^d$ which posses a finite limit as $|x| \uparrow \infty$. We write $\varphi \in \mathcal{C}_{\mathrm{l}}^{(2)} = \mathcal{C}_{\mathrm{l}}^{(2)}(\mathbb{R}^d)$, if $\varphi \in \mathcal{C}_{\mathrm{l}}$ has derivatives up to order 2 which belong to $\mathcal{C}_{\mathrm{l}}$. Additional superscripts "+" and "++" indicate the subspaces of all non-negative functions and all functions which have a positive infimum, respectively. The supremum norm is denoted by $\|\cdot\|_\infty$.

If E denotes a Polish space, write $\mathcal{D}(\mathbb{R}_+, E)$ for the Skorohod space of all E–valued càdlàg paths.

For $0<\alpha\le 2$, let S^α denote the semigroup associated with the d–dimensional fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, that is,

$$(2)\qquad S_t^\alpha\varphi(x) \ :=\ \int_{\mathbb{R}^d}\mathrm{p}_t^\alpha(y-x)\varphi(y)\,dy,\qquad t>0,\quad \varphi\in\mathcal{C}_1^{++},$$

where p^α is the continuous transition density function of the related symmetric α–stable motion $\xi=\{\xi_t:\,t\ge 0\}$ in $\mathbb{R}^d$ with càdlàg paths.

Write $\mathcal{M}_\mathrm{f}$ for the cone of all finite measures on $\mathbb{R}^d$, equipped it with the topology of weak convergence. The integral of a function φ with respect to a measure $\mu\in\mathcal{M}_\mathrm{f}$ is written as $\mu(\varphi)$. We set $\hat{\mu}:=\mu/\mu(1)$ for the normalized measure of $\mu\in\mathcal{M}_\mathrm{f}\backslash\{0\}$.

As usual, we write $f_t\sim g_t$ as $t\uparrow\infty$, if $f_t/g_t\to 1$ as $t\uparrow\infty$. Equality in law is denoted by $\overset{\mathcal{L}}{=}$, and convergence in law by $\overset{\mathcal{L}}{\Rightarrow}$.

1.3. **Super–α–stable motion X with Neveu's branching mechanism.** The *super–α–stable motion X with Neveu's branching mechanism* is a (time-homogeneous) Markov process with paths in $\mathcal{D}(\mathbb{R}_+,\mathcal{M}_\mathrm{f})$, described via its Laplace transition functional

$$(3)\qquad \mathbf{E}\Bigl\{e^{-X_t(\varphi)}\ \Big|\ X_0=\mu\Bigr\}\ =\ e^{-\mu\left(u_t[\varphi]\right)},\quad t\ge 0,\quad \varphi\in\mathcal{C}_1^{++},\quad \mu\in\mathcal{M}_\mathrm{f}\,,$$

where $u=u[\varphi]$ is the unique mild solution to the function-valued Cauchy problem

$$(4)\qquad \frac{d}{dt}u_t\ =\ \Delta_\alpha u_t-\varrho\,u_t\log u_t\quad\text{on}\quad(0,\infty)\quad\text{with}\quad u_{0+}=\varphi$$

(see [FS04, Theorems 1 and 2]).

1.4. **Large scale localization.** Recall that the "highly supercritical" process X has infinite expectation. So what method can be used to attack the open problem of large scale behavior in space?

The most general method to obtain limit theorems for "classical" supercritical, i.e. non-spatial supercritical branching processes, was proposed by Seneta [Sen68]. Let Z be a supercritical (discrete time) Galton-Watson process and f it's offspring generating function. This function has an inverse g, whose n-th iterate we shall denote by g_n. Clearly, for every $s\in[q,1]$, where q is the extinction probability of Z, the sequence $x_n(s):=\bigl(g_n(s)\bigr)^{Z_n}$, $n\ge 1$, is a non-negative martingale and, consequently, $x_\infty(s):=\lim_{n\to\infty}x_n(s)$ exists a.s. This property of the inverse of the generating function (or of a Laplace transform in more general "classical" situations) was also used in [Gre77] and [Nev92]. But in the present spatial case the described method fails. In fact, to get a martingale analogously to that used to prove (1), one would need to solve the log-Laplace equation (4) backwards, which in particular would require strong additional conditions on φ, which are not at all obvious.

The next observation is that a direct log-Laplace transform approach leads to difficult questions concerning the asymptotic behavior as $t\uparrow\infty$ of the solution $u_t[\varphi]$ of (4).

Let us try to consider the *randomly* normalized measures $\hat{X}_t=X_t/\bar{X}_t=X_t/X_t(1)$. Clearly, they will reflect the spatial structure of X_t as well. More precisely, for $k>0$ we will introduce the following *rescaled processes* $\hat{X}^{(k)}$:

$$(5)\qquad \hat{X}_t^{(k)}(B)\ :=\ \hat{X}_{kt}(k^{1/\alpha}B),\qquad t\ge 0,\quad B\in\mathcal{B}.$$

The following localization theorem is our *main result.* Recall that

Theorem 1 (**Large scale localization**). *Fix* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}} \backslash \{0\}$. *Let the (symmetric)* α*–stable motion* ξ *in* $\mathbb{R}^d$ *start from the origin.*

(a) (F.d.d. convergence): *For each finite collection of time points* $0 \le t_1 < \cdots < t_n$,

$$\big(\hat{X}^{(k)}_{t_1}, \ldots, \hat{X}^{(k)}_{t_n}\big) \underset{k\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \big(\delta_{\xi_{t_1}}, \ldots, \delta_{\xi_{t_n}}\big).$$

(b) (Convergence on path space): *If additionally* $\alpha = 2$, *then, in law on* $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{\mathrm{f}})$,

$$\hat{X}^{(k)} \underset{k\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \delta_\xi \,.$$

Consequently, if X is normalized by its total masses, speeded up time by a factor k, and contracted in space by $k^{1/\alpha}$, then the whole mass will finally be concentrated in a single *random* point, which fluctuates in macroscopic time according to the α–stable process ξ. In particular,

$$\hat{X}_t(t^{1/\alpha}B) \underset{t\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \delta_{\xi_1}(B), \qquad B \in \mathrm{b}\mathcal{B}_\ell \,. \tag{6}$$

Note that such limit behavior is not at all typical for supercritical spatial branching processes.

For example, Watanabe [Wat67] has shown the following local limit theorem for a supercritical branching Brownian motion Y in $\mathbb{R}^d$ with finite variance and starting from $Y_0 = \delta_0$:

$$e^{-at}\, t^{d/2}\, Y_t(B) \underset{t\uparrow\infty}{\longrightarrow} (2\pi)^{d/2}\, \ell(B)\, W, \quad \text{a.s.}, \quad B \in \mathrm{b}\mathcal{B}_\ell \,, \tag{7}$$

where a is the Malthusian parameter of the corresponding total mass process $\bar{Y}$ (non-spatial branching process), and

$$e^{-at}\, \bar{Y}_t \underset{t\uparrow\infty}{\longrightarrow} W, \quad \text{a.s.} \tag{8}$$

For supercritical spatially homogeneous branching particle systems Y in $\mathbb{R}^d$ in discrete time, with second moment assumptions, and starting from a homogenous Poisson point field, Fleischmann [Fle79] has derived a law of large numbers and a central limit theorem. This is based on the following global limit theorem for the process starting from a single ancestor:

$$e^{-at}\, Y_t(t^{1/2}B) \underset{t\uparrow\infty}{\longrightarrow} \Phi(B)\, W, \quad \text{a.s.}, \quad B \in \mathrm{b}\mathcal{B}_\ell \,, \tag{9}$$

where Φ is the standard Gaussian measure on $\mathbb{R}^d$.

Biggins [Big92] has proven a variant of (7) for supercritical branching random walks in discrete time under less restrictive conditions. From his result immediately a relation as (9) follows. Using Biggins' method one can verify that statements as in (7) and (9) are true for supercritical $(2, d, \beta)$–superprocesses Y (that is, measure-valued branching processes in $\mathbb{R}^d$ with Brownian migration and continuous-state branching of index $1+\beta$). From (8) and (9) we conclude that

$$\hat{Y}_t(t^{1/2}B) \underset{t\uparrow\infty}{\longrightarrow} \Phi(B), \quad \text{a.s.}, \quad B \in \mathrm{b}\mathcal{B}_\ell \,, \tag{10}$$

on the set of non-extinction. That is, the long-term limit on the set of non-extinction of the normalized $(2, d, \beta)$–superprocess is the deterministic Gaussian measure Φ.

However, for our X process the corresponding limit variable is the *random* δ–measure δ_{ξ_1} where ξ_1 is distributed according to Φ (in the present case $\alpha = 2$).

Remark 2 (**Open problem: tightness for** $\alpha < 2$)**.** The restriction to the Brownian case $\alpha = 2$ for the convergence on path space [Theorem 1(b)] comes from the fact that our tightness proof for one-dimensional marginals *fails* in the $\alpha < 2$ case (see Section 4.4). Since also our attempts failed to show tightness by using other tools, convergence on path space remains open in the non-Brownian case. ◇

1.5. **Approach.** Next we want to explain a bit our approach to the proof of Theorem 1. An essential tool will be some moment calculations. Clearly, the normalized processes $\hat{X}^{(k)}$ have moments of all orders. But how can they be computed? Surprisingly, they satisfy relatively simple formulas. We will state them for only the first two moments, although our method of proof actually allows to establish all of them. The following result is the key of our approach to the large scale behavior of X.

Proposition 3 (**First two moments**)**.** *Fix* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}\backslash\{0\}$. *Then, for* $0 < t_1 \leq t_2$ *and* $\varphi_1, \varphi_2 \in \mathcal{C}_1^+$,

$$\mathbf{E}\hat{X}_{t_1}(\varphi_1) \ = \ \hat{\mu}(S^{\alpha}_{t_1}\varphi_1) \tag{11}$$

and

$$\begin{aligned}\mathbf{E}\big(\hat{X}_{t_1}(\varphi_1)\,\hat{X}_{t_2}(\varphi_2)\big) \\ = \int_0^{t_1} \varrho\, e^{-\varrho s}\,\hat{\mu}\big(S^{\alpha}_{t_1-s}(S^{\alpha}_{s}\varphi_1\, S^{\alpha}_{s+t_2-t_1}\varphi_2)\big)\,ds \,+\, e^{-\varrho t_1}\,\hat{\mu}(S^{\alpha}_{t_1}\varphi_1)\,\hat{\mu}(S^{\alpha}_{t_2}\varphi_2).\end{aligned} \tag{12}$$

Remark 4 (**Moments involving indicator functions**)**.** Moment formulae (11) and (12) remain valid for functions $\varphi_i = 1_{B_i}$, $B_i \in \mathrm{b}\mathcal{B}_\ell$, $i = 1,2$. In fact, to each compact (or open bounded) $B \in \mathcal{B}$, there are compactly supported functions $\varphi^n \in \mathcal{C}_1^+$ such that $1 \geq \varphi^n \downarrow 1_B$ (or $0 \leq \varphi^n \uparrow 1_B$, respectively) as $n \uparrow \infty$ (see, for instance, Kallenberg [Kal97, A6.1]). ◇

Remark 5 (**Fleming-Viot super-Brownian motion**)**.** Note that in the case $\alpha = 2$ the moment formulas of Proposition 3 coincide with those of the Fleming-Viot super-Brownian motion, see, for instance, [Eth00, Proposition 2.27], although the processes are essentially different. (Recall the instantaneous propagation of mass instead of the compact support property, and the absolute continuity of states instead of singularity in dimensions $d \geq 2$.) Note also that for the Fleming-Viot super-Brownian motion one has also a large scale localization property as our Theorem 1, see Dawson and Hochberg [DH82, Theorem 8.1]. The mentioned coexistence of moment formulas suggests now to use our method of proof of Theorem 1 to get the corresponding Fleming-Viot superprocess result under weaker assumptions as in [DH82]. ◇

Here is our first consequence of the moment formulae. Recall that p^{α} denotes the α–stable transition kernel and ℓ the Lebesgue measure.

Corollary 6 (**Long-term behavior of moments**)**.** *For each* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}\backslash\{0\}$ *and* $\varphi_i \in \mathcal{C}_1^+$ *such that* $\ell(\varphi_i) < \infty$, $i = 1,2$,

$$\lim_{t\uparrow\infty} t^{d/\alpha}\,\mathbf{E}\hat{X}_t(\varphi_1) \ = \ \mathrm{p}_1^{\alpha}(0)\,\ell(\varphi_1) \tag{13}$$

and

$$\text{(14)}\quad \lim_{t\uparrow\infty} t^{d/\alpha}\,\mathbf{E}\big(\hat{X}_t(\varphi_1)\,\hat{X}_t(\varphi_2)\big) \;=\; \mathrm{p}_1^\alpha(0)\int_0^\infty \varrho\, e^{-\varrho s}\,\ell\big((S_s^\alpha\varphi_1)\,(S_s^\alpha\varphi_2)\big)\,ds \;<\; \infty.$$

(A formula for $\mathrm{p}_1^\alpha(0)$ is given in (A10) in the appendix.)

