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Integral manifolds for slow-fast differential systems loosing their attractivity in time

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Abstract

The work is devoted to the investigation of the integral manifolds of the nonautonomous slow-fast systems, which change their attractivity in time. The method used here is based on gluing attractive and repulsive integral manifolds by using an additional function.

1 Introduction.

Systems of differential equations with several time-scales play an important role in modeling processes in reaction kinetics [2], biophysics [6], and also in modern technology (e.g. dynamics of semiconductor lasers [7]). In the paper at hand we restrict ourselves to systems of ordinary differential equations with two-time scales in the slow-fast form

$$\begin{aligned}\frac{dy}{dt} &= \varepsilon\ f(t,y,z,\varepsilon),\\ \frac{dz}{dt} &= B(t)z+\tilde{g}(t,y,z,\varepsilon),\end{aligned}$$

(1.1)

where ε is a small parameter, $y\in R^n,\ z\in R^2$. We assume $\tilde{g}(t,y,0,0)\equiv 0$ so that $z\equiv 0$ is an integral manifold of (1.1) for $\varepsilon=0$. Our goal is to establish the existence of an integral manifold $\mathcal{M}_\varepsilon$ of (1.1) for sufficiently small ε with the representation

$$z=h(t,y,\varepsilon), \tag{1.2}$$

where h is uniformly bounded and tends to zero as $\varepsilon\to 0$. Under the crucial assumption that the linear system

$$\frac{dz}{dt}=B(t)z$$

exhibits an exponential dichotomy, the existence of an integral manifold of system (1.1) in the form (1.2) has been established in several papers (see e.g. the books

[3, 5, 11]). The peculiarity of this paper consists in proving the existence of such an integral manifold under the assumption that $B(t)$ has the form

$$B(t) = \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}. \tag{1.3}$$

We note that $B(t)$ has a pair of complex conjugate eigenvalues that cross the imaginary axis from left to right for increasing t at the moment $t = 0$. In that case, it can be checked easily that for $\varepsilon = 0$ the hyperplane $z \equiv 0$ is attracting for $t < 0$ and repelling for $t > 0$. Thus, we say that the integral manifold $z \equiv 0$ looses its attractivity for increasing t at $t = 0$. As a first step in treating this problem we consider in the next section the two-dimensional system

$$\frac{dz}{dt} = B(t)z + \eta(t, z) \tag{1.4}$$

where $B(t)$ is defined by (1.3). We will show that it has a solution bounded for all t only under a special condition on the function η. To be able to fulfil the corresponding condition for the existence of a bounded integral manifold $\mathcal{M}_\varepsilon$ for system (1.1) we include some control u into the function $\tilde{g}$, that is, we consider the slow-fast system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon\, f(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= B(t)z + g(t, y, z, u, \varepsilon), \end{aligned}$$

(1.5)

where u belongs to some control set U.

The paper is organized as follows. In the next section we derive a necessary condition for equation (1.4) to have a uniformly bounded solution. Section 3 contains the hypotheses on the right hand side of system (1.5), and also our main result. In section 4 we derive a necessary condition for the existence of a bounded integral manifold $\mathcal{M}_\varepsilon$ with the representation (1.2) for system (1.5). This condition will be used in section 5 to determine the control function u as a fixed point of some operator P in U. Section 6 is devoted to the existence of a unique fixed point of the operator T introduced in section 4. This fixed point yields the integral manifold $\mathcal{M}_\varepsilon$ to system (1.5) for sufficiently small ε. We close with some simple example.

2 Bounded solutions in case of missing dichotomy.

Let $G \in R^2$ be a connected set containing the origin. We consider the system of ordinary differential equations

$$\frac{dz}{dt} = B(t)z + \eta(t,z) \tag{2.1}$$

for $z \in G$, where the matrix $B(t)$ is defined by

$$B(t) := \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}. \tag{2.2}$$

Concerning the function η we assume

(H). $\eta : R \times G \to R^2$ is continuous and such that to any given (t_0, z_0) the Cauchy problem to (2.1) has a unique solution defined for $t \in R$.

