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Linear Sound Waves in Poroelastic Materials: Simple Mixtures vs. Biot's Model

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Abstract. The work contains the comparison of speeds and attenuations of P1-, S-, and P2-waves in poroelastic materials obtained within Biot's model and simple mixture model.

1 Introduction

Linear acoustic waves in saturated poroelastic materials are usually described by the model proposed by Biot in 1956 ([5], [18]). Numerous books on this subject concern primarily bulk waves ([7], [3], [19]).

This model is characterized by the following features:

– it is a linear mixture theory with two components – skeleton (solid) and fluid. The difference of their velocities describes diffusion,

– interactions of components are threefold: through the diffusion (permeability of the skeleton), through a coupling of partial stresses by volume changes of components, and through a relative acceleration usually prescribed to an influence of tortuosity.

Governing equations of the Biot's model in a chosen inertial frame of reference have the following form

$$\rho_0^S \frac{\partial \boldsymbol{v}^S}{\partial t} = \lambda^S \operatorname{grad} \operatorname{tr} \boldsymbol{e}^S + 2\mu^S \operatorname{div} \boldsymbol{e}^S + Q \operatorname{grad} \varepsilon + \pi \left(\boldsymbol{v}^F - \boldsymbol{v}^S \right) - \rho_{12} \left(\frac{\partial \boldsymbol{v}^F}{\partial t} - \frac{\partial \boldsymbol{v}^S}{\partial t} \right), \quad (1)$$

$$\rho_0^F \frac{\partial \boldsymbol{v}^F}{\partial t} = \kappa \rho_0^F \operatorname{grad} \varepsilon + Q \operatorname{grad} \operatorname{tr} \boldsymbol{e}^S - \pi \left(\boldsymbol{v}^F - \boldsymbol{v}^S \right) + \rho_{12} \left(\frac{\partial \boldsymbol{v}^F}{\partial t} - \frac{\partial \boldsymbol{v}^S}{\partial t} \right),$$

where

$$\frac{\partial \boldsymbol{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \boldsymbol{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = \operatorname{div} \boldsymbol{v}^F, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F} \equiv \operatorname{tr} \boldsymbol{e}^S - \frac{\zeta}{n_0}, \quad (2)$$

and $\boldsymbol{e}^S$ denotes the macroscopical Almansi-Hamel **deformation tensor of the skeleton**, its trace, $\operatorname{tr} \boldsymbol{e}^S$, is the **volume change** (small deformations!) **of the skeleton**, ε is the **volume change of the fluid** and this is related to the **increment of fluid content**, ζ, by the relation $(2)_3$. ρ_0^S, ρ_0^F are constant initial mass densities connected to the true mass densities ρ_0^{SR}, ρ_0^{FR} in the following way

$$\rho_0^S = (1 - n_0)\, \rho_0^{SR}, \quad \rho_0^F = n_0 \rho_0^{FR}, \quad (3)$$

where n_0 is the initial porosity. $\boldsymbol{v}^S, \boldsymbol{v}^F$ are **macroscopic velocities** of both components, i.e. $\boldsymbol{v}^F - \boldsymbol{v}^S$ is the seepage velocity. The material parameters $\lambda^S, \mu^S, \kappa, Q, \pi, \rho_{12}$ are constant and depend in a parametric way on the initial porosity n_0.

We use further the Biot's equations with all contributions of accelerations on the left hand side. Then it is convenient to introduce the notation

$$\rho_{11} = \rho_0^S - \rho_{12}, \quad \rho_{22} = \rho_0^F - \rho_{12}, \quad r = \frac{\rho_0^F}{\rho_0^S}. \quad (4)$$

The original Biot's model does not contain any information on changes of porosity. However, such a relation can be derived by means of gedankenexperiments proposed by Biot and Willis [6]. It has the following form

$$n = n_0 \left(1 + \delta \operatorname{tr} \boldsymbol{e}^S + \gamma \zeta \right), \quad (5)$$

where δ, γ are material parameters, specified by macro- and microcompressibilities.

The literature on Biot's model is far from being unique in relation to the notation and this creates a lot of confusion. The above material parameters which we shall use further in this work are characteristic for the formulation of a two-component mixture. Usually in soil mechanics use is being made of the total bulk stress $\boldsymbol{T} = \boldsymbol{T}^S + \boldsymbol{T}^F$, and the fluid partial stress is related solely to the pore pressure p. Namely $\boldsymbol{T}^F = -n_0 p \mathbf{1}$.