From the asymptotics of the mean of $\hat{X}_t(\varphi)$ together with Markov's inequality one can easily infer that for every $\varepsilon > 0$ and $\varphi \in \mathcal{C}_1^+$,

$$\text{(15)}\qquad \lim_{t\uparrow\infty} t^{d/\alpha-\varepsilon}\,\hat{X}_t(\varphi) \;=\; 0 \quad \text{in probability.}$$

Remark 7 (Open problem: local limit theorem). Is it true that $t^{d/\alpha}\hat{X}_t(\varphi)$ converges as $t \uparrow \infty$ in some sense? ◇

There is also the following consequence of the moment formulae.

Corollary 8 (Localization at all smaller scales). *Suppose* $X_0 = \mu \in \mathcal{M}_\mathrm{f}\backslash\{0\}$. *Consider a scaling function* $\sigma\colon (0,\infty) \to (0,\infty)$ *with*

$$\text{(16)}\qquad \sigma_t \uparrow \infty \quad \textit{and} \quad \sigma_t = o(t^{1/\alpha}) \quad \textit{as}\ t\uparrow\infty.$$

Then for every open $B \in \mathrm{b}\mathcal{B}_\ell$ *and* $\varepsilon \in (0,1)$,

$$\text{(17)}\qquad \Big(\frac{t^{1/\alpha}}{\sigma_t}\Big)^d\,\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon\big) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1^\alpha(0)\,\ell(B),$$

and

$$\text{(18)}\qquad \mathbf{P}\big(\hat{X}_t(\sigma_t B) \geq \varepsilon\big) \;\sim\; \mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon\big) \quad \textit{as} \quad t\uparrow\infty.$$

Relation (17) gives the asymptotics of the probability that the whole mass of our rescaled process at time t is in the set $\sigma_t B$. Relation (18) means, roughly speaking, if the whole normalized mass in $\sigma_t B$ is not very small then it is very large.

Recall that Theorem 1 in the reformulation (6) says that the total mass $\bar{X}_t$ concentrates asymptotically as $t\uparrow\infty$ in one point of the rescaled space (the scale $t^{1/\alpha}$ is related to the migration index). But Corollary 8 shows that this property remains valid for all smaller scales converging to infinity.

Recall also that for the state $\bar{X}_t$ at time $t > 0$ of Neveu's continuous state branching process there is the following *cluster representation*:

$$\text{(19)}\qquad \bar{X}_t \;=\; \sum_{i\geq 1} \vartheta_t^{(i)},$$

where $\vartheta_t^{(1)} > \vartheta_t^{(2)} > \cdots$ are the atoms of a Poisson point field π_t, say, on $(0,\infty)$ with intensity measure

$$\text{(20)}\qquad \lambda_t(dx) \;:=\; \frac{m e^{-\varrho t}}{\Gamma(1-e^{-\varrho t})}\,x^{-1-e^{-\varrho t}}\,dx$$

(cf. [BLG00]).

Proposition 9 (Localization in the main cluster). *We have the following convergence in probability:*

$$\text{(21)}\qquad \frac{\bar{X}_t - \vartheta_t^{(1)}}{\vartheta_t^{(1)}} \;\underset{t\uparrow\infty}{\overset{\mathbf{P}}{\longrightarrow}}\; 0.$$

This reminds a result of Darling [Dar52] saying that the sum of i.i.d. random variables with slowly varying tails behaves as the maximal element. Our $\bar{X}_t$ is stable of index $e^{-\varrho t}$, thus does not have slowly varying tails for finite $t > 0$, but the index goes to 0 as $t \uparrow \infty$, which explains the similar effect described in the former proposition.

Since $\vartheta_t^{(1)}$ is asymptotically equivalent to the whole mass $\bar{X}_t$, we can now reformulate Corollary 8 as follows. Statement (17) gives the asymptotic probability that $\vartheta_t^{(1)}$ locates in the set $\sigma_t B$, whereas (18) says that the subpopulation $\vartheta_t^{(1)}$ has no time to diffuse into a large subset of $\mathbb{R}^d$.

Remark 10 (**Intermittency**). Note that for each $n \geq 1$ and $B \in \mathrm{b}\mathcal{B}_\ell$ of positive Lebesgue measure we have

$$\frac{\log \mathbf{E}\big(t^{d/\alpha}\hat{X}_t(B)\big)^{n+1}}{n+1} - \frac{\log \mathbf{E}\big(t^{d/\alpha}\hat{X}_t(B)\big)^{n}}{n} \underset{t\uparrow\infty}{\longrightarrow} \infty. \tag{22}$$

In case of homogeneous random fields, such type of property of moments is known as *intermittency*, see Gärtner and Molchanov [GM90]. Indeed, as shown in [GM90], it is enough to verify it for $n = 1$, and this follows here from Corollary 6. ◇

Remark 11 (**Open problem: infinite measure states**). It would be interesting to construct X starting from the Lebesgue measure $X_0 = \ell$, and to study its large scale behavior. Although then the normalization $\hat{X}_t = X_t/\bar{X}_t$ would not be possible since $\bar{X}_t \equiv \infty$, one still would expect some intermittency effects, i.e. the relative localization of masses in remote locations. ◇

The rest of this paper is laid out as follows. In Section 2 we recall with Lemma 12 some known properties of the Cauchy problem (4) and prove with Lemmas 13 and 15 two technical results about its solutions. The proofs of Proposition 3 as well as of Corollaries 6 and 8 will be provided in Section 3. The final section is devoted to the proof of Theorem 1. In an appendix we collect some remarks on stable distributions and prove Proposition 9.

2. Related log-Laplace equation

An essential step in our procedure is to establish a log-Laplace product formula (Lemma 13) and a small-ε-asymptotics of log-Laplace functions (Lemma 15).

2.1. **Basic setting.** A *continuous-state branching process* with branching mechanism $u \mapsto g(u)$ is a time-homogeneous Markov process, whose Laplace transition functional can be characterized as follows. For every $\lambda \geq 0$,

$$\mathbf{E}\Big\{e^{-\lambda \bar{X}_t} \,\Big|\, \bar{X}_0 = m\Big\} \;=\; e^{-m\bar{u}_t[\lambda]}, \qquad t, m \geq 0, \tag{23}$$

where $\bar{u} = \bar{u}[\lambda]$ solves

$$\frac{d}{dt}\bar{u}_t \;=\; -g(\bar{u}_t) \quad \text{on} \quad (0,\infty) \quad \text{with} \quad \bar{u}_{0+} = \lambda. \tag{24}$$

Actually, we restrict our consideration to Neveu's special case

$$g(u) := \varrho\, u \log u, \qquad u \geq 0, \tag{25}$$

(for the fixed $\varrho > 0$). Then necessarily,

$$\bar{u}_t[\lambda] \;=\; \lambda^{(e^{-\varrho t})}, \tag{26}$$

which demonstrates that for every $t > 0$ fixed, $\bar{X}_t$ has a stable distribution with index $e^{-\varrho t} < 1$. In particular, in this case the random variable $\bar{X}_t$ is non-zero and finite with probability one.

Next we rewrite log-Laplace equation (4) in integral form:

$$u_t \;=\; S_t^\alpha \varphi - \int_0^t S_{t-s}^\alpha \big(g(u_s)\big)\, ds, \qquad t \geq 0, \tag{27}$$

where we used notation (25). The following result is taken from [FS04, Theorem 1].

Lemma 12 (**Well-posedness of log-Laplace equation**). *To* $\varphi \in \mathcal{C}_1^{++}$ *there is a unique (pointwise) solution* $u[\varphi]$ *to equation* (27), *and*

$$\min\Big\{ \inf_{x \in \mathbb{R}^d} \varphi(y), 1\Big\} \;\leq\; u_t[\varphi] \;\leq\; \max\Big\{ \sup_{x \in \mathbb{R}^d} \varphi(y), 1\Big\}. \tag{28}$$

Moreover, if $\varphi \in \mathcal{C}_1^{(2)++}$, *then* $u_t[\varphi]$ *solves the related function-valued Cauchy problem as in* (4). *Further, if* $\varphi_n \in \mathcal{C}_1^{++}$ *pointwise satisfy* $\varphi_n \downarrow \varphi \in \mathcal{C}_1^{+}$ *as* $n \uparrow \infty$, *then pointwise* $u[\varphi_n] \downarrow u[\varphi]$ *holds, and the limit function* $u[\varphi]$ *is a solution to equation* (27), *satisfies* (28), *and is independent of the choice of the approximating sequence* $\{\varphi_n\}_{n \geq 1}$.

From now on, for $\varphi \in \mathcal{C}_1^{+}$ fixed, under $u[\varphi]$ we mean the solution to (27), which can be obtained as such limit of some $u[\varphi_n]$.

From the expression (26) for $\bar{u}[\lambda]$ one can easily infer that $\bar{u}_t[\lambda\theta] = \bar{u}_t[\lambda]\,\bar{u}_t[\theta]$ for all positive constants λ and θ. In the following lemma we generalize this identity for the solutions $u[\varphi]$ to equation (27).

Lemma 13 (**Log-Laplace product formula**). *For* $t \geq 0$, $\lambda > 0$, *and* $\varphi \in \mathcal{C}_1^{+}$,

$$u_t[\lambda\varphi] \;=\; \bar{u}_t[\lambda]\, u_t[\varphi]. \tag{29}$$

Proof. Let us first assume, that $\varphi \in \mathcal{C}_1^{(2)++}$. Then, by Lemma 12, $u[\varphi]$ is the unique solution to the Cauchy problem (4). Clearly,

$$\frac{\partial}{\partial t}\big(\bar{u}_t[\lambda]\, u_t[\varphi]\big) \;=\; \bar{u}_t[\lambda]\, \Big(\Delta_\alpha u_t[\varphi] - g\big(u_t[\varphi]\big)\Big) - u_t[\varphi]\, g\big(\bar{u}_t[\lambda]\big). \tag{30}$$

Therefore, in view of $\bar{u}_t[\lambda]\,\Delta_\alpha u_t[\varphi] = \Delta_\alpha\big(\bar{u}_t[\lambda]\, u_t[\varphi]\big)$, and

$$g\big(u_t[\varphi]\big)\, \bar{u}_t[\lambda] + u_t[\varphi]\, g\big(\bar{u}_t[\lambda]\big) \;=\; g\big(u_t[\varphi]\, \bar{u}_t[\lambda]\big), \tag{31}$$

we conclude, that $u[\varphi]\,\bar{u}[\lambda]$ solves the Cauchy problem (4) with initial condition $\lambda\varphi$. Uniqueness of the solution to (4) gives the proof of (29) in the case $\varphi \in \mathcal{C}_1^{(2)++}$. To finish the proof, approximate $\varphi \in \mathcal{C}_1^{+}$ monotonously from above by appropriate $\varphi_n \in \mathcal{C}_1^{(2),++}$ and use Lemma 12. □

2.2. **A distributional relation.** Using log-Laplace product formula (29) one can establish a simple connection in law between the random variables $X_t(\varphi)$ and $\bar{X}_t$. Indeed, for t, λ, φ, as in Lemma 13,

$$\begin{aligned} \mu\big(u_t[\lambda\varphi]\big) \;&=\; \bar{u}_t[\lambda]\, \mu\big(u_t[\varphi]\big) \;=\; \mu(1)\, \bar{u}_t[\lambda]\, \bar{u}_t\Big(\Big(\hat{\mu}\big(u_t[\varphi]\big)\Big)^{(e^{\varrho t})}\Big) \\ &=\; \mu(1)\, \bar{u}_t\Big(\lambda\Big(\hat{\mu}\big(u_t[\varphi]\big)\Big)^{(e^{\varrho t})}\Big). \end{aligned} \tag{32}$$

Hence, from these equalities and the Laplace transition functional (3) we conclude that

$$\mathbf{E}\Big\{e^{-\lambda X_t(\varphi)}\;\Big|\;X_0=\mu\Big\} \;=\; \mathbf{E}\Big\{e^{-\lambda\theta_t\bar{X}_t}\;\Big|\;\bar{X}_0=\mu(1)\Big\} \quad\text{with}\quad \theta_t:=\Big(\hat{\mu}\big(u_t[\varphi]\big)\Big)^{(e^{\varrho t})}.$$

This means that

$$X_t(\varphi) \;\overset{\mathcal{L}}{=}\; \Big(\hat{\mu}\big(u_t[\varphi]\big)\Big)^{(e^{\varrho t})}\bar{X}_t\,, \qquad t\geq 0,\quad \varphi\in\mathcal{C}_1^+. \tag{33}$$

Now we show one possible application of this equality in law. From (33) it follows that for every $\varphi\in\mathcal{C}_1^+$,

$$e^{-\varrho t}\,\mathbf{E}\log X_t(\varphi) \;=\; e^{-\varrho t}\,\mathbf{E}\log\bar{X}_t \;+\;\log\Big(\hat{\mu}\big(u_t[\varphi]\big)\Big). \tag{34}$$

Since $X_t(c\varphi)=cX_t(\varphi)$, for any constant c, we may assume without loss of generality that $\|\varphi\|_\infty\leq 1$. Then $X_t(\varphi)\leq\bar{X}_t$ and, consequently,

$$e^{-\varrho t}\,\mathbf{E}\big|\log\bar{X}_t-\log X_t(\varphi)\big| \;=\; -\log\Big(\hat{\mu}\big(u_t[\varphi]\big)\Big). \tag{35}$$

Thus,

$$e^{-\varrho t}\,\mathbf{E}\big|\log\bar{X}_t-\log X_t(\varphi)\big| \underset{t\uparrow\infty}{\longrightarrow} 0 \quad\text{if and only if}\quad \hat{\mu}\big(u_t[\varphi]\big)\underset{t\uparrow\infty}{\longrightarrow} 1. \tag{36}$$

On the other hand, by Proposition A1 in the appendix,

$$\mathbf{E}\Big|e^{-\varrho t}\log\bar{X}_t-(-\log V)\Big|^r \underset{t\uparrow\infty}{\longrightarrow} 0, \qquad r>0. \tag{37}$$

Combining (36) and (37), we obtain the following result.