First we consider the linear system

$$\frac{dz}{dt} = B(t)z, \tag{2.3}$$

which has the fundamental matrix

$$V(t,t_0) := e^{\frac{1}{2}(t^2-t_0^2)}W(t-t_0), \tag{2.4}$$

where $W(t)$ is defined by

$$W(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2.5}$$

If we denote by $|\cdot|$ the Euclidean norm and by $||\cdot||$ the corresponding matrix norm, then we get from (2.4), (2.5)

$$||V^{-1}(t,t_0)|| = ||e^{\frac{1}{2}(t_0^2-t^2)}W^{-1}(t-t_0)|| \le e^{\frac{1}{2}(t_0^2-t^2)},$$

that is, we have

$$\lim_{t\to\pm\infty} ||V^{-1}(t,t_0)|| = 0. \tag{2.6}$$

Furthermore, the general solution $z(t;t_0,z_0) = V(t,t_0)z_0$ of (2.3) satisfies

$$|z(t;t_0,z_0)| \le |z_0|e^{\frac{1}{2}(t^2-t_0^2)}.$$

Hence, the solution $z \equiv 0$ of the linear system (2.3) is exponentially attracting for $t < 0$ and exponentially repelling $t > 0$. Moreover, the following canard-like effect can be observed: The trajectory of system (2.3) starting for $t = t_0 < 0$ at any initial point $z_0 \neq 0$ enters after a short time interval a small neighbourhood of the solution $z \equiv 0$ and stays in it until some time $t = t^* > 0$. For $t > |t_0|$ the trajectory grows exponentially.

A solution $z(t; t_0, z_0)$ of the nonlinear system (2.1) satisfying $z(t_0; t_0, z_0) = z_0$ is a solution of the integral equation

$$z(t) = V(t, t_0)\left(z_0 + \int_{t_0}^{t} V^{-1}(s, t_0)\eta(s, z(s))ds\right) \tag{2.7}$$

and vice versa. If we look for an initial value z_0 such that the solution $z(t; z_0)$ of (2.7) obeys

$$|z(t; t_0, z_0)| \leq c \quad \forall t \in R, \tag{2.8}$$

where c is some positive constant, then we get from (2.6), (2.7) that z_0 has to fulfil the conditions

$$\begin{aligned} z_0 &= \int_{t_0}^{\infty} V^{-1}(s, t_0)\eta(s, z(s))ds, \\ z_0 &= \int_{t_0}^{-\infty} V^{-1}(s, t_0)\eta(s, z(s))ds. \end{aligned} \tag{2.9}$$

Therefore, a solution $z(t; t_0, z_0)$ of (2.7) satisfying (2.8) has to fulfil the condition

$$\int_{-\infty}^{\infty} V^{-1}(s, t_0)\eta(s, z(s))ds = 0. \tag{2.10}$$

Using (2.4) and (2.5) and the fact that $V(t - t_0) = V(t)V^{-1}(t_0)$, we can rewrite (2.10) as

$$\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} W^{-1}(s)\eta(s, z(s))ds = 0. \tag{2.11}$$

If the condition (2.11) is fulfilled, then any solution of (2.1) satisfying (2.8) is a solution of the integral equation

$$z(t) = e^{\frac{t^2}{2}} W(t) \int_{-\infty}^{t} e^{-\frac{s^2}{2}} W^{-1}(s)\eta(s, z(s))ds \quad \text{for} \quad t \leq 0, \tag{2.12}$$

and of the integral equation

$$z(t) = e^{\frac{t^2}{2}} W(t) \int_{\infty}^{t} e^{-\frac{s^2}{2}} W^{-1}(s)\eta(s, z(s))ds \quad \text{for} \quad t \geq 0. \tag{2.13}$$

Consequently, we have the result

Lemma 2.1 *Suppose the function η satisfies hypothesis (H) and the matrix $B(t)$ is defined by* (2.2). *Then, for equation* (2.1) *to have a solution $\bar{z}(t)$ uniformly bounded for all t, it is necessary that the relation*

$$\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} W^{-1}(s)\eta(s,\bar{z}(s))ds = 0 \tag{2.14}$$

holds. Moreover, $\bar{z}(t)$ is a solution of the integral equations (2.12) *and* (2.13).

A similar result has been obtained in [9].

As an example we consider the differential system

$$\frac{dz}{dt} = B(t) + \tilde{\eta}(t) + u, \tag{2.15}$$

where

$$\tilde{\eta}(t) = (\cos t,\ 0)^T \tag{2.16}$$

and u is a constant two-dimensional vector to be determined. The function $\eta := \tilde{\eta} + u$ satisfies hypothesis (H). The necessary condition (2.14) for a uniformly bounded solution of (2.15) takes the form

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \Big(\cos^2 s + u_1 \cos s + u_2 \sin s \Big) ds = 0, \\ \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \Big(-\frac{1}{2}\sin 2s - u_1 \sin s + u_2 \cos s \Big) ds = 0. \end{aligned} \tag{2.17}$$