For this reason the material parameters are introduced, for instance, in the following way [16]

$$K := \lambda^S + \frac{2}{3}\mu^S + \rho_0^F \kappa + 2Q, \quad G := \mu^S, \tag{6}$$
$$C := \frac{1}{n_0}\left(Q + \rho_0^F \kappa\right), \quad M := \frac{\rho_0^F \kappa}{n_0^2}.$$

Let us return to the set (1). The parameter ρ_{12} describing the contribution of the relative acceleration is usually related to the **tortuosity** of the porous material. For example, in the works [4], [12] the following approximate relation between this parameter, the porosity n_0, and the tortuosity parameter $a \in [1, \infty)$, is proposed

$$\rho_{12} = \rho_0^F (1 - a), \quad a = \frac{1}{2}\left(\frac{1}{n_0} + 1\right). \tag{7}$$

In spite of its popularity, the Biot's model possesses a number of weak points which are ignored by the Biot's community. We mention here the three most important weaknesses. First of all, the contribution of relative accelerations violates the principle of material objectivity. Secondly, the coupling of partial stresses violates the second law of thermodynamics. Thirdly, the added mass effect which is identified with the influence of tortuosity yields an unnatural reduction of attenuation of acoustic waves.

The lack of material objectivity of the model (1) is immediately visible. The change of the reference frame to a noninertial system (a time dependent change of observer) is described by the relation

$$\boldsymbol{x}^* = \mathbf{O}(t)\,\boldsymbol{x} + \boldsymbol{c}(t), \quad \boldsymbol{O}^T = \boldsymbol{O}^{-1}, \tag{8}$$

where $\boldsymbol{O}, \boldsymbol{c}$ are arbitrary. This transformation performed in Biot's equations yields constitutive contributions in these equations following from the presence of the relative acceleration. They are additional to the usual centrifugal, Coriolis, Euler and translational accelerations which are characteristic for the continuum mechanics in noninertial frames (e.g. I-Shih Liu [15]). The problem has been investigated in the paper [20].

The question arises if one could overcome this difficulty by assuming that the nonobjectivity follows from the linearization of some objective nonlinear equations. If this was the case, one would have to describe porous materials by Biot's equations solely in inertial reference systems and a time dependent change of reference would require an addition of classical acceleration terms and ignoring contributions from the relative acceleration. Such a procedure is indeed possible. One can define an objective relative acceleration which contains some nonlinear contributions similar to those appearing in objective time derivatives of, say, Jaumann, Oldroyd, or Truesdell. Then it can be shown [28] that a nonlinear poroelastic two-component model with the contribution of this objective acceleration yields the Biot's added mass term by linearization.

Let us mention in passing that the lack of relative accelerations in a model does not mean that the influence of tortuosity is neglected. Certainly, the permeability of the material described by the parameter π in our notation contains an influence of the morphology of the porous materials and this includes an influence of tortuosity.

The second flaw of the Biot's model, the violation of the second law by the coupling of partial stresses, has been extensively discussed in the work [23]. It has been shown that, as in the case of classical mixtures of fluids, such a coupling may appear solely in models in which a list of constitutive variables contains gradients of fundamental fields. In the theory of mixture of fluids, it was a dependence on gradients of mass densities. In the case of porous materials, it is sufficient to introduce, for example, a dependence on the gradient of porosity. Otherwise, the second law of thermodynamics reduces the model to the so-called **simple mixture model** in which the coupling between partial stresses must vanish: $Q = 0$. However, one can also show [27] that the extension on the gradient of porosity yields very small contributions in the linear model. Consequently, the Biot's model in its linear form quoted above can be considered to be an acceptable approximation of the thermodynamically admissible model.

However, nonlinear extensions of the Biot's model describing such phenomena as dilatancy, swelling, coupling of shearing with expansion, fluidization, plastic deformations cannot be based on *ad hoc* terms in the above equations as it is done sometimes in the literature. When we start the analysis of such nonlinear problems with the simplest poroelastic model we can construct a simple mixture model without added mass effects and coupling of partial stresses but with an additional balance equation for porosity [24], [21], [26]. Linearization of such a model yields equations (1) with $Q = 0$, $\rho_{12} = 0$ and with the balance equation of porosity. The latter can be formally solved and yields viscous effects related to relaxation properties of porosity. In soil mechanics, one can usually neglect them (i.e one can assume the limit $\tau \to \infty$, where τ is the relaxation time of porosity) and then this formal solution is identical with (5). It was demonstrated in a series of works (for review see e.g. [22], [25]) that such a model yields the behavior of bulk waves which is in a good qualitative agreement with observations. Simultaneously, the technical complexity of the simple mixture model is much less than this of Biot's model what provides the possibility of an investigation of such problems as surface waves for various boundaries and their parameter analysis in the whole range of frequencies (e.g. [10], [1]).

In this work, we present some results of comparison of the Biot's model and of the simple mixture model of poroelastic materials in application to the analysis of acoustic bulk waves.