Proposition 14 (Equivalent formulations). *Consider $\varphi\in\mathcal{C}_1^+$. Then condition $\hat{\mu}\big(u_t[\varphi]\big)\to 1$ as $t\uparrow\infty$ is necessary and sufficient for the convergence*

$$\mathbf{E}\Big|e^{-\varrho t}\log X_t(\varphi)-(-\log V)\Big| \underset{t\uparrow\infty}{\longrightarrow} 0. \tag{38}$$

Clearly, one expects that the convergence $\hat{\mu}\big(u_t[\varphi]\big)\to 1$ holds for all non-vanishing $\varphi\in\mathcal{C}_1^+$. Then, comparing with (1), the proposition would say, roughly speaking, that on a logarithmic scale, $X_t(\varphi)$ behaves just as $\bar{X}_t$. Since this statement is not very informative, we do not insist to settle the statement $\hat{\mu}\big(u_t[\varphi]\big)\to 1$ and follow instead another route.

2.3. **Small ε–asymptotics.** A crucial step in our development is the following perturbation result.

Lemma 15 (Small ε–asymptotics). *Let $\varphi\in\mathcal{C}_1^{(2)++}$ with $\|\varphi\|_\infty\leq 1$. Then for fixed $t>0$,*

$$u_t[1+\varepsilon\varphi] \;=\; 1\;+\;\varepsilon\,e^{-\varrho t}\,S_t^\alpha\varphi\;-\;\frac{\varepsilon^2}{2}\,e^{-\varrho t}\int_0^t\varrho\,e^{-\varrho s}\,S_{t-s}^\alpha(S_s^\alpha\varphi)^2\,ds\;+\;O(\varepsilon^3e^{-\varrho t})$$

as $\varepsilon\downarrow 0$.

Proof. Fix $0<\varepsilon\leq 1$. We define the function $v=v[\varepsilon\varphi]:=u[1+\varepsilon\varphi]-1\geq 0$, which is the unique solution to the Cauchy problem

$$\frac{d}{dt}v_t \;=\; \Delta_\alpha v_t-\varrho\,(1+v_t)\log(1+v_t) \quad\text{on}\quad (0,\infty) \quad\text{with}\quad v_{0+}=\varepsilon\varphi \tag{39}$$

(note that $v \mapsto \varrho\,(1+v)\log(1+v)$ is locally Lipschitz on $\mathbb{R}_+$). It follows that the function $t \mapsto w_t = w_t[\varepsilon\varphi] := e^{\varrho t} v_t$ solves the equation

$$\frac{d}{dt} w_t \;=\; \Delta_\alpha w_t - \varrho\, e^{\varrho t}(1+e^{-\varrho t} w_t)\log(1+e^{-\varrho t} w_t) + \varrho w_t \quad \text{with} \quad w_{0+} \;=\; \varepsilon\varphi,$$

which in integral form reads as

$$w_t \;=\; \varepsilon\, S_t^\alpha \varphi - \int_0^t \varrho\, S_{t-s}^\alpha \Big(e^{\varrho s}(1+e^{-\varrho s} w_s)\log(1+e^{-\varrho s} w_s) - w_s\Big)\, ds. \tag{40}$$

Hence,

$$v_t[\varepsilon\varphi] \;=\; \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi - e^{-\varrho t} \int_0^t \varrho\, e^{\varrho s}\, S_{t-s}^\alpha \Big((1+v_s)\log(1+v_s) - v_s\Big)\, ds. \tag{41}$$

Using Taylor expansion for $\log(1+x)$, we get for $0 < x < 1$,

$$(1+x)\log(1+x) \;=\; x \,+\, \sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)}\, x^k. \tag{42}$$

From this identity it follows that for $0 < x < 1$,

$$x + \frac{x^2}{2} - \frac{x^3}{6} \;<\; (1+x)\log(1+x) \;<\; x + \frac{x^2}{2}\,. \tag{43}$$

Moreover, by (28),

$$0 \;\le\; v_t[\varepsilon\varphi] \;\le\; \varepsilon\, \|\varphi\|_\infty \;\le\; \varepsilon. \tag{44}$$

Applying these bounds to the right-hand side of (41), we have

$$v_t[\varepsilon\varphi] \;\ge\; \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi \;-\; \frac{e^{-\varrho t}}{2} \int_0^t \varrho\, e^{\varrho s}\, S_{t-s}^\alpha v_s^2\, ds \tag{45}$$

and

$$v_t[\varepsilon\varphi] \;\le\; \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi \;-\; \frac{e^{-\varrho t}}{2} \int_0^t \varrho\, e^{\varrho s}\, S_{t-s}^\alpha v_s^2\, ds \;+\; \frac{e^{-\varrho t}}{6} \int_0^t \varrho\, e^{\varrho s}\, S_{t-s}^\alpha v_s^3\, ds. \tag{46}$$

Note next that from (41),

$$v_t[\varepsilon\varphi] \;\le\; \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi. \tag{47}$$

Combining (45) and (47) gives the following lower bound

$$v_t[\varepsilon\varphi] \;\ge\; \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi \;-\; \frac{e^{-\varrho t}}{2}\, \varepsilon^2 \int_0^t \varrho\, e^{-\varrho s}\, S_{t-s}^\alpha (S_s^\alpha \varphi)^2\, ds. \tag{48}$$

Comparing (48) with the claim in Lemma 15, we infer that it remains to find a suitable upper bound for $v_t[\varepsilon\varphi]$. Applying estimates (48) and (47) to the first and second integrals at the right hand side of (46), respectively, we obtain

$$\begin{aligned} v_t[\varepsilon\varphi] \;\le\; & \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi \\ & -\; \frac{e^{-\varrho t}}{2}\, \varepsilon^2 \int_0^t \varrho\, e^{-\varrho s}\, S_{t-s}^\alpha \Big(S_s^\alpha \varphi - \frac{\varepsilon}{2} \int_0^s \varrho\, e^{-\varrho r}\, S_{s-r}^\alpha (S_r^\alpha \varphi)^2 dr\Big)^2 ds \\ & +\; \frac{e^{-\varrho t}}{6}\, \varepsilon^3 \int_0^t \varrho\, e^{\varrho s}\, S_{t-s}^\alpha (S_s^\alpha \varphi)^3\, ds \\ =\; & \varepsilon\, e^{-\varrho t}\, S_t^\alpha \varphi - \frac{e^{-\varrho t}}{2}\, \varepsilon^2 \int_0^t \varrho\, e^{-\varrho s}\, S_{t-s}^\alpha (S_s^\alpha \varphi)^2\, ds + O(\varepsilon^3 e^{-\varrho t}). \end{aligned} \tag{49}$$

This finishes the proof. $\square$

Remark 16 (Asymptotic expansion). It is easy to see that we can expand $u_t[1+\varepsilon\varphi]$ in a power series

$$u_t[1+\varepsilon\varphi]\,(x) \;=\; 1+\sum_{i=1}^{\infty} H_i(t,x)\,\varepsilon^i, \tag{50}$$

where H_i are some functions which could be expressed in terms of the semigroup S^α and the initial condition φ. ◇

3. Asymptotics for moments

Here we derive the needed moment formulae (Proposition 3) and study their asymptotic properties (Corollaries 6 and 8).

3.1. **Moment formulae (proof of Proposition 3).** First we will show that formulae (11) and (12) hold for $\varphi_1=\varphi_2=:\varphi\in\mathcal{C}_1^{(2)++}$ and $t_1=t_2=:t>0$. Recall that without loss of generality we may assume that $\|\varphi\|_\infty\le 1$. By (33), for every $\varepsilon>0$,

$$\bar{X}_t+\varepsilon X_t(\varphi) \;\overset{\mathcal{L}}{=}\; \Big(\hat{\mu}\big(u_t[1+\varepsilon\varphi]\big)\Big)^{(e^{\varrho t})}\bar{X}_t\,. \tag{51}$$

Taking first logarithm at both sides and then expectations, we obtain

$$\mathbf{E}\log\big(\bar{X}_t+\varepsilon X_t(\varphi)\big) \;=\; \mathbf{E}\log\bar{X}_t \;+\; e^{\varrho t}\log\Big(\hat{\mu}\big(u_t[1+\varepsilon\varphi]\big)\Big). \tag{52}$$

Therefore,

$$\mathbf{E}\log\big(1+\varepsilon\,\hat{X}_t(\varphi)\big) \;=\; e^{\varrho t}\log\Big(\hat{\mu}\big(u_t[1+\varepsilon\varphi]\big)\Big). \tag{53}$$

Evidently, $\hat{X}_t(\varphi)\le\|\varphi\|_\infty\;\le\;1$. Hence, from the Taylor expansion $\log(1+x)=x-x^2/2+O(x^3)$ as $x\downarrow 0$, and the boundedness of $\hat{X}_t(\varphi)$, it follows that

$$\mathbf{E}\log\big(1+\varepsilon\,\hat{X}_t(\varphi)\big) \;=\; \varepsilon\,\mathbf{E}\hat{X}_t(\varphi)-\frac{\varepsilon^2}{2}\,\mathbf{E}\big(\hat{X}_t(\varphi)\big)^2+O(\varepsilon^3)\quad\text{as}\quad\varepsilon\downarrow 0. \tag{54}$$

By Lemma 15,

$$\begin{aligned}
&e^{\varrho t}\log\Big(\hat{\mu}\big(u_t[1+\varepsilon\varphi]\big)\Big) \;=\; e^{\varrho t}\log\Big(1+\hat{\mu}\big(u_t[1+\varepsilon\varphi]-1\big)\Big)\\
&=\; e^{\varrho t}\hat{\mu}\big(v_t[\varepsilon\varphi]\big)-\frac{e^{\varrho t}}{2}\Big(\hat{\mu}\big(v_t[\varepsilon\varphi]\big)\Big)^2+O\bigg(e^{\varrho t}\Big(\hat{\mu}\big(v_t[\varepsilon\varphi]\big)\Big)^3\bigg)\\
&=\; \varepsilon\,\hat{\mu}(S_t^\alpha\varphi)-\frac{\varepsilon^2}{2}\int_0^t\varrho\,e^{-\varrho s}\,\hat{\mu}\big(S_{t-s}^\alpha(S_s^\alpha\varphi)^2\big)\,ds-\frac{\varepsilon^2e^{-\varrho t}}{2}\,\big(\hat{\mu}(S_t^\alpha\varphi)\big)^2+O(\varepsilon^3).
\end{aligned}$$

Combining with (53) and (54), we conclude that

$$\begin{aligned}
\varepsilon\,\big(\mathbf{E}\hat{X}_t(\varphi)-\hat{\mu}(S_t^\alpha\varphi)\big) \;=\; \frac{\varepsilon^2}{2}\Big[&\mathbf{E}\big(\hat{X}_t(\varphi)\big)^2-\int_0^t\varrho\,e^{-\varrho s}\,\hat{\mu}\big(S_{t-s}^\alpha(S_s^\alpha\varphi)^2\big)\,ds\\
&-e^{-\varrho t}\,\big(\hat{\mu}(S_t^\alpha\varphi)\big)^2\Big]+O(\varepsilon^3).
\end{aligned} \tag{55}$$

Dividing by ε and letting $\varepsilon\downarrow 0$, we obtain (11) in the case $\varphi\in\mathcal{C}_1^{(2)++}$. Therefore,

$$\frac{\varepsilon^2}{2}\Big[\mathbf{E}\big(\hat{X}_t(\varphi)\big)^2-\int_0^t\varrho\,e^{-\varrho s}\,\hat{\mu}\big(S_{t-s}^\alpha(S_s^\alpha\varphi)^2\big)\,ds-e^{-\varrho t}\,\big(\hat{\mu}(S_t^\alpha\varphi)\big)^2\Big] \;=\; O(\varepsilon^3). \tag{56}$$

Dividing now by ε^2 and letting again $\varepsilon \downarrow 0$, we arrive at (12) in the case $\varphi_1 = \varphi_2 = \varphi \in \mathcal{C}_1^{(2)++}$ and $t_1 = t_2 = t > 0$.