Using the relations

$$\int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \cos s\, ds = \sqrt{\frac{2\pi}{e}}, \quad \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \sin ks\, ds = 0, \quad k = 1, 2, \tag{2.18}$$

$$\int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \cos^2 s\, ds = \frac{\sqrt{2\pi}}{2}(1 + e^{-2}), \tag{2.19}$$

we get from (2.17)

$$u_1 = -\frac{\sqrt{e}(e^2+1)}{2e^2}, \quad u_2 = 0. \tag{2.20}$$

According to (2.12), (2.13), the uniformly bounded solution of (2.15), where u_1 and u_2 are determined by (2.20), can be represented by

$$z(t) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} W(t-s)\Big(\tilde{\eta}(s)+u\Big)ds & \text{for} \quad t \le 0, \\ -\int\limits_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} W(t-s)\Big(\tilde{\eta}(s)+u\Big)ds & \text{for} \quad t \ge 0. \end{cases}$$

Let us return to the slow-fast system (1.1). If we assume that this system has an integral manifold $z = h^*(t,y,\varepsilon)$ which is uniformly bounded for all $(t,y,\varepsilon) \in R \times R^n \times I_{\varepsilon_0}$ and if we suppose that $y = \varphi(t;t_0,y_0,\varepsilon)$ is a solution of the Cauchy problem

$$\frac{dy}{dt} = \varepsilon f(t,y,h^*(t,y,\varepsilon),\varepsilon), \quad y(t_0) = y_0,$$

defined for $\forall t \in R$, then $z(t,y_0,\varepsilon) := h^*(t,\varphi(t;t_0,y_0,\varepsilon),\varepsilon)$ represents a uniformly bounded solution of the system

$$\frac{dz}{dt} = B(t)z + \tilde{g}(t,z,h^*(t,\varphi(t;t_0,y_0,\varepsilon),\varepsilon),\varepsilon).$$

According to Lemma 2.1, this solution satisfies the relation

$$\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} W^{-1}(s)\tilde{g}(s,\varphi(s;t_0,y,\varepsilon),h^*(s,\varphi(s;t_0,y_0,\varepsilon),\varepsilon),\varepsilon)\,ds = 0 \qquad (2.21)$$

for any $t_0 \in R$, $y_0 \in R^n$ and $\forall\varepsilon \in I_{\varepsilon_0}$. In order to be able to fulfill relation (2.21) without imposing the condition $\tilde{g} \equiv 0$ we include a control $u = u(y,\varepsilon)$ into the function $\tilde{g}$, that is, we will consider slow-fast systems of the type (1.5), where the control belongs to some admissible set U. If we suppose $g(t,y,0,0,0) \equiv 0$ for all $(t,y) \in R \times R^n$, then any admissible control u must tend to zero as $\varepsilon \to 0$.

3 Notation. Assumptions. Formulation of the problem.

We consider the slow-fast system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t,y,z,\varepsilon), \\ \frac{dz}{dt} &= B(t)z + Z(t,y,z,u,\varepsilon) + u, \end{aligned} \qquad (3.1)$$

where the matrix $B(t)$ is defined in (2.2), and ε is a small parameter. Let $\Omega_z \subset R^2$

and $\Omega_u \in R^2$ be bounded connected regions containing the origin, let I_{ε_0} be the interval $I_{\varepsilon_0} := \{\varepsilon \in R : 0 \le \varepsilon \le \varepsilon_0 \ll 1\}$.

We study system (3.1) under the assumptions

(A$_0$). $Y \in C(R \times R^n \times \Omega_z \times I_{\varepsilon_0}, R^n)$, $Z \in C(R \times R^n \times \Omega_z \times \Omega_u \times I_{\varepsilon_0}, R^2)$.

(A$_1$). There are positive constants b_1, b_2, l_1, l_2 such that for $t \in R,\ y, \bar{y} \in R^n,\ z, \bar{z} \in \Omega_z,\ u, \bar{u} \in \Omega_u$ the following relations hold

$$|Y(t, y, z, \varepsilon)| \le b_1, \tag{3.2}$$

$$|Z(t, y, z, u, \varepsilon)| \le b_2 \left(\varepsilon + \varepsilon|z| + |z|^2\right), \tag{3.3}$$

$$|Y(t, y, z, \varepsilon) - Y(t, \bar{y}, \bar{z}, \varepsilon)| \le l_1 \left(|y - \bar{y}| + |z - \bar{z}|\right), \tag{3.4}$$

$$\begin{gathered}|Z(t, y, z, u, \varepsilon) - Z(t, \bar{y}, \bar{z}, \bar{u}, \varepsilon)| \le \\ l_2 \left((\varepsilon + \varepsilon|\tilde{z}| + |\tilde{z}|^2)|y - \bar{y}| + (\varepsilon + |\tilde{z}|)|z - \bar{z}| + \varepsilon|u - \bar{u}|\right),\end{gathered} \tag{3.5}$$

where $|\tilde{z}| := \max\{|z|, |\bar{z}|\}$.