2 Propagation of fronts of acoustic waves in Biot's model

We begin the analysis of the system (1) by proving its hyperbolicity. To this aim we consider the propagation of the front $\mathcal{S}$ of the weak discontinuity wave, i.e. of a singular surface on which

$$\left[\!\left[\boldsymbol{v}^S\right]\!\right] = 0, \quad \left[\!\left[\boldsymbol{v}^F\right]\!\right] = 0, \tag{9}$$

where $[\![\ldots]\!]$ denotes the jump of the quantity. On such a surface the accelerations may be discontinuous and we call their jumps the **amplitudes of discontinuity**

$$\boldsymbol{a}^S := \left[\!\left[\frac{\partial \boldsymbol{v}^S}{\partial t}\right]\!\right], \quad \boldsymbol{a}^F := \left[\!\left[\frac{\partial \boldsymbol{v}^F}{\partial t}\right]\!\right]. \tag{10}$$

Then the following compatibility conditions hold

$$\left[\!\left[\operatorname{grad} \boldsymbol{v}^S\right]\!\right] = -\frac{1}{c}\boldsymbol{a}^S \otimes \boldsymbol{n}, \quad \left[\!\left[\operatorname{grad} \boldsymbol{v}^F\right]\!\right] = -\frac{1}{c}\boldsymbol{a}^F \otimes \boldsymbol{n}, \tag{11}$$

$$\left[\!\left[\operatorname{grad} \boldsymbol{e}^S\right]\!\right] = -\frac{1}{c}\left[\!\left[\frac{\partial \boldsymbol{e}^S}{\partial t}\right]\!\right] \otimes \boldsymbol{n}, \quad \left[\!\left[\operatorname{grad} \varepsilon\right]\!\right] = -\frac{1}{c}\left[\!\left[\frac{\partial \varepsilon}{\partial t}\right]\!\right] \boldsymbol{n},$$

where c is the speed of propagation of the surface $\mathcal{S}$ and $\boldsymbol{n}$ its unit normal vector. The latter gives, of course, the direction of propagation of the wave.

Bearing (2) in mind we obtain immediately

$$\begin{aligned} \left[\!\left[\operatorname{grad} \boldsymbol{e}^S\right]\!\right] &= \frac{1}{2c^2}\left(\boldsymbol{a}^S \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{a}^S\right) \otimes \boldsymbol{n}, \\ \left[\!\left[\operatorname{grad} \varepsilon\right]\!\right] &= \frac{1}{c^2}\boldsymbol{a}^S \cdot \boldsymbol{n}\boldsymbol{n}. \end{aligned} \tag{12}$$

We evaluate the jump of field equations (1) on the surface $\mathcal{S}$. We obtain easily

$$\begin{aligned} \left[\rho_{11}c^2\mathbf{1} - \lambda^S\boldsymbol{n} \otimes \boldsymbol{n} - \mu^S\left(\mathbf{1} + \boldsymbol{n} \otimes \boldsymbol{n}\right)\right]\boldsymbol{a}^S + \left[\rho_{12}c^2\mathbf{1} - Q\boldsymbol{n} \otimes \boldsymbol{n}\right]\boldsymbol{a}^F = 0, \\ \left[\rho_{12}c^2\mathbf{1} - Q\boldsymbol{n} \otimes \boldsymbol{n}\right]\boldsymbol{a}^S + \left[\rho_{22}c^2\mathbf{1} - \kappa\rho_0^F\boldsymbol{n} \otimes \boldsymbol{n}\right]\boldsymbol{a}^F = 0. \end{aligned} \tag{13}$$

We say that the system (1) is **hyperbolic** if the eigenvalues c of the above eigenvalue problem are real and the corresponding eigenvectors $\left[\boldsymbol{a}^S, \boldsymbol{a}^F\right]$ linearly independent. We check these conditions.

It is convenient to separate the transversal and longitudinal parts of the problem (13). The **transversal** part follows if we take the scalar product of the equations with a vector $\boldsymbol{n}_\perp$ perpendicular to $\boldsymbol{n}$. We obtain

$$\begin{aligned} \left(\rho_{11}c^2 - \mu^S\right)a_\perp^S + \rho_{12}c^2 a_\perp^F = 0, \quad \rho_{12}a_\perp^S + \rho_{22}a_\perp^F = 0, \\ a_\perp^S := \boldsymbol{a}^S \cdot \boldsymbol{n}_\perp, \quad a_\perp^F := \boldsymbol{a}^F \cdot \boldsymbol{n}_\perp. \end{aligned} \tag{14}$$

Hence we have for the speed of the front

$$c^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\mu^S. \tag{15}$$

As $\rho_{22} > 0$, $\mu^S > 0$, the **first condition for hyperbolicity** of the set (1) follows

$$a - r\left(1 - a\right) > 0. \tag{16}$$

This condition is obviously fulfilled because a is not smaller than 1.

The speed of propagation (15) describes the shear wave. It is not influenced by the coupling parameter Q. It is easy to see that in the particular case without the influence of tortuosity $a = 1$ (simple mixture model) this relation reduces to the classical formula $c = \sqrt{\mu^S/\rho_0^S}$. In this case, according to $(14)_2$, the amplitude in the fluid $a_\perp^F$ is zero, i.e. the shear wave is carried solely by the skeleton.