The case of possibly different $\varphi_1, \varphi_2 \in \mathcal{C}_1^{(2)++}$ follows by polarization. To extent to $\varphi_1, \varphi_2 \in \mathcal{C}_1^{+}$, approximate monotonously from above by functions in $\mathcal{C}_1^{(2)++}$, and use Lemma 12 as well as monotone and bounded convergence. This completes the proof of expectation formula (11).

Finally, second moment formula (12) in the case $t_1 < t_2$ follows by using the Markov property and (11). □

3.2. **Long-term behavior of moments (proof of Corollary 6).** Take $\mu, \varphi_1, \varphi_2$ as in Corollary 6. By polarization, we may assume that $\varphi_1 = \varphi_2 =: \varphi$. We will again additionally suppose that $\|\varphi\|_\infty \leq 1$. Recall the following scaling property of the stable density function: For every $k > 0$,

$$\text{(57)} \qquad \mathrm{p}_t^\alpha(x) \;=\; k^{d/\alpha}\, \mathrm{p}_{kt}^\alpha(k^{1/\alpha}x), \qquad t > 0, \quad x \in \mathsf{R}^d.$$

Using this identity with $k = t^{-1}$, we have

$$\text{(58)} \qquad S_t^\alpha \varphi(x) \;=\; t^{-d/\alpha} \int_{\mathbb{R}^d} \mathrm{p}_1^\alpha\big(t^{-1/\alpha}(y-x)\big)\, \varphi(y)\, dy.$$

In view of $\mathrm{p}_1^\alpha\big(t^{-1/\alpha}(y-x)\big) \to \mathrm{p}_1^\alpha(0)$ as $t \uparrow \infty$, we obtain

$$\text{(59)} \qquad t^{d/\alpha}\, \hat{\mu}(S_t^\alpha \varphi) \;=\; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{p}_1^\alpha\big(t^{-1/\alpha}(y-x)\big)\, \hat{\mu}(dx)\, \varphi(y)\, dy \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0)\, \ell(\varphi).$$

Combining this relation with the expectation formula (11) gives (13).

Using the same arguments one can show that for every fixed $s \geq 0$,

$$\text{(60)} \qquad t^{d/\alpha}\, \hat{\mu}\big(S_{t-s}^\alpha (S_s^\alpha \varphi)^2\big) \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0)\, \ell\big((S_s^\alpha \varphi)^2\big) \;\leq\; \mathrm{p}_1^\alpha(0)\, \ell(\varphi).$$

Here we used $\|S_s^\alpha \varphi\|_\infty \leq 1$. Hence, by dominated convergence, for every fixed $s_0 > 0$,

$$\text{(61)} \qquad \lim_{t\uparrow\infty} t^{d/\alpha} \int_0^{s_0} \varrho\, e^{-\varrho s}\, \hat{\mu}\big(S_{t-s}^\alpha (S_s^\alpha \varphi)^2\big)\, ds \;=\; \mathrm{p}_1^\alpha(0) \int_0^{s_0} \varrho\, e^{-\varrho s} \ell\big((S_s^\alpha \varphi)^2\big)\, ds.$$

Using again $\|S_s^\alpha \varphi\|_\infty \leq 1$, we arrive at the bound

$$\text{(62)} \qquad \int_{s_0}^t \varrho\, e^{-\varrho s}\, \hat{\mu}\big(S_{t-s}^\alpha (S_s^\alpha \varphi)^2\big)\, ds \;\leq\; \hat{\mu}(S_t^\alpha \varphi) \int_{s_0}^t \varrho\, e^{-\varrho s}\, ds, \qquad t \geq s_0\,.$$

Therefore, by (59),

$$\text{(63)} \qquad \limsup_{t\uparrow\infty}\, t^{d/\alpha} \int_{s_0}^t \varrho\, e^{-\varrho s}\, \hat{\mu}\big(S_{t-s}^\alpha (S_s^\alpha \varphi)^2\big)\, ds \;\leq\; \mathrm{p}_1^\alpha(0)\, \ell(\varphi)\, e^{-\varrho s_0}.$$

Since (61) and (63) are valid for any $s_0 > 0$, we can combine them and let $s_0 \uparrow \infty$ to get

$$\text{(64)} \qquad \lim_{t\uparrow\infty} t^{d/\alpha} \int_0^t \varrho\, e^{-\varrho s}\, \hat{\mu}\big(S_{t-s}^\alpha (S_s^\alpha \varphi)^2\big)\, ds \;=\; \mathrm{p}_1^\alpha(0) \int_0^\infty \varrho\, e^{-\varrho s}\, \ell\big((S_s^\alpha \varphi)^2\big)\, ds.$$

Together with the second moment formula (12), the proof of Corollary 6 is complete. □

3.3. **Localization at all scales (proof of Corollary 8).** Fix μ,σ,B,ε as in Corollary 8.

1° [*Expectation asymptotics*]. For $t>0$, let be given open $B_t\in \mathrm{b}\mathcal{B}_\ell$ with $B_t\subseteq B$, and such that $\ell(B_t)\uparrow \ell(B)$ as $t\uparrow\infty$. For the moment, fix $s\geq 0$. Then, for $t>s$, from (11) (and Remark 4) and (58),

$$(65)\quad \mathbf{E}\hat{X}_{t-s}(\sigma_t B_t) \;=\; \Big(\frac{\sigma_t}{(t-s)^{1/\alpha}}\Big)^d \int_{B_t}\int_{\mathbb{R}^d} \mathrm{p}_1^\alpha\big((t-s)^{-1/\alpha}(\sigma_t z - x)\big)\,\hat{\mu}(dx)\,dz.$$

Using assumption (16) one can easily infer that

$$(66)\qquad \int_{\mathbb{R}^d} \mathrm{p}_1^\alpha\big((t-s)^{-1/\alpha}(\sigma_t z - x)\big)\,\hat{\mu}(dx) \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0),\qquad z\in B.$$

Therefore, setting

$$(67)\qquad c_t \;:=\; \Big(\frac{t^{1/\alpha}}{\sigma_t}\Big)^d,\qquad t>0,$$

we obtain

$$(68)\qquad c_t\,\mathbf{E}\hat{X}_{t-s}(\sigma_t B_t) \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0)\,\ell(B),\qquad s\geq 0.$$

2° [*Second moment asymptotics*]. Our next purpose is to prove the convergence

$$(69)\qquad c_t\,\mathbf{E}\big(\hat{X}_t(\sigma_t B)\big)^2 \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0)\,\ell(B).$$

Since $\hat{X}_t(\sigma_t B)\leq 1$, by (68) it suffices to show that the limit inferior as $t\uparrow\infty$ of a suitable lower estimate of the left hand side in claim (69) equals the right hand side of (69).

Fix $s_0>0$. Choose a number R such that $\int_{|y|<R}\mathrm{p}_s^\alpha(y)\,dy\geq 1-\varepsilon$ for all $s\leq s_0$. Define $B_t:=\{y\in B:\ |y-\partial B|>R/\sigma_t\}$. Trivially, $\sigma_t B_t:=\{y\in\sigma_t B:\ |y-\partial(\sigma_t B)|>R\}$, since $\sigma_t\partial B=\partial(\sigma_t B)$. Then, for every $x\in\sigma_t B_t$,

$$(70)\qquad S_s^\alpha 1_{\sigma_t B}\,(x) \;\geq\; \int_{\sigma_t B_t}\mathrm{p}_s^\alpha(y-x)\,dy \;\geq\; \int_{|y-x|<R}\mathrm{p}_s^\alpha(y-x)\,dy \;\geq\; 1-\varepsilon.$$

Hence, for $t>s_0$,

$$(71)\qquad \int_0^t \varrho\, e^{-\varrho s}\,\hat{\mu}\big(S_{t-s}^\alpha(S_s^\alpha 1_{\sigma_t B})^2\big)\,ds \;\geq\; (1-\varepsilon)^2\int_0^{s_0}\varrho\, e^{-\varrho s}\,\hat{\mu}(S_{t-s}^\alpha 1_{\sigma_t B_t})\,ds.$$

By (68) and (11), we conclude that for every $s\leq s_0$,

$$(72)\qquad c_t\,\hat{\mu}(S_{t-s}^\alpha 1_{\sigma_t B_t}) \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0)\,\ell(B).$$

By dominated convergence,

$$(73)\qquad c_t\int_0^{s_0}\varrho\, e^{-\varrho s}\,\hat{\mu}(S_{t-s}^\alpha 1_{\sigma_t B_t})\,ds \;\underset{t\uparrow\infty}{\longrightarrow}\; \mathrm{p}_1^\alpha(0)\,\ell(B)(1-e^{-\varrho s_0}).$$

Combining (73) and (71), we arrive at

$$\liminf_{t\uparrow\infty} c_t\int_0^t \varrho\, e^{-\varrho s}\,\hat{\mu}\big(S_{t-s}^\alpha(S_s^\alpha 1_{\sigma_t B})^2\big)\,ds \;\geq\; (1-\varepsilon)^2\,\mathrm{p}_1^\alpha(0)\,\ell(B)(1-e^{-\varrho s_0}).$$

Letting $\varepsilon\downarrow 0$ and $s_0\uparrow\infty$, as well as using the second moment formula (12), we get (69).

3° [*Verifying* (17)]. Based on (68) in the case $B_t \equiv B$ and (69) we find $\varepsilon \geq \varepsilon_t \downarrow 0$ as $t \uparrow \infty$ so that

$$\varepsilon_t^{-1}\Big(\mathbf{E}\hat{X}_t(\sigma_t B) - \mathbf{E}\big(\hat{X}_t(\sigma_t B)\big)^2\Big) \;=\; o\big(\mathbf{E}\hat{X}_t(\sigma_t B)\big). \tag{74}$$

Now, since $\hat{X}_t(\sigma_t B) \leq 1$,

$$\begin{aligned}\mathbf{E}\big(\hat{X}_t(\sigma_t B)\big)^2 \;\leq\; & (1-\varepsilon_t)\,\mathbf{E}\Big\{\hat{X}_t(\sigma_t B);\ \hat{X}_t(\sigma_t B) \leq 1-\varepsilon_t\Big\} \\ & +\ \mathbf{E}\Big\{\hat{X}_t(\sigma_t B);\ \hat{X}_t(\sigma_t B) > 1-\varepsilon_t\Big\}.\end{aligned} \tag{75}$$

Rearranging gives

$$\mathbf{E}\Big\{\hat{X}_t(\sigma_t B);\ \hat{X}_t(\sigma_t B) \leq 1-\varepsilon_t\Big\} \;\leq\; \varepsilon_t^{-1}\Big(\mathbf{E}\hat{X}_t(\sigma_t B) - \mathbf{E}\big(\hat{X}_t(\sigma_t B)\big)^2\Big). \tag{76}$$

Hence, by (74),

$$\mathbf{E}\Big\{\hat{X}_t(\sigma_t B);\ \hat{X}_t(\sigma_t B) \leq 1-\varepsilon_t\Big\} \;=\; o\big(\mathbf{E}\hat{X}_t(\sigma_t B)\big) \quad \text{as} \quad t \uparrow \infty. \tag{77}$$

Again by $1 \geq \hat{X}_t(\sigma_t B)$,

$$\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon_t\big) \;\geq\; (1-\varepsilon_t)\,\mathbf{E}\Big\{\hat{X}_t(\sigma_t B);\ \hat{X}_t(\sigma_t B) > 1-\varepsilon_t\Big\}. \tag{78}$$

Combining (78), (77), and (68) (in the case $B_t \equiv B$) gives

$$\liminf_{t\uparrow\infty}\, c_t\,\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon_t\big) \;\geq\; \mathrm{p}_1^{\alpha}(0)\,\ell(B). \tag{79}$$

On the other hand, from Markov's inequality,

$$\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon_t\big) \;\leq\; (1-\varepsilon_t)^{-1}\,\mathbf{E}\hat{X}_t(\sigma_t B). \tag{80}$$

Therefore, again by (68),

$$\limsup_{t\uparrow\infty}\, c_t\,\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon_t\big) \;\leq\; \mathrm{p}_1^{\alpha}(0)\,\ell(B). \tag{81}$$

Combining (79) and (81), we arrive at (17) with ε replaced by ε_t [which was chosen for (74)].