A manifold $\mathcal{M}_\varepsilon$ in the space of motion $R \times R^n \times \Omega_z$ is called an integral manifold of (3.1) if a solution of (3.1) passing for $t = t_0$ a point on $\mathcal{M}_\varepsilon$ stays for all t on $\mathcal{M}_\varepsilon$.

From (3.3) we get

$$Z(t, y, 0, u, 0) \equiv 0. \tag{3.6}$$

Hence, for $\varepsilon = 0, u = 0$, system (3.1) coincides with the linear system (2.3) and has the integral manifold $z \equiv 0$, which is attracting for $t < 0$, and repelling for $t > 0$. In the sequel we characterize such behavior by saying that the integral manifold $z \equiv 0$ loses its attractivity with increasing t.

From (3.6) we conclude that any admissible control u must tend to zero as ε tends to zero. Hence, we suppose that the set U of admissible control functions consists of all function u mapping $R^n \times I_{\varepsilon_0}$ continuously into Ω_u and satisfy for all $y, \bar{y} \in R^n,\ \varepsilon \in I_{\varepsilon_0}$

$$|u(y, \varepsilon)| \le \varepsilon b_3, \quad |u(y, \varepsilon) - u(\bar{y}, \varepsilon)| \le \varepsilon l_3 |y - \bar{y}|, \tag{3.7}$$

where b_3 and l_3 are some positive numbers to be determined later. If we endow U with the metric

$$\varrho(u, \bar{u}) := \sup_{y \in R^n,\ \varepsilon \in I_{\varepsilon_0}} |u(y, \varepsilon) - \bar{u}(y, \varepsilon)|, \tag{3.8}$$

then U is a complete metric space.

Our goal is, for sufficiently small ε, to establish the existence of a control function $u \in U$ such that the slow-fast system (3.1) has an integral manifold $\mathcal{M}_\varepsilon := \{(t,y,z) \in R \times R^n \times \Omega_z : z = h(t,y,\varepsilon)\}$, where h is continuous and satisfies for $t \in R, \varepsilon \in I_{\varepsilon_0}, y, \bar{y} \in R^n$ the inequalities

$$|h(t,y,\varepsilon)| \le \varepsilon b_4, \quad |h(t,y,\varepsilon) - h(t,\bar{y},\varepsilon)| \le \varepsilon l_4 |y - \bar{y}|, \tag{3.9}$$

where b_4 and l_4 will be determined later. We denote the space of these functions by H. With respect to the metric

$$d(h,\bar{h}) := \sup_{t\in R,\ y\in R^n,\ \varepsilon\in I_{\varepsilon_0}} |h(t,y,\varepsilon) - \bar{h}(t,y,\varepsilon)|$$

H is a complete metric space.
Our main result is the following:

Theorem 3.1 *Under the assumptions $(A_0), (A_1)$ there exists an $\varepsilon^* \in I_{\varepsilon_0}$ such that for all $0 \le \varepsilon \le \varepsilon^*$ there is a control function $u \in U$ ensuring that system* (3.1) *has an integral manifold $z = h(t,y,\varepsilon)$ with $h \in H$.*

Remark 3.2 If for sufficiently small ε system (3.1) has an integral manifold $z = h(t,y,\varepsilon)$ with $h \in H$, then we know that for $\varepsilon = 0$ the integral manifold $z \equiv 0$ loses its attractivity for increasing t. Therefore, it follows from the continuous dependence of the trajectories of (3.1) on the parameter ε that also the integral manifold $z = h(t,y,\varepsilon)$ loses its attractivity for increasing t. In this case for sufficiently small ε the trajectories of system (3.1) starting for $t_0 < 0$ at any initial point after a short time interval enter a small neighbourhood of the attracting part of the integral manifold $z = h(t,y,\varepsilon)$ and follow it until the time $t = 0$. For $t > 0$ the trajectories stay in this small neighbourhood of the repelling part of the integral manifold until some time $t = t^* > 0$. For $t > |t_0|$ the trajectory grows exponentially. We note that this property reminds of the phenomenon of delayed loss of stability in the theory of singularly perturbed systems [1, 4, 10].