We proceed to the **longitudinal** part. To this aim, we take the scalar product of the relations (13) with the vector $\boldsymbol{n}$. It follows

$$\begin{aligned} \left[\rho_{11}c^2 - \left(\lambda^S + 2\mu^S\right)\right]\boldsymbol{a}^S \cdot \boldsymbol{n} + \left[\rho_{12}c^2 - Q\right]\boldsymbol{a}^F \cdot \boldsymbol{n} = 0, \\ \left[\rho_{12}c^2 - Q\right]\boldsymbol{a}^S \cdot \boldsymbol{n} + \left[\rho_{22}c^2 - \kappa\rho_0^F\right]\boldsymbol{a}^F \cdot \boldsymbol{n} = 0, \end{aligned} \tag{17}$$

and the dispersion relation is as follows

$$r\left[\left(1 - r\left(1 - a\right)\right)c^2 - c_{P1}^2\right]\left[ac^2 - c_{P2}^2\right] - \left[r\left(1 - a\right)c^2 - \frac{Q}{\rho_0^S}\right]^2 = 0, \tag{18}$$

where

$$c_{P1}^2 := \frac{\lambda^S + 2\mu^S}{\rho_0^S}, \quad c_{P2}^2 := \kappa. \tag{19}$$

The eigenvalues of this problem have the form

$$c^2 = \frac{1}{2r\left[a - r\left(1-a\right)\right]}\left[A \pm \sqrt{B}\right], \tag{20}$$

where

$$A := rac_{P1}^2 + \left[1 - r\left(1-a\right)\right] rc_{P2}^2 - 2\frac{Q}{\rho_0^S} r\left(1-a\right), \tag{21}$$
$$B := A^2 - 4r\left[a - r\left(1-a\right)\right]\left[c_{P1}^2 c_{P2}^2 r - \frac{Q^2}{\rho_0^{S2}}\right].$$

It can be easily shown that under the condition (16) $B > 0$ for all $a \geq 1$, $Q \geq 0$. However, c^2 defined by (20) is positive solely if the additional condition on Q is satisfied

$$Q \leq \rho_0^S \sqrt{r} c_{P1} c_{P2} \equiv \sqrt{\rho_0^F \kappa\left(\lambda^S + 2\mu^S\right)}. \tag{22}$$

This is the **second condition for hyperbolicity** of Biot's equations.

In the particular case $a = 1$, $Q = 0$ we have c equal to either c_{P1} or c_{P2} which means that the set is unconditionally hyperbolic.

The two solutions for c^2 define two longitudinal modes of propagation, P1 and P2. The P2-mode, called the Biot's wave or the **slow wave** in the theory of porous materials, is also known as the **second sound** and it appears in all two-component systems described by hyperbolic field equations. For instance, it is known in the theory of binary mixtures of fluids in which it is applied to describe dynamical properties of liquid helium as discovered by L. Tisza in 1938 [17]. For porous materials, it has been discovered by Ya. Frenkel in 1944 [11].

3 Monochromatic acoustic waves

The above analysis yields solely the propagation properties of the wave front $\mathcal{S}$. In order to analyze the attenuation we investigate monochromatic waves. We shall see that speeds of fronts follow in the limit of frequency $\omega \to \infty$.

We seek solutions of equations (1) which have the form of the following monochromatic waves

$$\boldsymbol{v}^S = \boldsymbol{V}^S \mathcal{E}, \quad \boldsymbol{v}^F = \boldsymbol{V}^F \mathcal{E}, \quad \boldsymbol{e}^S = \boldsymbol{E}^S \mathcal{E}, \quad \varepsilon = E^F \mathcal{E}, \tag{23}$$
$$\mathcal{E} := \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right],$$

where $\boldsymbol{V}^S, \boldsymbol{V}^F, \boldsymbol{E}^S, E^F$ are constant amplitudes, $\boldsymbol{k}$ is the wave vector, ω real frequency.

Substitution of this ansatz in field equations yields the following compatibility conditions

$$\left[\rho_{11}\omega^2 \mathbf{1} - \lambda^S \boldsymbol{k}\otimes\boldsymbol{k} - \mu^S\left(k^2\mathbf{1} + \boldsymbol{k}\otimes\boldsymbol{k}\right) + \mathrm{i}\pi\omega\mathbf{1}\right]\boldsymbol{V}^S +$$
$$+\left[\rho_{12}\omega^2\mathbf{1} - Q\boldsymbol{k}\otimes\boldsymbol{k} - \mathrm{i}\pi\omega\mathbf{1}\right]\boldsymbol{V}^F = 0, \tag{24}$$
$$\left[\rho_{12}\omega^2\mathbf{1} - Q\boldsymbol{k}\otimes\boldsymbol{k} - \mathrm{i}\pi\omega\mathbf{1}\right]\boldsymbol{V}^S + \left[\rho_{22}\omega^2\mathbf{1} - \kappa\rho_0^F\boldsymbol{k}\otimes\boldsymbol{k} + \mathrm{i}\pi\omega\mathbf{1}\right]\boldsymbol{V}^F = 0.$$

As usual, the problem of existence of such waves reduces to the eigenvalue problem with the eigenvector $\left[\boldsymbol{V}^S, \boldsymbol{V}^F\right]$. We split again the problem into two parts: in the direction $\boldsymbol{k}_\perp$ perpendicular to $\boldsymbol{k}$ (transversal modes) and in the direction of the wave vector $\boldsymbol{k}$ (longitudinal modes).