Clearly, from $\varepsilon_t \leq \varepsilon$ we get

$$\liminf_{t\uparrow\infty}\, c_t\,\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon\big) \;\geq\; \mathrm{p}_1^{\alpha}(0)\,\ell(B). \tag{82}$$

On the other hand,

$$\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon\big) \;=\; \mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon_t\big) + \mathbf{P}\big(1-\varepsilon < \hat{X}_t(\sigma_t B) \leq 1-\varepsilon_t\big). \tag{83}$$

By Markov's inequality,

$$\mathbf{P}\big(1-\varepsilon < \hat{X}_t(\sigma_t B) \leq 1-\varepsilon_t\big) \;\leq\; (1-\varepsilon)^{-1}\,\mathbf{E}\Big\{\hat{X}_t(\sigma_t B);\ \hat{X}_t(\sigma_t B) \leq 1-\varepsilon_t\Big\}.$$

Inserting into (83), from (77), (68), and (81) we get

$$\limsup_{t\uparrow\infty}\, c_t\,\mathbf{P}\big(\hat{X}_t(\sigma_t B) > 1-\varepsilon\big) \;\leq\; \mathrm{p}_1^{\alpha}(0)\,\ell(B). \tag{84}$$

Together with (82) the proof of (17) is finished.

4° [*Verifying* (18)]. Using Markov's inequality, we have

$$\begin{aligned}\mathbf{P}\Big(\hat{X}_t(\sigma_t B)\in[\varepsilon_t,1-\varepsilon_t]\Big) \;&=\; \mathbf{P}\Big(\hat{X}_t(\sigma_t B)\big(1-\hat{X}_t(\sigma_t B)\big)\geq\varepsilon_t(1-\varepsilon_t)\Big)\\ &\leq\; \varepsilon_t^{-1}(1-\varepsilon_t)^{-1}\Big(\mathbf{E}\hat{X}_t(\sigma_t B)-\mathbf{E}\big(\hat{X}_t(\sigma_t B)\big)^2\Big). \qquad (85)\end{aligned}$$

Recalling (74) we conclude

$$\limsup_{t\uparrow\infty}\; c_t\,\mathbf{P}\Big(\hat{X}_t(\sigma_t B)\in[\varepsilon_t,1-\varepsilon_t]\Big) \;=\; 0. \qquad (86)$$

But

$$\frac{\mathbf{P}\big(\hat{X}_t(\sigma_t B)\geq\varepsilon\big)}{\mathbf{P}\big(\hat{X}_t(\sigma_t B)>1-\varepsilon\big)} \;=\; 1+\frac{c_t\,\mathbf{P}\big(\hat{X}_t(\sigma_t B)\in[\varepsilon,1-\varepsilon]\big)}{c_t\,\mathbf{P}\big(\hat{X}_t(\sigma_t B)>1-\varepsilon\big)}\,,$$

and (18) follows from (86) and (17). This completes the proof of Corollary 8. □

4. Large scale localization (proof of Theorem 1)

We start with the convergence of finite-dimensional distributions (Section 4.1). Compact containment is provided in Section 4.2, and tightness of one-dimensional marginals in the Brownian case in Section 4.3. The proof of Theorem 1 is then completed in the end of Section 4.3. That our tightness proof fails in the non-Brownian case is explained in Section 4.4.

4.1. **Convergence of finite-dimensional marginals.**

To prepare for the proof of convergence of finite-dimensional distributions, we first derive the following simple result.

Lemma 17 (0-1-valued limits). *For* $k,n\geq 1$, *consider* $[0,1]$*–valued random variables* $\pi_{k,i}\,,\ 1\leq i\leq n$, *such that*

$$\lim_{k\uparrow\infty}\,\mathbf{E}\pi_{k,i}(1-\pi_{k,i})=0,\qquad 1\leq i\leq n. \qquad (87)$$

Moreover, suppose

$$\lim_{k\uparrow\infty}\mathbf{E}\prod_{i=1}^{n}\pi_{k,i}\quad \textit{exists},\qquad n\geq 1. \qquad (88)$$

Then for each $n\geq 1$ *and as* $k\uparrow\infty$, *the random vectors* $\boldsymbol{\pi}_k:=(\pi_{k,1},\dots,\pi_{k,n})$ *converge in law to some random vector* $\boldsymbol{\pi}_\infty$ *of* 0-1*-valued random variables satisfying*

$$\mathbf{P}(\boldsymbol{\pi}_\infty=\boldsymbol{\varepsilon}) \;=\; \lim_{k\uparrow\infty}\mathbf{E}\prod_{i=1}^{n}\pi_{k,i}^{1-\varepsilon_i}(1-\pi_{k,i})^{\varepsilon_i},\qquad \boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_n)\in\{0,1\}^n. \qquad (89)$$

Proof. First we prove that condition (87) implies that to each $n\geq 1$ there exist $1\geq\delta_k\downarrow 0$ as $k\uparrow\infty$, such that

$$\mathbf{P}\Big(\bigcap_{i=1}^{n}\big\{\pi_{k,i}^{\varepsilon_i}(1-\pi_{k,i})^{1-\varepsilon_i}\leq\delta_k\big\}\Big) \;=\; \mathbf{E}\prod_{i=1}^{n}\pi_{k,i}^{1-\varepsilon_i}(1-\pi_{k,i})^{\varepsilon_i}\;+\;o(1), \qquad (90)$$

$\boldsymbol{\varepsilon}\in\{0,1\}^n$. To do this, note that after a change $\pi_{k,i}\to 1-\pi_{k,i}$ for some $i\in\{1,\dots,n\}$ and $k\geq 1$, we get a sequence of vectors $\boldsymbol{\pi}_k$ which also satisfies (87). Thus, for the proof of (90) without loss of generalities we may assume that $\boldsymbol{\varepsilon}=\mathbf{0}$.

Then the left hand side of (90) can be written as and afterwards obviously estimated by

$$\mathbf{P}\Big(\bigcap_{i=1}^{n}\{\pi_{k,i}\geq 1-\delta_k\}\Big) \ \leq \ (1-\delta_k)^{-n}\,\mathbf{E}\prod_{i=1}^{n}\pi_{k,i} \ = \ \mathbf{E}\prod_{i=1}^{n}\pi_{k,i}+o(1). \tag{91}$$

On the other hand,

$$\begin{aligned}\mathbf{P}\Big(\bigcap_{i=1}^{n}\{\pi_{k,i}\geq 1-\delta_k\}\Big) \ &\geq \ \mathbf{E}\Big\{\prod_{i=1}^{n}\pi_{k,i}\,;\ \bigcap_{i=1}^{n}\{\pi_{k,i}\geq 1-\delta_k\}\Big\} \\ &\geq \ \mathbf{E}\prod_{i=1}^{n}\pi_{k,i} \ - \ \mathbf{E}\Big\{\prod_{i=1}^{n}\pi_{k,i}\,;\ \bigcup_{i=1}^{n}\{\pi_{k,i}<1-\delta_k\}\Big\}.\end{aligned} \tag{92}$$

Choose now $\delta_k\in(0,1]$ such that $\sum_{i=1}^{n}\mathbf{E}\pi_{k,i}(1-\pi_{k,i})\leq\delta_k^2$ for all k. Then by Markov's inequality the second term in (92) is bounded from above by δ_k. Thus, for (92) we get the lower estimate $\mathbf{E}\prod_{i=1}^{n}\pi_{k,i}+o(1)$, too, altogether giving (90).

To verify the claim on the existence of a limiting random variable $\boldsymbol{\pi}_\infty$, it suffices to show that for each $n\geq 1$ and $\boldsymbol{\varepsilon}\in\{0,1\}^n$,

$$\lim_{k\uparrow\infty}\mathbf{P}\Big(\bigcap_{i=1}^{n}\{|\pi_{k,i}-\varepsilon_i|\leq\delta_k\}\Big) \ =: \ p_{\boldsymbol{\varepsilon}} \quad \text{exists}, \tag{93}$$

and

$$\sum_{\boldsymbol{\varepsilon}}p_{\boldsymbol{\varepsilon}}=1. \tag{94}$$

Since the $\pi_{k,i}$ are $[0,1]$–valued, we can rewrite $|\pi_{k,i}-\varepsilon_i|\leq\delta_k$ as $\pi_{k,i}^{1-\varepsilon_i}(1-\pi_{k,i})^{\varepsilon_i}\leq\delta_k$. Then, by using (90), instead of (93) it is enough to verify that for each $\boldsymbol{\varepsilon}\in\{0,1\}^n$,

$$\lim_{k\uparrow\infty}\mathbf{E}\prod_{i=1}^{n}\pi_{k,i}^{1-\varepsilon_i}(1-\pi_{k,i})^{\varepsilon_i} \ =: \ p_{\boldsymbol{\varepsilon}} \quad \text{exists}. \tag{95}$$

But here again without loss of generality we can take $\boldsymbol{\varepsilon}=\mathbf{0}$, and then (95) follows from assumption (88). To finish the proof, it remains to show (94). However, by dominated convergence, from (95),

$$\sum_{\boldsymbol{\varepsilon}}p_{\boldsymbol{\varepsilon}} \ = \ \lim_{k\uparrow\infty}\mathbf{E}\sum_{\boldsymbol{\varepsilon}}\prod_{i=1}^{n}\pi_{k,i}^{1-\varepsilon_i}(1-\pi_{k,i})^{\varepsilon_i} \ = \ 1, \tag{96}$$

since the sum under the expectation sign is identical to 1. This finishes the proof. □

To get the convergence of finite-dimensional distributions it is enough to prove convergence in law of finite vectors as $\boldsymbol{\pi}_k:=\big(\hat{X}_{t_1}^{(k)}(B_1),\hat{X}_{t_2}^{(k)}(B_2),\ldots,\hat{X}_{t_n}^{(k)}(B_n)\big)$, where $B_1,\ldots,B_n$ are open (bounded) parallelepipeds in $\mathbb{R}^d$, and $0=:t_0<t_1<\cdots<t_n$.

Lemma 18 (F.d.d. convergence). *We have the following convergence in law on* $\mathbb{R}_+^n$:

$$\boldsymbol{\pi}_k \ \underset{k\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \ \big(\delta_{\xi_{t_1}}(B_1),\ldots,\delta_{\xi_{t_n}}(B_n)\big). \tag{97}$$

Proof. It is easy to see that

$$\int_{\mathbb{R}^d} \mathrm{p}_1^\alpha(t^{-1/\alpha}x - z)\,\hat{\mu}(dx) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1^\alpha(z), \qquad z \in \mathbb{R}^d. \tag{98}$$

Proceeding as in the proof of (68) and (69), but using (98) instead of (66), we get

$$\lim_{t\uparrow\infty} \mathbf{E}\hat{X}_t(t^{1/\alpha}B_1) \;=\; \lim_{t\uparrow\infty} \mathbf{E}\big(\hat{X}_t(t^{1/\alpha}B_1)\big)^2 \;=\; \int_{B_1} \mathrm{p}_1^\alpha(z)\,dz. \tag{99}$$

Hence,

$$\lim_{k\uparrow\infty} \mathbf{E}\hat{X}_{t_i}^{(k)}(B_i)\big(1-\hat{X}_{t_i}^{(k)}(B_i)\big) \;=\; 0, \qquad 1\le i\le n. \tag{100}$$

We claim that for each $n\ge 1$,

$$\lim_{k\uparrow\infty} \mathbf{E}\prod_{i=1}^n \hat{X}_{t_i}^{(k)}(B_i) \;=\; S_{\tau_1}^\alpha\Big(1_{B_1}S_{\tau_2}^\alpha\big(1_{B_2}\ldots(1_{B_{n-1}}S_{\tau_n}^\alpha 1_{B_n})\ldots\big)\Big)(0), \tag{101}$$

where $\tau_j := t_j - t_{j-1}$, $1\le j\le n$. Since the right hand side obviously equals $\mathbf{E}\prod_{i=1}^n \delta_{\xi_{t_i}}(B_i)$, then with Lemma 17 the proof of Lemma 18 will be finished. In order to verify (101), note that the indicator function 1_{B_i} of the parallelepiped B_i can monotonously be approximated from both sides by compactly supported functions (recall Remark 4). Therefore, it suffices to demonstrate that

$$\lim_{k\uparrow\infty} \mathbf{E}\prod_{i=1}^n \hat{X}_{t_i}^{(k)}(\varphi_i) \;=\; S_{\tau_1}^\alpha\Big(\varphi_1 S_{\tau_2}^\alpha\big(\varphi_2\ldots(\varphi_{n-1}S_{\tau_n}^\alpha\varphi_n)\ldots\big)\Big)(0), \qquad n\ge 1, \tag{102}$$

where $\varphi_1,\ldots,\varphi_j \le 1$ are compactly supported functions in $\mathcal{C}_1^+$.