4 A necessary condition for the existence of the integral manifold $\mathcal{M}_\varepsilon$.

We assume that system (3.1) has for $u = u^*(y,\varepsilon)$ an integral manifold $\mathcal{M}_\varepsilon$ with the representation $z = h^*(t,y,\varepsilon)$, where h^* belongs to the space H. The dynamics of (3.1) on $\mathcal{M}_\varepsilon$ is described by the differential system

$$\frac{dy}{dt} = \varepsilon Y(t, y, h^*(t, y, \varepsilon), \varepsilon). \tag{4.1}$$

Under the hypotheses (A_0), (A_1), the Cauchy problem $y(t_0) = y_0$ to (4.1) has for any $t_0 \in R$ $y_0 \in R^n$ and $\varepsilon \in I_{\varepsilon_0}$ a solution $y = \varphi(t; t_0, y_0, \varepsilon)$ defined for all $t \in R$. Thus, the function $z(t, y, \varepsilon) = h^*(t, \varphi(t; t_0, y_0, \varepsilon), \varepsilon)$ is a solution of the two-dimensional system

$$\frac{dz}{dt} = B(t)z + Z(t, \varphi(t; t_0, y_0, \varepsilon), z, u^*(\varphi(t; t_0, y_0, \varepsilon), \varepsilon), \varepsilon) + u^*(\varphi(t; t_0, y_0, \varepsilon), \varepsilon),$$

which is bounded for all t. According to (2.21), the following relation must be valid for any $(t_0, y_0, \varepsilon) \in R \times R^n \times I_{\varepsilon_0}$.

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} W^{-1}(s) \Big[Z(s, \varphi(s; t_0, y_0, \varepsilon), h^*(s, \varphi(s; t_0, y_0, \varepsilon), \varepsilon), \\ &u^*(\varphi(s; t_0, y_0, \varepsilon), \varepsilon), \varepsilon) + u^*(\varphi(s; t_0, y_0, \varepsilon), \varepsilon) \Big] ds = 0. \end{aligned} \tag{4.2}$$

Our idea is to use the necessary condition (4.2) for the existence of the integral manifold $\mathcal{M}_\varepsilon$ in order to determine the control function $u^* \in U$. For this purpose we consider for any $h \in H$ the Cauchy problem

$$\frac{dy}{dt} = \varepsilon Y(t, y, h(y, t, \varepsilon), \varepsilon), \quad y(t_0) = y_0. \tag{4.3}$$

Under our assumptions, it has a unique solution denoted by $\varphi_h(t; t_0, y_0, \varepsilon)$ which is defined for all t. Using this solution we will employ the relation

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} W^{-1}(s) \Big[Z(s, \varphi_h(s; t_0, y_0, \varepsilon),\ h(s, \varphi_h(s; t_0, y_0, \varepsilon), \varepsilon), \\ &u(\varphi_h(s; t_0, y_0, \varepsilon), \varepsilon), \varepsilon) + u(\varphi_h(s; t_0, y_0, \varepsilon), \varepsilon) \Big] ds = 0 \end{aligned} \tag{4.4}$$

to determine $u \in U$ as a function of (y, h, ε).

Using the fact that

$$\varphi_h(t; t_0, y_0, \varepsilon) = \varphi_h(t; 0, \tilde{y}_0, \varepsilon),$$

we rewrite (4.4) in the form

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} W^{-1}(s) \Big[Z(s, \varphi_h(s; 0, \tilde{y}_0, \varepsilon),\ h(s, \varphi_h(s; 0, \tilde{y}_0, \varepsilon), \varepsilon), \\ &u(\varphi_h(s; 0, \tilde{y}_0, \varepsilon), \varepsilon), \varepsilon) + u(\varphi_h(s; 0, \tilde{y}_0, \varepsilon), \varepsilon) \Big] ds = 0. \end{aligned} \tag{4.5}$$

In the following section we will show that to given $h \in H$ and for sufficiently small ε, equation (4.5) determines $u \in U$ as a unique function of (h, y, ε). We denote this function by $u_h(y, \varepsilon)$.

Since t_0, y_0 are arbitrary, we put $t_0 = t$, $y_0 = y$. Then, by means of the function $u_h(y,\varepsilon)$ we define on H the operator T by

$$(Th)(t,y,\varepsilon) := \begin{cases} e^{\frac{t^2}{2}}W(t)\displaystyle\int_{-\infty}^{t} e^{\frac{-s^2}{2}}W^{-1}(s)\Big[Z(s,\varphi_h(s;t,y,\varepsilon),h(s,\varphi_h(s;t,y,\varepsilon),\varepsilon), \\ u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon),\varepsilon) + u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon)\Big]ds \quad \text{for} \quad t \le 0, \\ -e^{\frac{t^2}{2}}W(t)\displaystyle\int_{t}^{\infty} e^{\frac{-s^2}{2}}W^{-1}(s)\Big[Z(s,\varphi_h(s;t,y,\varepsilon),h(s,\varphi_h(s;t,y,\varepsilon),\varepsilon), \\ u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon),\varepsilon) + u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon)\Big]ds \quad \text{for} \quad t \ge 0. \end{cases} \tag{4.6}$$