For transversal modes (monochromatic shear waves) we have

$$\begin{gathered}\left[\rho_{11}\omega^2 - \mu^S k^2 + \mathrm{i}\pi\omega\right] V_\perp^S + \left[\rho_{12}\omega^2 - \mathrm{i}\pi\omega\right] V_\perp^F = 0, \quad k^2 = \boldsymbol{k}\cdot\boldsymbol{k}, \\ \left[\rho_{12}\omega^2 - \mathrm{i}\pi\omega\right] V_\perp^S + \left[\rho_{22}\omega^2 + \mathrm{i}\pi\omega\right] V_\perp^F = 0, \\ V_\perp^S = \boldsymbol{V}^S \cdot \boldsymbol{k}_\perp, \quad V_\perp^F = \boldsymbol{V} n^F \cdot \boldsymbol{k}_\perp. \end{gathered} \tag{25}$$

The solution of the dispersion relation follows in this case in the following form

$$\left(\frac{\omega}{k}\right)^2 = \frac{\omega r a + \mathrm{i}\frac{\pi}{\rho_0^S}}{\omega r\left[a - r(1-a)\right] + \mathrm{i}\frac{\pi}{\rho_0^S}(1+r)} c_S^2, \quad c_S^2 = \frac{\mu^S}{\rho_0^S}. \tag{26}$$

Consequently, neither the phase speed $\omega/\operatorname{Re}k$ nor the attenuation $\operatorname{Im}k$ of monochromatic shear waves is dependent on the coupling coefficient Q.

In the two limits of frequencies we have then the following solutions

$$\omega \to 0: \quad \lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re}k}\right)^2 = \frac{\mu^S}{\rho_0^S + \rho_0^F}, \quad \lim_{\omega\to 0}\left(\operatorname{Im}k\right) = 0,$$

$$\begin{aligned} \omega \to \infty: \quad & \lim_{\omega\to\infty}\left(\frac{\omega}{\operatorname{Re}k}\right)^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\mu^S, \\ & \lim_{\omega\to\infty}\left(\operatorname{Im}k\right) = \frac{\pi}{2\sqrt{\rho_0^S\mu^S}}\frac{1}{a^2}\sqrt{\frac{a}{a - r(1-a)}}. \end{aligned} \tag{27}$$

The first result checks with the results of the classical one-component model commonly used in soil mechanics. The speed in the second one is identical with this of formula (15). Hence the propagation of the front of shear waves is identical with the propagation of monochromatic waves of infinite frequency. The attenuation in this limit is finite.

For longitudinal modes we obtain the dispersion relation

$$\begin{gathered} \omega\left\{\left[1 - r(1-a)\right]\left(\frac{\omega}{k}\right)^2 - c_{P1}^2\right\}\left\{a\left(\frac{\omega}{k}\right)^2 - c_{P2}^2\right\} + \\ + \frac{1}{r}\mathrm{i}\frac{\pi}{\rho_0^S}\left(\frac{\omega}{k}\right)^2\left\{(1+r)\left(\frac{\omega}{k}\right)^2 - r c_{P2}^2 - c_{P1}^2 - 2\frac{Q}{\rho_0^S}\right\} - \\ - \frac{1}{r}\omega\left\{r(1-a)\left(\frac{\omega}{k}\right)^2 - \frac{Q}{\rho_0^S}\right\}^2 = 0. \end{gathered} \tag{28}$$

Let us check again two limits of frequencies: $\omega \to 0$, and $\omega \to \infty$.

In the first case we obtain

$$\begin{aligned} & \omega \to 0: \quad c_0 := \lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re}k}\right), \\ & c_0^2\left\{(1+r)c_0^2 - r c_{P2}^2 - c_{P1}^2 + 2\frac{Q}{\rho_0^S}\right\} = 0, \quad \lim_{\omega\to 0}\left(\operatorname{Im}k\right) = 0. \end{aligned} \tag{29}$$

Obviously, we obtain two real solutions of this equation

$$\lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re}k}\right)^2\Big|_1 := c_{oP1}^2 = \frac{c_{P1}^2 + r c_{P2}^2 + 2\frac{Q}{\rho_0^S}}{1+r} \equiv \frac{\lambda^S + 2\mu^S + \rho_0^F\kappa + 2\frac{Q}{\rho_0^S}}{\rho_0^S + \rho_0^F}, \tag{30}$$
$$\lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re}k}\right)^2\Big|_2 := c_{oP2}^2 = 0.$$

These are squares of speeds of propagation of two longitudinal modes in the limit of zero frequency. Clearly, the second mode, P2-wave, does not propagate in this limit. Both limits are independent of tortuosity. The result (30) checks with the relation for the speed of longitudinal waves used in the classical one-component model of soil mechanics provided $Q = 0$.