Recall from expectation formula (11) that

$$\mathbf{E}\hat{X}_{t_1}^{(k)}(\varphi_1) \;=\; \hat{\mu}\big(S_{kt_1}^\alpha\varphi_1^{(k)}\big) \;=\; \hat{\mu}\big(S_{t_1}^\alpha\varphi_1\,(k^{-1/\alpha}\,\cdot)\big) \underset{k\uparrow\infty}{\longrightarrow} S_{t_1}^\alpha\varphi_1\,(0), \tag{103}$$

where we used the abbreviation $\varphi^{(k)} := \varphi(k^{-1/\alpha}\,\cdot)$, and

$$S_t^\alpha\varphi \;=\; S_{kt}^\alpha\varphi^{(k)}\,(k^{1/\alpha}\,\cdot), \tag{104}$$

which follows from scaling (57). Similarly, by second moment formula (12),

$$\begin{aligned}\mathbf{E}\hat{X}_{t_1}^{(k)}(\varphi_1)\,\hat{X}_{t_2}^{(k)}(\varphi_2) \;&=\; o(1)\\ &+\int_0^{kt_1} \varrho\,e^{-\varrho s}\,\hat{\mu}\Big(\big(S_{t_1-s/k}^\alpha(S_{s/k}^\alpha\varphi_1\,S_{s/k+\tau_2}^\alpha\varphi_2)\big)^{(k)}\Big)ds\end{aligned} \tag{105}$$

as $k\uparrow\infty$, where the $o(1)$–term is bounded by 1.

Because of (103), for the proof of (102) we may assume that $n\ge 2$. Then, by the Markov property and (105), the expectation at the left hand side of (102) can be written as

$$\begin{aligned}\int_0^{k\tau_{n-1}} \varrho\,e^{-\varrho s}\,\mathbf{E}\Big(\prod_{1\le i\le n-2} \hat{X}_{t_i}^{(k)}(\varphi_i)\Big) \times\\ \hat{X}_{t_{n-1}}^{(k)}\Big(\big(S_{\tau_{n-1}-s/k}^\alpha(S_{s/k}^\alpha\varphi_{n-1}\,S_{s/k\,+\,\tau_n}^\alpha\varphi_n)\big)^{(k)}\Big)ds\end{aligned} \tag{106}$$

except an $o(1)$–term, bounded by 1. It is well-known that

$$\big\|S_q^\alpha\varphi - \varphi\big\|_\infty \to 0 \quad\text{as}\quad q\downarrow 0, \qquad \varphi\in\mathcal{C}_1^+. \tag{107}$$

Therefore,

$$\left\| \left(S^{\alpha}_{\tau_{n-1}-s/k}\left(S^{\alpha}_{s/k}\varphi_{n-1}\, S^{\alpha}_{s/k+\tau_n}\varphi_n\right)\right)^{(k)} - \left(S^{\alpha}_{\tau_{n-1}}\left(\varphi_{n-1} S^{\alpha}_{\tau_n}\varphi_n\right)\right)^{(k)} \right\|_{\infty} \underset{k\uparrow\infty}{\longrightarrow} 0. \tag{108}$$

Inserting into (106), instead of (102) we need to show that

$$\int_0^{k\tau_{n-1}} \varrho\, e^{-\varrho s}\, \mathbf{E}\Big(\prod_{1\le i\le n-2} \hat{X}^{(k)}_{t_i}(\varphi_i)\Big) \hat{X}^{(k)}_{t_{n-1}}\Big(\left(S^{\alpha}_{\tau_{n-1}}\left(\varphi_{n-1} S^{\alpha}_{\tau_n}\varphi_n\right)\right)^{(k)}\Big)\, ds$$

$$\underset{k\uparrow\infty}{\longrightarrow} S^{\alpha}_{\tau_1}\Big(\varphi_1 S^{\alpha}_{\tau_2}\big(\varphi_2 \ldots (\varphi_{n-1} S^{\alpha}_{\tau_n}\varphi_n)\ldots\big)\Big)(0), \qquad n\ge 2. \tag{109}$$

But this can easily seen by induction on n. This finishes the proof. □

4.2. **Compact containment.** As a preparation of the tightness proof we establish here the following result.

Lemma 19 (Compact containment condition). *To all $\varepsilon\in(0,1]$ and $T>0$, there exists a relatively compact set $K_{\varepsilon,T}\subset\mathcal{M}_{\mathrm{f}}$ such that*

$$\inf_{k>0} \mathbf{P}\big(\hat{X}^{(k)}_t \in K_{\varepsilon,T} \text{ for all } t\le T\big) \;\ge\; 1-\varepsilon. \tag{110}$$

Proof. Recall (see Kallenberg [Kal97, A7.5]) that a subset K of $\mathcal{M}_{\mathrm{f}}$ is relatively compact if and only if

$$\sup_{\nu\in K} \nu(\mathbb{R}^d) < \infty \quad \text{and} \quad \inf_{B\in \mathrm{b}\mathcal{B}} \sup_{\nu\in K} \nu(B^{\mathrm{c}}) = 0. \tag{111}$$

Since $\hat{X}^{(k)}_t(\mathbb{R}^d)\equiv 1$, to prove lemma it is enough to show that

$$\lim_{n\uparrow\infty} \sup_{k>0} \mathbf{P}\Big(\sup_{t\le T} \hat{X}^{(k)}_t(A_n) > \varepsilon\Big) \;=\; 0, \tag{112}$$

where $A_n := \big\{x\in\mathbb{R}^d:\ |x|>n\big\}$. Let r_n denote a function in the domain of Δ_α such that $r_n(x)\le 1$ for all x, and $r_n(x)=0$ if $|x|<n-1$, as well as $r_n(x)=1$ if $|x|\ge n$. For every $k>0$ define a function $r^{(k)}_n(x) := r_n(k^{-1/\alpha}x)$. It is not difficult to see that

$$\begin{aligned} \mathbf{P}\Big(\sup_{t\le T} \hat{X}^{(k)}_t(A_n) > \varepsilon\Big) \;&\le\; \mathbf{P}\Big(\sup_{t\le kT} \hat{X}_t(r^{(k)}_n) > \varepsilon\Big) \\ &\le\; \mathbf{P}\Big(\sup_{t\le kT} \Big(\hat{X}_t(r^{(k)}_n) - \int_0^t \hat{X}_s(\Delta_\alpha r^{(k)}_n)\, ds\Big) > \frac{\varepsilon}{2}\Big) \\ &\qquad\qquad + \mathbf{P}\Big(\int_0^{kT} \hat{X}_s\big(|\Delta_\alpha r^{(k)}_n|\big)\, ds > \frac{\varepsilon}{2}\Big). \end{aligned} \tag{113}$$

Using Proposition 3, one can easily verify that

$$t \;\mapsto\; \hat{X}_t(r^{(k)}_n) - \int_0^t \hat{X}_s(\Delta_\alpha r^{(k)}_n)\, ds, \qquad t\ge 0, \tag{114}$$

is a martingale with deterministic initial position $\hat{\mu}(r^{(k)}_n)$. Hence, applying well-known Doob's inequality to the first probability expression at the right-hand side of (113), we obtain

$$\mathbf{P}\Big(\sup_{t\le kT}\Big(\hat{X}_t(r_n^{(k)})-\int_0^t \hat{X}_s(\Delta_\alpha r_n^{(k)})\,ds\Big)>\frac{\varepsilon}{2}\Big)$$
$$\le \frac{2}{\varepsilon}\,\mathbf{E}\Big|\hat{X}_{kT}(r_n^{(k)})-\int_0^{kT}\hat{X}_s(\Delta_\alpha r_n^{(k)})\,ds\Big|$$
$$\le \frac{2}{\varepsilon}\Big(\mathbf{E}\hat{X}_{kT}(r_n^{(k)})+\mathbf{E}\int_0^{kT}\hat{X}_s\big(|\Delta_\alpha r_n^{(k)}|\big)\,ds\Big). \tag{115}$$

For the other probability expression, by Markov's inequality,

$$\mathbf{P}\Big(\int_0^{kT}\hat{X}_s\big(|\Delta_\alpha r_n^{(k)}|\big)\,ds>\frac{\varepsilon}{2}\Big)\;\le\;\frac{2}{\varepsilon}\,\mathbf{E}\int_0^{kT}\hat{X}_s\big(|\Delta_\alpha r_n^{(k)}|\big)\,ds. \tag{116}$$

Exploiting expectation formula (11) to the right-hand terms of (115) and (116), we have

$$\mathbf{P}\Big(\sup_{t\le T}\hat{X}_t^{(k)}(A_n)>\varepsilon\Big)\;\le\;\frac{2}{\varepsilon}\Big(\hat{\mu}(S^\alpha_{kT}r_n^{(k)})+\int_0^{kT}\hat{\mu}\big(S^\alpha_s|\Delta_\alpha r_n^{(k)}|\big)\,ds\Big). \tag{117}$$

Obviously, $\hat{\mu}(S^\alpha_{kT}r_n^{(k)})=\hat{\mu}(S^\alpha_T r_n)\to 0$ as $n\uparrow\infty$. Further, from the self-similarity of Δ_α it follows that

$$\Delta_\alpha r_n^{(k)}(x)\;=\;k^{-1}\Delta_\alpha r_n\,(k^{-1/\alpha}x). \tag{118}$$

Consequently, $S^\alpha_s|\Delta_\alpha r_n^{(k)}|=k^{-1}S^\alpha_{s/k}|\Delta_\alpha r_n|$ and

$$\int_0^{kT}\hat{\mu}\big(S^\alpha_s|\Delta_\alpha r_n^{(k)}|\big)\,ds\;=\;\int_0^T\hat{\mu}\big(S^\alpha_z|\Delta_\alpha r_n|\big)\,dz. \tag{119}$$

By Fleischmann and Mytnik [FM03, Corollary A6], this integral converges to zero as $n\uparrow\infty$. So we have shown that the right-hand side of (117) is independent of k and goes to 0 as $n\uparrow\infty$. Thus, the proof of the lemma is finished. □

4.3. **Tightness of one-dimensional marginals.** Another prerequisite for tightness is the following lemma.

Lemma 20 (Tightness of one-dimensional marginals). *Suppose $\alpha=2$. For each $\varphi\in\mathcal{C}_1^{(2)++}$, the family $\{\hat{X}_t^{(k)}(\varphi):\,k>0\}$ is tight in law on $\mathcal{D}(\mathbb{R}_+,\mathbb{R}_+)$.*

Proof. Fix $\varphi\in\mathcal{C}_1^{(2)++}$ with $\varphi\le 1$, and $T\ge 1$. Since $\hat{X}_t^{(k)}(\varphi)\le\|\varphi\|_\infty\le 1$, by Theorem 15.2 of [Bil68] it suffices to check the following condition:

For $\varepsilon,\eta>0$, there exists a $\delta\in(0,1)$ and a $k_0>0$ such that

$$\mathbf{P}\big(w'_{\hat{X}^{(k)}(\varphi)}(\delta)\ge\varepsilon\big)\;\le\;\eta,\qquad k\ge k_0\,. \tag{120}$$

Here the modulus $w'_{\hat{X}^{(k)}(\varphi)}(\delta)$ is defined by

$$w'_x(\delta)\;:=\;\inf_{\mathbf{t}}\max_{0<i\le n}w_x\big([t_{i-1},t_i)\big)\quad\text{with}\quad w_x(I)\;:=\;\sup_{s,t\in I}|x_s-x_t| \tag{121}$$

where $\mathbf{t}$ refers to any decomposition of $[0,T]$ by means of $0=:t_0<t_1<\cdots<t_n:=T$ with the property that $t_i-t_{i-1}>\delta,\ 1\le i\le n$. Obviously,

$$\Big\{w'_{\hat{X}^{(k)}(\varphi)}(\delta)\ge\varepsilon\Big\}\;\subseteq\;\bigcup_{i=0}^{[T/\delta]+1}\Big\{w_{\hat{X}^{(k)}(\varphi)}\Big(\big[i\delta,(i+1)\delta\big)\Big)\ge\varepsilon\Big\}. \tag{122}$$

Hence,

$$\text{(123)}\qquad \mathbf{P}\Big(w'_{\hat{X}^{(k)}(\varphi)}(\delta)\geq\varepsilon\Big) \;\leq\; 2\sum_{i=0}^{[T/\delta]+1}\mathbf{P}\bigg(\sup_{0\leq t\leq\delta}\big|\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)\big|\geq\frac{\varepsilon}{2}\bigg).$$

Now, for each i,

$$\begin{aligned}\big|\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)\big| \;&\leq\; \Big|\int_{i\delta}^{i\delta+t}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds\Big| \\ \text{(124)}\qquad &\quad+\;\Big|\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)-\int_{i\delta}^{i\delta+t}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds\Big|.\end{aligned}$$

Clearly,

$$\text{(125)}\qquad \sup_{0\leq t\leq\delta}\Big|\int_{i\delta}^{i\delta+t}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds\Big| \;\leq\; \delta\,\|\Delta_\alpha\varphi\|_\infty \quad \text{a.s.}$$