In section 6 we will prove that under the hypotheses $(A_0),(A_1)$ the operator T maps H into itself and is strictly contractive for sufficiently small ε. That is, T has a unique fixed point h^* in H. It is then easy to see that the relation

$$z = h^*(t,y,\varepsilon) \tag{4.7}$$

defines an integral manifold to system (3.1) in the (t,y,z)-space. If we replace in the right hand side of (4.7) y by the trajectory $\varphi_{h^*}(t;t_0,y_0,\varepsilon)$, then it is easy to prove that $z(t;t_0,y_0,h^*,\varepsilon) := h^*(t,\varphi_{h^*}(t;t_0,y_0,\varepsilon),\varepsilon)$ satisfies the differential equation

$$\frac{dz}{dt} = B(t)z + Z(t,\varphi_{h^*}(t;t_0,y_0,\varepsilon),z,u_{h^*}(\varphi_{h^*}(t;t_0,y_0,\varepsilon),\varepsilon),\varepsilon) + u_{h^*}(\varphi_{h^*}(t;t_0,y_0,\varepsilon),\varepsilon).$$

5 Determination of the control function

At first we describe the dependence of the solution $\varphi_h(s;t,y,\varepsilon)$ of (4.3) on the initial value y and on the function $h \in H$.

Lemma 5.1 *Under the assumptions $(A_0),(A_1)$ the following inequalities are valid for any $y,\bar{y} \in R^n$, $h,\bar{h} \in H$*

$$|\varphi_h(s;t,y,\varepsilon) - \varphi_h(s;t,\bar{y},\varepsilon)| \le |y-\bar{y}|e^{\varepsilon l_1(1+\varepsilon l_4)|s-t|},$$

$$|\varphi_h(s;t,y,\varepsilon) - \varphi_{\bar{h}}(s;t,y,\varepsilon)| \le \frac{1}{1+\varepsilon l_4}d(h,\bar{h})\left(e^{\varepsilon l_1(1+\varepsilon l_4)|s-t|} - 1\right).$$

Proof. By (4.3) it holds

$$\begin{aligned}
\varphi_h(s;t,y,\varepsilon) &= y+\varepsilon\int_t^s Y(\eta,\varphi_h(\eta;t,y,\varepsilon),h(\eta,\varphi_h(\eta;t,y,\varepsilon),\varepsilon),\varepsilon)d\eta,\\
\varphi_h(s;t,\bar y,\varepsilon) &= \bar y+\varepsilon\int_t^s Y(\eta,\varphi_h(\eta;t,\bar y,\varepsilon),h(\eta,\varphi_h(\eta;t,\bar y,\varepsilon),\varepsilon),\varepsilon)d\eta,\\
\varphi_{\bar h}(s;t,y,\varepsilon) &= y+\varepsilon\int_t^s Y(\eta,\varphi_{\bar h}(\eta;t,y,\varepsilon),\bar h(\eta,\varphi_{\bar h}(\eta;t,y,\varepsilon),\varepsilon),\varepsilon)d\eta.
\end{aligned} \tag{5.8}$$

Using (5.8) and the inequalities (3.2), (3.4) and (3.9) we obtain for $s\geq t$

$$\begin{gathered}
|\varphi_h(s;t,y,\varepsilon)-\varphi_h(s;t,\bar y,\varepsilon)|\leq |y-\bar y|+\\
+\int_t^s \varepsilon|Y(\eta,\varphi_h(\eta;t,y,\varepsilon),h(\eta,\varphi_h(\eta;t,y,\varepsilon),\varepsilon),\varepsilon)-\\
-Y(\eta,\varphi_h(\eta;t,\bar y,\varepsilon),h(\eta,\varphi_h(\eta;t,\bar y,\varepsilon),\varepsilon),\varepsilon)|d\eta\leq\\
\leq |y-\bar y|+\int_t^s \varepsilon l_1\left(|\varphi_h(\eta;t,y,\varepsilon)-\varphi_h(\eta;t,\bar y,\varepsilon)|+\right.\\
\left.+|h(\eta,\varphi_h(\eta;t,y,\varepsilon),\varepsilon)-h(\eta,\varphi_h(\eta;t,\bar y,\varepsilon),\varepsilon)|\right)d\eta\leq\\
\leq |y-\bar y|+\int_t^s \varepsilon l_1(1+\varepsilon l_4)|\varphi_h(\eta;t,y,\varepsilon)-\varphi_h(\eta;t,\bar y,\varepsilon)|d\eta.
\end{gathered}$$