In the second case we have

$$\omega \to \infty: \quad c_\infty := \lim_{\omega\to\infty}\left(\frac{\omega}{\operatorname{Re}k}\right),$$
$$r\left\{[1 - r(1-a)]\, c_\infty^2 - c_{P1}^2\right\}\left\{a c_\infty^2 - c_{P2}^2\right\} - \left\{r(1-a)\, c_\infty^2 - \frac{Q}{\rho_0^S}\right\}^2 = 0. \tag{31}$$

This coincides with the relation (18), i.e. speeds in the limit $\omega \to \infty$ are identical with speeds of fronts.

Simultaneously, we obtain the following attenuation in the limit of infinite frequencies

$$\lim_{\omega\to\infty}(\operatorname{Im}k) = \frac{\pi\varGamma_1}{2\rho_0^S r\varGamma_2}, \tag{32}$$

$$\varGamma_1 = c_\infty\left[1 + r - \frac{1}{c_\infty^2}\left(c_{P1}^2 + r c_{P2}^2 + 2\frac{Q}{\rho_0^S}\right)\right],$$
$$\varGamma_2 = c_{P1}^2\left(a - \frac{c_{P2}^2}{c_\infty^2}\right) + c_{P2}^2\left(1 - r(1-a) - \frac{c_{P1}^2}{c_\infty^2}\right) + 2\frac{Q}{\rho_0^S}\left(1 - a - \frac{Q}{r\rho_0^S c_\infty^2}\right).$$

Hence both limits of attenuation for the P1-wave and P2-wave are finite.

We proceed to the presentation of a numerical result in the whole range of frequencies $\omega \in [0,\infty)$. We use the following numerical data

$$c_{P1} = 2500\,\frac{\mathrm{m}}{\mathrm{s}}, \quad c_{P2} = 1000\,\frac{\mathrm{m}}{\mathrm{s}}, \quad c_S = 1500\,\frac{\mathrm{m}}{\mathrm{s}},$$
$$\rho_0^S = 2500\,\frac{\mathrm{kg}}{\mathrm{m}^3}, \quad r = 0.1, \quad \pi = 10^8\,\frac{\mathrm{kg}}{\mathrm{m}^3\mathrm{s}}, \tag{33}$$
$$Q = 0.8\ \mathrm{GPa}, \quad n_0 = 0.4, \quad a = 1.75.$$

Speeds c_{P1}, c_{P2}, c_S, the mass density ρ_0^S (i.e. $\rho_0^{SR} = 4167\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ for the porosity $n_0 = 0.4$) and the fraction $r = \rho_0^F/\rho_0^S$ possess values typical for many granular materials under a confining pressure of a few atmospheres and saturated by water. In units standard for soil mechanics the permeability π corresponds to app. 0.1 Darcy. The coupling coefficient Q has been estimated by means of the Gassmann relation (e.g. [27]). The tortuosity coefficient $a = 1.75$ follows from Berryman formula $(7)_2$.

Transversal waves described by the relation (26) are characterized by the following distribution of speeds and attenuation in function of frequency (Fig. 1). The solid lines correspond to the solution of Biot's model and the dashed lines to the solution of the simple mixture model.

It is clear that the qualitative behavior of the speed of propagation is the same in both models. It is a few percent smaller in Biot's model than this in the simple mixture model in the range of high frequencies. A large quantitative difference between these models appears for the attenuation. In the range of higher frequencies it is much smaller in the Biot's model, i.e. tortuosity decreases the dissipation of shear waves.