Then, for $\delta\leq\varepsilon/4\|\Delta_\alpha\varphi\|_\infty$,

$$\begin{aligned}&\mathbf{P}\Big(\sup_{0\leq t\leq\delta}\big|\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)\big|\geq\frac{\varepsilon}{2}\Big)\\ \text{(126)}\qquad &\leq\;\mathbf{P}\bigg(\sup_{0\leq t\leq\delta}\Big|\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)-\int_{i\delta}^{i\delta+t}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds\Big|\geq\frac{\varepsilon}{4}\bigg).\end{aligned}$$

But $t\mapsto\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)-\int_{i\delta}^{i\delta+t}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds$ is a martingale, hence by well-known Doob's inequality,

$$\begin{aligned}&\mathbf{P}\bigg(\sup_{0\leq t\leq\delta}\Big|\hat{X}^{(k)}_{i\delta+t}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)-\int_{i\delta}^{i\delta+t}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds\Big|\geq\frac{\varepsilon}{4}\bigg)\\ \text{(127)}\qquad &\leq\;\Big(\frac{\varepsilon}{4}\Big)^{-4}\mathbf{E}\bigg(\hat{X}^{(k)}_{(i+1)\delta}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)-\int_{i\delta}^{(i+1)\delta}\hat{X}^{(k)}_s(\Delta_\alpha\varphi)\,ds\bigg)^4.\end{aligned}$$

Since $(a+b)^4\;\leq\;\big(2a^2+2b^2\big)^2\;\leq\;8a^4+8b^4$, the whole expression (127) can be estimated from above by

$$\text{(128)}\qquad c\,\varepsilon^{-4}\bigg(\mathbf{E}\Big(\hat{X}^{(k)}_{(i+1)\delta}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)\Big)^4\;+\;\delta^4\,\|\Delta_\alpha\varphi\|^4_\infty\bigg),$$

where we used (125), and c is a certain (later changing) constant. From the f.d.d. convergence (Lemma 18) and dominated convergence it follows that

$$\text{(129)}\qquad \lim_{k\uparrow\infty}\mathbf{E}\Big(\hat{X}^{(k)}_{(i+1)\delta}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)\Big)^4\;=\;\mathbf{E}\Big(\varphi(\xi_{(i+1)\delta})-\varphi(\xi_{i\delta})\Big)^4.$$

Thus, there is a $k_0>0$ such that

$$\text{(130)}\qquad \mathbf{E}\Big(\hat{X}^{(k)}_{(i+1)\delta}(\varphi)-\hat{X}^{(k)}_{i\delta}(\varphi)\Big)^4\;\leq\;2\,\mathbf{E}\Big(\varphi(\xi_{(i+1)\delta})-\varphi(\xi_{i\delta})\Big)^4,\qquad k\geq k_0\,.$$

The latter moment can actually be computed:

$$\begin{aligned}\text{(131)}\qquad &\mathbf{E}\Big(\varphi(\xi_{(i+1)\delta})-\varphi(\xi_{i\delta})\Big)^4\\ &=\;S^\alpha_{i\delta}\Big(S^\alpha_\delta\varphi^4-4\varphi S^\alpha_\delta\varphi^3+6\varphi^2S^\alpha_\delta\varphi^2-4\varphi^3S^\alpha_\delta\varphi+\varphi^4\Big)(0).\end{aligned}$$

Since S^α has generator Δ_α, to $\beta>0$ one can find $\delta_0=\delta_0(\beta)>0$ such that

$$\text{(132)}\qquad \big\|S^\alpha_\delta\varphi^j-\varphi^j-\delta\Delta_\alpha\varphi^j\big\|_\infty\;\leq\;\beta\delta,\qquad 0<\delta<\delta_0\,,\quad 1\leq j\leq 4.$$

Applying this repeatedly to (131), we get

$$
\begin{aligned}
&\mathbf{E}\Big(\varphi(\xi_{(i+1)\delta}) - \varphi(\xi_{i\delta})\Big)^4 \\
(133) \qquad &\le\ \delta\, S^{\alpha}_{i\delta}\Big|\Delta_\alpha\varphi^4 - 4\varphi\Delta_\alpha\varphi^3 + 6\varphi^2\Delta_\alpha\varphi^2 - 4\varphi^3\Delta_\alpha\varphi\Big|(0)\ +\ 4\beta\delta
\end{aligned}
$$

[note that $(1-4+6-4+1)\varphi^4 \equiv 0$]. Now we use our assumption $\alpha = 2$, since in this case $\Delta_\alpha\varphi^4 - 4\varphi\Delta_\alpha\varphi^3 + 6\varphi^2\Delta_\alpha\varphi^2 - 4\varphi^3\Delta_\alpha\varphi \equiv 0$. Consequently,

$$
(134) \qquad \mathbf{E}\Big(\varphi(\xi_{(i+1)\delta}) - \varphi(\xi_{i\delta})\Big)^4\ \le\ 4\beta\delta
$$

(actually, this moment is of order δ^2). Then from (123), (126)–(128), and (130),

$$
(135) \qquad \mathbf{P}\Big(w'_{\hat{X}^{(k)}(\varphi)}(\delta) \ge \varepsilon\Big)\ \le\ c\,T\,\delta^{-1}\varepsilon^{-4}(\beta\delta + \delta^4)\ =\ c\,T\,\varepsilon^{-4}(\beta + \delta^3).
$$

Choosing now β and δ sufficiently small, the latter probability expression can be made smaller than η, as required for (120). This finishes the proof. □

Completion of the proof of Theorem 1. Part (a) was provided by Lemma 18. Since $\{\mu \mapsto \mu(\varphi) : \varphi \in \mathcal{C}_1^{(2)++}\}$ is a family of continuous functions on $\mathcal{M}_\mathrm{f}$ that separates points, Lemmas 19 and 20 together with Jakubowski's criterion (see Theorem 3.1 of [Jak86]) yield that in the case $\alpha = 2$ the $\hat{X}^{(k)}$ are tight in law in $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_\mathrm{f})$, giving also part (b). □

4.4. **Failure of our method in non-Brownian situations.** Our method of proving tightness of one-dimensional marginals does not work if $\alpha < 2$. In fact, similarly to (129), we have for even $q \ge 2$,

$$
(136) \qquad \lim_{k\uparrow\infty} \mathbf{E}\Big(\hat{X}^{(k)}_{(i+1)\delta}(\varphi) - \hat{X}^{(k)}_{i\delta}(\varphi)\Big)^q\ =\ \mathbf{E}\Big(\varphi(\xi_{(i+1)\delta}) - \varphi(\xi_{i\delta})\Big)^q.
$$

Also, as in (131),

$$
(137) \qquad \mathbf{E}\Big(\varphi(\xi_{(i+1)\delta}) - \varphi(\xi_{i\delta})\Big)^q\ =\ S^{\alpha}_{i\delta}\bigg(\sum_{j=0}^{q}\binom{q}{j}(-\varphi)^j S^{\alpha}_{\delta}\varphi^{q-j}\bigg)(0).
$$

By (132),

$$
(138) \qquad \mathbf{E}\Big(\varphi(\xi_{(i+1)\delta}) - \varphi(\xi_{i\delta})\Big)^q\ =\ \delta\ (S^{\alpha}_{i\delta}\psi)\,(0)\ +\ o(\delta),
$$

where

$$
(139) \qquad \psi\ :=\ \sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j\Delta_a\varphi^{q-j}.
$$

Now, since $\alpha < 2$,

$$
(140) \qquad \Delta_\alpha\varphi\,(x)\ =\ \int_{\mathbb{R}^d}\bigg[\varphi(y) - \varphi(x) - \frac{\nabla\varphi\,(x)\cdot(y-x)}{1+|y-x|^2}\bigg]\,\frac{dy}{|y-x|^{d+\alpha}}\,,
$$

see, for instance, Dawson and Gorostiza [DG90, p.245]. Hence,

$$\psi(x) \;=\; \int_{\mathbb{R}^d}\left[\sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j(x)\left(\varphi^{q-j}(y)-\varphi^{q-j}(x)\right) - \frac{\left[\sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j(x)\nabla\varphi^{q-j}\,(x)\right]\cdot(y-x)}{1+|y-x|^2}\right]\frac{dy}{|y-x|^{d+\alpha}}\,, \tag{141}$$

But $\nabla\varphi^{q-j}=(q-j)\varphi^{q-j-1}\nabla\varphi$. Therefore,

$$\sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j(x)\nabla\varphi^{q-j} \;=\; \varphi^{q-1}(\nabla\varphi)\sum_{j=0}^{q-1}\binom{q}{j}(-1)^j(q-j)\;\equiv 0. \tag{142}$$

On the other hand,

$$\begin{aligned}
&\sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j(x)\left(\varphi^{q-j}(y)-\varphi^{q-j}(x)\right)\\
&\quad= \sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j(x)\varphi^{q-j}(y)-\varphi^{q}(x)\sum_{j=0}^{q-1}\binom{q}{j}(-1)^j\\
&\quad= \sum_{j=0}^{q-1}\binom{q}{j}(-\varphi)^j(x)\varphi^{q-j}(y)+\varphi^{q}(x)\;=\;\left(\varphi(y)-\varphi(x)\right)^q.
\end{aligned} \tag{143}$$

Inserting both into (141) gives

$$\psi(x) \;=\; \int_{\mathbb{R}^d}\frac{\left(\varphi(y)-\varphi(x)\right)^q}{|y-x|^{d+\alpha}}\,dy \;\;\geq 0, \tag{144}$$

which in general is different from 0 for any choice of an even q. Hence, (138) is not of a smaller order than δ as $\delta\downarrow 0$ [opposed to (134)]. Thus, for $\alpha<2$ our method of proof cannot lead to (120).

Appendix

In this section we will recall some facts about stable distributions and prove results on the total mass process $\bar{X}$.

A.1. **On stable distributions.** First of all we want to relate non-negative stable random variables to exponentially distributed ones. For this purpose, for fixed $m>0$ and $0<\gamma\leq 1$, let $\zeta_m^\gamma\geq 0$ denote a random variable with Laplace transform

$$\mathbf{E}e^{-\lambda\zeta_m^\gamma} \;=\; \exp\{-m\lambda^\gamma\},\qquad \lambda\geq 0. \tag{A1}$$

In the stable case $\gamma<1$, write q_m^γ for the density function corresponding to ζ_m^γ. Moreover, let η_m be independent of ζ_m^γ and exponentially distributed with mean $1/m$. Then

$$\mathbf{P}(\eta_1>\lambda\zeta_m^\gamma) \;=\; \mathbf{E}e^{-\lambda\zeta_m^\gamma}, \tag{A2}$$

thus, from (A1),

$$\mathbf{P}\big((\eta_1/\zeta_m^\gamma)^\gamma>\lambda^\gamma\big) \;=\; \exp\{-m\lambda^\gamma\},\qquad \lambda\geq 0. \tag{A3}$$

Consequently,

$$\text{(A4)} \qquad \Big(\frac{\eta_1}{\zeta_m^\gamma}\Big)^\gamma \stackrel{\mathcal{L}}{=} \eta_m\,.$$

(This method was proposed by Williams in [Wil77]; using this trick he obtained a representation of stable distribution as a convolution of gamma distributions.)

Obviously, the Laplace transforms (A1) are continuous in γ. That is, $\gamma_n \to \gamma$ as $n \uparrow \infty$ in $(0,1]$ implies the convergence in law $\zeta_m^{\gamma_n} \stackrel{\mathcal{L}}{\Rightarrow} \zeta_m^\gamma$. On the other hand, $\gamma \downarrow 0$ leads only to the limit law $e^{-m}\delta_0 + (1-e^{-m})\delta_\infty$ of ζ_m^γ. But under logarithmic scaling in this case

$$\text{(A5)} \qquad \gamma \log \zeta_m^\gamma \underset{\gamma\downarrow 0}{\stackrel{\mathcal{L}}{\Longrightarrow}} -\log \eta_m\,,$$

which follows from (A4).

As another consequence of (A4) we express all moments of negative order of the random variable ζ_m^γ. Indeed, (A4) and the independence of ζ_m^γ and η_1 gives

$$\text{(A6)} \qquad \mathbf{E}\eta_1^r\,\mathbf{E}(\zeta_m^\gamma)^{-r} \;=\; \mathbf{E}\eta_m^{r/\gamma}, \qquad r>0.$$

Hence, using the well-known formula $\mathbf{E}\eta_m^r = \Gamma(1+r)/m^r$, $r>0$, we get

$$\text{(A7)} \qquad \mathbf{E}(\zeta_m^\gamma)^{-r} \;=\; m^{-r/\gamma}\,\frac{\Gamma(1+r/\gamma)}{\Gamma(1+r)}\,, \qquad r>0,$$

where Γ denotes the Gamma function.