Using the Gronwall-Bellman inequality we get

$$|\varphi_h(s;t,y,\varepsilon)-\varphi_h(s;t,\bar y,\varepsilon)|\leq |y-\bar y|e^{\varepsilon l_1(1+\varepsilon l_4)(s-t)} \quad \text{for} \quad s\geq t. \tag{5.9}$$

For the difference $|\varphi_h(s;t,y,\varepsilon)-\varphi_{\bar h}(s;t,y,\varepsilon)|$ we have

$$\begin{gathered}
|\varphi_h(s;t,y,\varepsilon)-\varphi_{\bar h}(s;t,y,\varepsilon)|\leq \int_t^s \varepsilon|Y(\eta,\varphi_h(\eta;t,y,\varepsilon),h(\eta,\varphi_h(\eta;t,y,\varepsilon),\varepsilon),\varepsilon)-\\
-Y(\eta,\varphi_{\bar h}(\eta;t,y,\varepsilon),\bar h(\eta,\varphi_{\bar h}(\eta;t,y,\varepsilon),\varepsilon),\varepsilon)|d\eta\leq\\
\leq\int_t^s \varepsilon l_1\left((1+\varepsilon l_4)|\varphi_h(\eta;t,y,\varepsilon)-\varphi_{\bar h}(\eta;t,y,\varepsilon)|+d(h,\bar h)\right)d\eta.
\end{gathered}$$

Using the Gronwall-Bellman inequality we obtain

$$|\varphi_h(s;t,y,\varepsilon)-\varphi_{\bar h}(s;t,y,\varepsilon)|\leq \frac{1}{1+\varepsilon l_4}d(h,\bar h)\left(e^{\varepsilon l_1(1+\varepsilon l_4)(s-t)}-1\right) \quad \text{for} \quad s\geq t. \tag{5.10}$$

In the same way we get for $s\leq t$

$$|\varphi_h(s;t,y,\varepsilon)-\varphi_h(s;t,\bar y,\varepsilon)|\leq |y-\bar y|e^{\varepsilon l_1(1+\varepsilon l_4)(t-s)},$$

$$|\varphi_h(s;t,y,\varepsilon)-\varphi_{\bar h}(s;t,y,\varepsilon)|\leq \frac{1}{1+\varepsilon l_4}d(h,\bar h)\left(e^{\varepsilon l_1(1+\varepsilon l_4)(t-s)}-1\right).$$

This completes the proof.

□

Now we consider equation (4.5). In what follows we prove that to any given $h \in H$ this equation determines uniquely a function $u \in U$ which we denote by $u_h(y, \varepsilon)$.

Theorem 5.2 *Suppose the hypotheses* $(A_0), (A_1)$, *to be valid. If we choose* $b_3 = 4b_2$ *and* $l_3 = 32l_2$, *then there is a sufficiently small* $\varepsilon_1 \in I_{\varepsilon_0}$ *such that to given* $h \in H$ *equation (4.5) defines uniquely a function* $u_h(y, \varepsilon) \in U$ *for* $\varepsilon \in I_{\varepsilon_1}$.

Proof. To given $h \in H$ we define on U the linear operator A_h and the nonlinear operator Q_h by

$$(A_h u)(y, \varepsilon) := \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) u(\varphi(s; 0, y, h, \varepsilon), \varepsilon)\, ds,$$

$$(Q_h u)(y, \varepsilon) := -\sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) Z(\cdot)\, ds, \tag{5.11}$$

where

$$Z(\cdot) = Z(s, \varphi_h(s; 0, y, \varepsilon), h(s, \varphi_h(s; 0, y, \varepsilon), \varepsilon), u(\varphi_h(s; 0, y, \varepsilon), \varepsilon), \varepsilon). \tag{5.12}$$

By means of these operators we can rewrite equation (4.5) in the form

$$A_h u = Q_h u. \tag{5.13}$$

In order to be able to prove that A_h is invertible it is convenient to represent the operator A_h in the form $A_h = I + R_h$, where I is the identity and R_h is defined by

$$(R_h u)(y, \varepsilon) := \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s)[u(\varphi_h(s; 0, y, \varepsilon), \varepsilon) - u(y, \varepsilon)] ds. \tag{5.14}$$