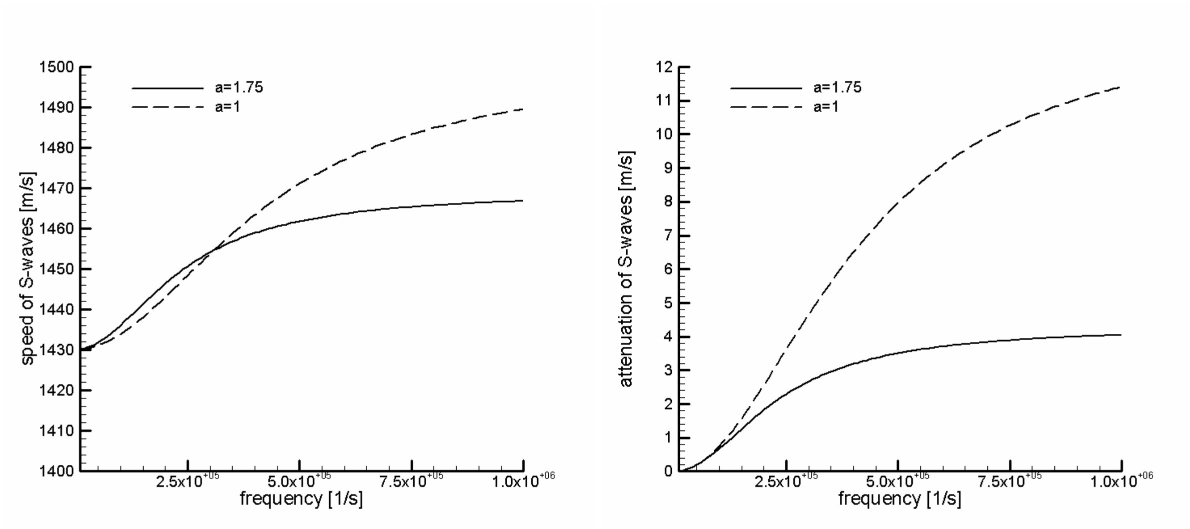

Fig. 1. Speed of propagation and attenuation of monochromatic S-waves for two values of the tortuosity coefficient $a:\ 1.75$ (Biot), 1.00 (simple mixture)

The latter property is illustrated in Fig. 2 where we plot the attenuation of the front of shear waves, i.e. $\lim_{\omega\to\infty} \operatorname{Im} k$, as a function of the tortuosity coefficient a. This behavior of attenuation indicates that damping of waves created by the tortuosity, which is connected in the macroscopic model to the relative velocity of components, is not related to scattering of waves on the microstructure. It is rather related to the decrease of the macroscopic diffusion velocity in comparison with the difference of velocities on the microscopic level due to the curvature of channels and volume averaging. Fluctuations are related solely to this averaging and not to temporal deviation from time averages (lack of ergodicity!).

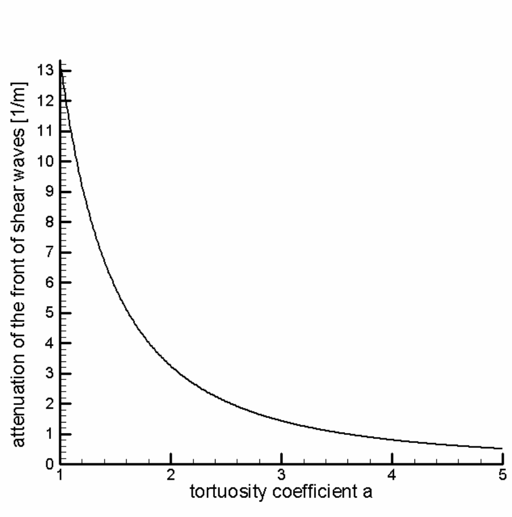

Fig. 2. Attenuation of the front of shear waves in function of the tortuosity coefficient a

We proceed to longitudinal waves. The solid lines on the following figures correspond again to Biot's model, the dashed lines to the simple mixture model. In order to show separately the influence of tortuosity a and of the coupling Q we plot as well the solutions with $a = 1$ (dashed dotted lines) and the solutions with $Q = 0$ (dashed double dotted lines).

Even though similar again the quantitative differences are much more substantial for P1-waves (Fig. 3). This is primarily an influence of the coupling through partial stresses described by the parameter Q. The simple mixture model ($Q = 0, a = 1$) as well as Biot's model with $Q = 0$ yield speeds of these waves different only a few percent (lower curves in the left diagram). The coupling Q shifts the curves to higher values and reduces the difference caused by the tortuosity. This result does not seem to be very realistic because the real differences between low frequency and high frequency speeds were measured in soils to be rather as big as indicated by the simple mixture model. This may be an indication that Gassmann relations give much too big values of the coupling parameter Q with respect to these indeed appearing in real granular materials.

Both the tortuosity a and the coupling Q reduce the attenuation quite considerably as indicated in the right figure.

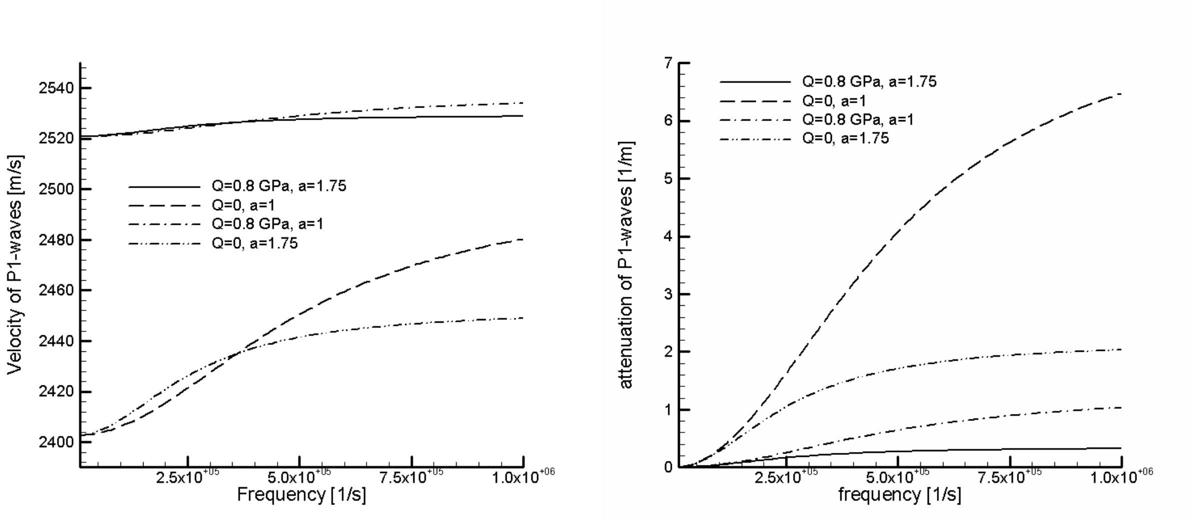

Fig. 3. Speed of propagation and attenuation of monochromatic P1-waves for various coupling parameters Q and tortuosity coefficients a