Recall the symmetric α–stable transition density functions p^α occurring in (2). We want to calculate the quantity $\mathrm{p}_1^\alpha(0)$ (which occurs in Corollary 6). For $\alpha<2$, from subordination (see, e.g., [FG86]),

$$\text{(A8)} \qquad \mathrm{p}_t^\alpha(x) \;=\; \int_0^\infty \mathrm{p}_s^2(x)\, q_t^{\alpha/2}(s)\, ds$$

(recall that $q_t^{\alpha/2}$ is the density function of the random variable $\zeta_t^{\alpha/2}$ with index $\gamma=\alpha/2$, and p^2 the heat kernel). Therefore,

$$\text{(A9)} \qquad \mathrm{p}_1^\alpha(0) \;=\; (4\pi)^{-d/2}\int_0^\infty s^{-d/2}\, q_1^{\alpha/2}(s)\, ds \;=\; (4\pi)^{-d/2}\,\mathbf{E}(\zeta_1^{\alpha/2})^{-d/2},$$

and (A7) gives

$$\text{(A10)} \qquad \mathrm{p}_1^\alpha(0) \;=\; (4\pi)^{-d/2}\,\frac{\Gamma(1+d/\alpha)}{\Gamma(1+d/2)}$$

(which is trivially true also for $\alpha=2$).

Another possible application of (A4) is the calculation of $\mathbf{E}(\log\zeta_m^\gamma)^n$ for $n = 0,1,\ldots$ In fact, taking logarithm from both sides of (A4), we have

$$\text{(A11)} \qquad \gamma\log\eta_1 - \gamma\log\zeta_m^\gamma \stackrel{\mathcal{L}}{=} \log\eta_m\,.$$

Therefore,

$$\text{(A12)} \qquad \mathbf{E}(\log\eta_1 - \log\zeta_m^\gamma)^n \;=\; \frac{1}{\gamma^n}\,\mathbf{E}(\log\eta_m)^n, \qquad n=0,1,\ldots$$

Using this relation we can express $\mathbf{E}(\log\zeta_m^\gamma)^n$ via moments $\mathbf{E}(\log\eta_1)^i$ with $i\le n$ and $\mathbf{E}(\log\eta_m)^n$ for every natural n. An alternative method was proposed by Zolotarev [Zol86, §3.6]. He has shown, that the n-th logarithmic moment of the stable random variable ζ_m^γ can be calculated as a value of the Bell polynomial $C_n(u_1,\dots,u_n)$, where $u_i := c_i\gamma^{-i}$ with c_i some absolute constants.

A.2. **Localization in the main cluster (proof of Proposition 9).** From the cluster representation (19) we have

$$\begin{aligned}\mathbf{P}(\vartheta_t^{(1)}<y) &= \mathbf{P}\Big(\pi_t\big([y,\infty)\big)=0\Big) = e^{-\lambda_t([y,\infty))}\\ &= \exp\Big[-\frac{m}{\Gamma(1-e^{-\varrho t})}\,y^{-(e^{-\varrho t})}\Big], \qquad y>0. \end{aligned}\tag{A13}$$

Substituting $y=\exp[e^{\varrho t}z]$ gives

$$\mathbf{P}\big(e^{-\varrho t}\log\vartheta_t^{(1)}<z\big) = \exp\Big[-\frac{m}{\Gamma(1-e^{-\varrho t})}\,e^{-z}\Big], \qquad z\in\mathbb{R},\tag{A14}$$

hence

$$\lim_{t\uparrow\infty}\mathbf{P}\big(e^{-\varrho t}\log\vartheta_t^{(1)}<z\big) = \exp[-m\,e^{-z}], \qquad z\in\mathbb{R}.\tag{A15}$$

Comparing with Neveu's limit theorem (1) we see that $e^{-\varrho t}\log\vartheta_t^{(1)}$ and $e^{-\varrho t}\log\bar{X}_t$ have the same limiting distribution.

Next we want to deal with $\frac{\vartheta_t^{(i)}}{\vartheta_t^{(1)}}$ for all $i\geq 2$. Clearly, for $x>y>0$,

$$\begin{aligned}\mathbf{P}\big(\vartheta_t^{(1)}\in dx,\ \vartheta_t^{(i)}\in dy\big) &= e^{-\lambda_t([x,\infty))}\,\frac{m\,e^{-\varrho t}}{\Gamma(1-e^{-\varrho t})}\,x^{-1-e^{-\varrho t}}\,dx\ \times\\ &\quad \frac{\lambda_t^{i-2}\big([y,x)\big)}{(i-2)!}\,e^{-\lambda_t([y,x))}\,\frac{m\,e^{-\varrho t}}{\Gamma(1-e^{-\varrho t})}\,y^{-1-e^{-\varrho t}}\,dy.\end{aligned}\tag{A16}$$

Hence, for $0<z\leq 1$,

$$\begin{aligned}&\mathbf{P}\Big(\frac{\vartheta_t^{(i)}}{\vartheta_t^{(1)}}<z\Big)\\ &= \frac{m^2\,e^{-2\varrho t}}{(i-2)!\,\Gamma^2(1-e^{-\varrho t})}\int_0^\infty e^{-\lambda_t([y,\infty))}\,y^{-1-e^{-\varrho t}}\int_{y/z}^\infty \lambda_t^{i-2}\big([y,x)\big)\,x^{-1-e^{-\varrho t}}\,dx\,dy.\end{aligned}$$

But by (20),

$$\lambda_t\big([y,x)\big) = \frac{m}{\Gamma(1-e^{-\varrho t})}\,\big[y^{-(e^{-\varrho t})}-x^{-(e^{-\varrho t})}\big],\tag{A17}$$

giving

$$\begin{aligned}&\int_{y/z}^\infty \lambda_t^{i-2}\big([y,x)\big)\,x^{-1-e^{-\varrho t}}\,dx\\ &= \Big(\frac{m}{\Gamma(1-e^{-\varrho t})}\Big)^{i-2}\int_{y/z}^\infty \big[y^{-(e^{-\varrho t})}-x^{-(e^{-\varrho t})}\big]^{i-2}x^{-1-e^{-\varrho t}}\,dx\\ &= \frac{1}{e^{-\varrho t}}\Big(\frac{m}{\Gamma(1-e^{-\varrho t})}\Big)^{i-2}y^{-(i-1)e^{-\varrho t}}\int_0^{z^{(e^{-\varrho t})}}(1-\tau)^{i-2}\,d\tau\\ &= \frac{1}{e^{-\varrho t}}\Big(\frac{m}{\Gamma(1-e^{-\varrho t})}\Big)^{i-2}y^{-(i-1)e^{-\varrho t}}\,\frac{1-(1-z^{(e^{-\varrho t})})^{i-1}}{(i-1)}\,.\end{aligned}\tag{A18}$$

Inserting this yields

$$\text{(A19)}\quad \mathbf{P}\Big(\tfrac{\vartheta_t^{(i)}}{\vartheta_t^{(1)}} < z\Big) = \frac{m^i\, e^{-\varrho t}}{(i-1)!\;\Gamma^i(1-e^{-\varrho t})}\Big[1-\big(1-z^{(e^{-\varrho t})}\big)^{i-1}\Big]\int_0^\infty e^{-\lambda_t([y,\infty))}\, y^{-1-e^{-\varrho t}}\,dy.$$

Now the latter integral equals

$$\text{(A20)}\qquad -\frac{1}{e^{-\varrho t}}\int_0^\infty \exp\Big[-\frac{m\,y^{-(e^{-\varrho t})}}{\Gamma(1-e^{-\varrho t})}\Big]\, y^{-(i-1)e^{-\varrho t}}\, d(y^{-(e^{-\varrho t})}) = \frac{\Gamma^i(1-e^{-\varrho t})}{m^i\, e^{-\varrho t}}\int_0^\infty e^{-x}\, x^{i-1}\,dx \;=\; \frac{\Gamma^i(1-e^{-\varrho t})}{m^i\, e^{-\varrho t}}\;(i-1)!\,.$$

Putting into (A19) gives

$$\text{(A21)}\qquad \mathbf{P}\Big(\tfrac{\vartheta_t^{(i)}}{\vartheta_t^{(1)}} \ge z\Big) \;=\; \big(1-z^{(e^{-\varrho t})}\big)^{i-1}, \qquad 0 < z \le 1.$$

Finally, substituting $z = \exp[-y\, e^{\varrho t}]$ we arrive at

$$\text{(A22)}\qquad \mathbf{P}\Big(e^{-\varrho t}\log \tfrac{\vartheta_t^{(1)}}{\vartheta_t^{(i)}} \le y\Big) \;=\; (1-e^{-y})^{i-1}, \qquad y \ge 0, \quad i \ge 2, \quad t > 0.$$

By the way, this means that the distribution of $e^{-\varrho t}\log \frac{\vartheta_t^{(1)}}{\vartheta_t^{(i)}}$ is independent(!) of t and equals the law of the maximum of $i-1$ i.i.d. standard exponentially distributed random variables. From (A22), for $0 < \varepsilon \le 1$,

$$\text{(A23)}\quad \mathbf{P}\Big(\tfrac{\vartheta_t^{(i)}}{\vartheta_t^{(1)}} \ge \frac{\varepsilon}{i^2}\Big) \;=\; \mathbf{P}\Big(e^{-\varrho t}\log \tfrac{\vartheta_t^{(1)}}{\vartheta_t^{(i)}} \;\le\; \log\big(\tfrac{i^2}{\varepsilon}\big)^{(e^{-\varrho t})}\Big) \;=\; \Big(1-\big(\tfrac{\varepsilon}{i^2}\big)^{(e^{-\varrho t})}\Big)^{i-1}.$$

Since $\sum_{i=2}^\infty i^{-2} < 1$,

$$\text{(A24)}\qquad \mathbf{P}\Big(\tfrac{\bar X_t-\vartheta_t^{(1)}}{\vartheta_t^{(1)}} \ge \varepsilon\Big) \;\le\; \sum_{i=2}^\infty \mathbf{P}\Big(\tfrac{\vartheta_t^{(i)}}{\vartheta_t^{(1)}} \ge \frac{\varepsilon}{i^2}\Big) \;=\; \sum_{i=2}^\infty \Big(1-\big(\tfrac{\varepsilon}{i^2}\big)^{(e^{-\varrho t})}\Big)^{i-1}.$$

But each summand tends to zero as $t\uparrow\infty$ and is dominated by

$$\exp\Big[-(i-1)\big(\tfrac{\varepsilon}{i^2}\big)^{(e^{-\varrho t})}\Big] \;\le\; \exp\Big[-\frac{1}{2}\, i^{1-2e^{-\varrho t}}\,\varepsilon^{(e^{-\varrho t})}\Big] \;\le\; \exp\Big[-\frac{1}{2}\, i^{1/2}\,\frac{1}{2}\Big]$$

for all sufficiently large t (for $i \ge 2$ and the fixed ε). Then the claim follows by dominated convergence, and the proof of Proposition 9 is finished. □

A.3. **More on Neveu's branching process.** Consider $\bar X$ with $\bar X_0 = m > 0$. Recall that in the notation of (A1),

$$\text{(A25)}\qquad \bar X_t \overset{\mathcal{L}}{=} \zeta_m^\gamma \quad \text{with} \quad \gamma = e^{-\varrho t}, \qquad t \ge 0.$$

Then, from (A5) we get the following weak form of Neveu's limit theorem (1):

$$\text{(A26)}\qquad e^{-\varrho t}\log \bar X_t \underset{t\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} -\log V \quad \text{with} \quad V \overset{\mathcal{L}}{=} \eta_m\,.$$

Besides (1), the following statement holds.

Proposition A1 (**Convergence in moments of all positive orders**). *For every* $m,r > 0$,

$$\text{(A27)}\qquad \mathbf{E}\Big|e^{-\varrho t}\log \bar X_t - (-\log V)\Big|^r \underset{t\uparrow\infty}{\longrightarrow} 0.$$

Proof. Fix $m > 0$. Rewriting (A11) as $\gamma \log \zeta_m^\gamma + \log \eta_m \overset{\mathcal{L}}{=} \gamma \log \eta_1$, from (A25) it follows that for some constant c_r,

$$\text{(A28)} \qquad \begin{aligned} e^{-r\varrho t}\,\mathbf{E}|\log \bar{X}_t|^r \;&\le\; c_r \Big(\mathbf{E}\,\big|\gamma \log \zeta_m^\gamma + \log \eta_m\big|^r + \mathbf{E}\,|\log \eta_m|^r\Big) \\ &=\; c_r \Big(e^{-r\varrho t}\,\mathbf{E}\,|\log \eta_1|^r + \mathbf{E}\,|\log \eta_m|^r\Big). \end{aligned}$$

Thus, the function $t \mapsto e^{-r\varrho t}\,\mathbf{E}\,|\log \bar{X}_t|^r$ is bounded on $\mathbb{R}_+$, for each $r > 0$. It means that the family $\big\{(e^{-r\varrho t} \log \bar{X}_t)^r : t \ge 0\big\}$ is uniformly integrable, for each $r > 0$. This together with Neveu's limit theorem (1) gives (A27), finishing the proof. $\square$

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