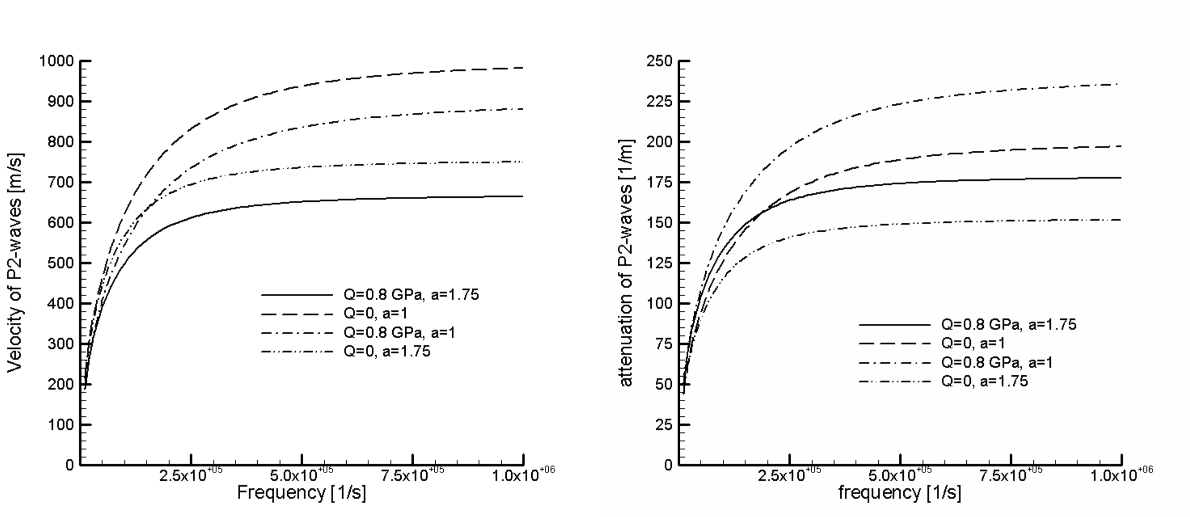

Fig. 4. Speed of propagation and attenuation of monochromatic P2-waves for various coupling parameters Q and tortuosity coefficients a

In spite of some claims in the literature the tortuosity a does not influence the existence of the slow (P2-) wave (Fig. 4). Speeds of this wave are again qualitatively similar in Biot's model and in the simple mixture model. The maximum differences appear in the range of high frequencies and reach some 35 percent. The same concerns the attenuation even though quantitative differences are not so big (app. 8 percent).

4 Conclusions

The analysis presented in this work yields the following conclusions.

1° We have demonstrated on properties of acoustic waves that relative accelerations and coupling through partial stresses in the Biot's model have a quantitative but not qualitative influence on results. We have compared results for Biot's model with these for the simple mixture model in which the tortuosity $a = 1$ and the coupling parameter $Q = 0$. We have proven that both models are hyperbolic provided the parameter Q satisfies a condition bounding this parameter from above. In particular, both models predict the existence of the P2-wave. Speeds and attenuations of monochromatic P1-, P2- and S-waves are qualitatively the same but there are quantitative discrepancies.

2° Tortuosity introduced to the model through the relative acceleration yields dissipation solely due to the modification of the relative motion. Namely if we assume the permeability coefficient $\pi = 0$ the dissipation in isothermal processes without relaxation of porosity vanishes. This is due to the fact that tortuosity, in contrast to porosity, is not introduced as a field described by its own field equation. It explains of a rather unexpected behavior of attenuation of monochromatic waves within the Biot's model. Inspection of figures shown in this work makes clear that the presence of tortuosity $a \neq 1$ yields a smaller attenuation rather than bigger as it would be in the case of a dissipative field. A dependence of the permeability π on tortuosity would eliminate this paradoxon. Then the added mass effect could be left out in the model as it is done in the simple mixture model.

3° We have demonstrated that a rather moderate value of the parameter Q suggested by the classical Gassmann relation for granular materials leads to an unreasonable increment of speeds of propagation and reduction of attenuation. In addition, the speed of propagation of monochromatic P1-waves becomes very flat as a function of frequency. This contradicts observations in soil mechanics and geotechnics and indicates that the Gassmann relations predict too big values of this parameter.

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