By (2.5), (3.7) we obtain

$$|(R_h u)(y, \varepsilon)| \le \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} |\ u(\varphi_h(s; 0, y, \varepsilon), \varepsilon) - u(y, \varepsilon)| ds \le$$

$$\le \varepsilon l_3 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} |\varphi_h(s; 0, y, \varepsilon) - y| ds \le$$

$$\le 2\varepsilon^2 l_3 \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} e^{\frac{-s^2}{2}} \int_{0}^{s} |Y(r, \varphi_h(r; 0, y, \varepsilon), h(r, \varphi_h(r; 0, y, \varepsilon), \varepsilon), \varepsilon)| dr\, ds \ \le$$

$$\leq 2\varepsilon^2 l_3 b_1 \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{\frac{-s^2}{2}} s\, ds = 2\varepsilon^2 l_3 b_1 \sqrt{\frac{2}{\pi}}.$$

Thus, if we choose ε sufficiently small such that

$$\varepsilon^2 l_3 b_1 \sqrt{\frac{2}{\pi}} < \frac{1}{4},$$

then the operator norm of R_h is less than $\frac{1}{2}$, and there exists the linear inverse operator $(I+R_h)^{-1}$ satisfying

$$||(I+R_h)^{-1}|| \leq 2. \tag{5.15}$$

Let us introduce the operator P_h with domain U by

$$P_h u := (I+R_h)^{-1} Q_h u. \tag{5.16}$$

Then the operator equation (5.13) is equivalent to the fixed point problem

$$u = P_h u.$$

In the sequel we prove that the operator P_h maps U into itself and is strictly contractive. Thereby, the error integral

$$\operatorname{erf}(r) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^r e^{-\frac{s^2}{2}} ds \tag{5.17}$$

satisfying

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(-r) = \operatorname{erf}(r), \quad \operatorname{erf}'(r) > 0, \quad \operatorname{erf}(+\infty) = 1 \tag{5.18}$$

will be used.

From (3.3), (3.9), (5.11), (5.12) we get

$$|(Q_h u)(y,\varepsilon)| \leq \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} |Z(\cdot)| ds \leq$$

$$\leq \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} b_2(\varepsilon + \varepsilon|h| + |h|^2) ds \leq \varepsilon b_2 (1 + \varepsilon b_4 + \varepsilon b_4^2).$$

Using this estimate and inequality (5.15), we obtain from (5.16)

$$|(P_h u)(y,\varepsilon)| \leq 2\varepsilon b_2 (1 + \varepsilon b_4 + \varepsilon b_4^2).$$

If we set

$$b_3 := 4 b_2, \tag{5.19}$$

then the estimate

$$|P_h u(y,\varepsilon)| \le \varepsilon b_3$$

is valid for sufficiently small ε.

By Lemma 5.1 and inequality (3.5) we obtain

$$|(Q_h u)(y,\varepsilon) - (Q_h u)(\bar{y},\varepsilon)| \le$$

$$\le \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} l_2 e^{\frac{-s^2}{2}} [(\varepsilon + \varepsilon|h| + |h|^2)|\varphi_h(s;0,y,\varepsilon) - \varphi_h(s;0,\bar{y},\varepsilon)| +$$

$$+(\varepsilon + |h|)\,|h(s,\varphi_h(s;0,y,\varepsilon),\varepsilon) - h(s,\varphi_h(s;0,\bar{y},\varepsilon),\varepsilon)| +$$

$$+\varepsilon|u(\varphi_h(s;0,y,\varepsilon),\varepsilon) - u(\varphi_h(s;0,\bar{y},\varepsilon),\varepsilon)|]ds \le$$

$$\le \frac{\varepsilon\sqrt{2} l_2 l_5(\varepsilon)}{\sqrt{\pi}} \int_{\infty}^{+\infty} e^{\frac{-s^2}{2}} |\varphi(s;0,y,h,\varepsilon) - \varphi(s;0,\bar{y},h,\varepsilon)| ds \le$$

$$\le \frac{\varepsilon\sqrt{2} l_2 l_5(\varepsilon)}{\sqrt{\pi}} |y-\bar{y}| \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} e^{\varepsilon l_1(1+\varepsilon l_4)|s|} ds, \tag{5.20}$$

where

$$l_5(\varepsilon) := 1 + \varepsilon b_4 + \varepsilon b_4^2 + \varepsilon l_4(1+b_4) + \varepsilon l_3. \tag{5.21}$$

For sufficiently small ε we have

$$l_5(\varepsilon) \le 2. \tag{5.22}$$

The integral in the last line of (5.20) can be rewritten as

$$\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} e^{\varepsilon l_1(1+\varepsilon l_4)|s|} ds = 2\int_0^{+\infty} e^{-\frac{s^2}{2} + \varepsilon l_1(1+\varepsilon l_4)s} ds. \tag{5.23}$$

From the relation

$$-\sigma^2 + 2\varepsilon l_1(1+\varepsilon l_4)\sigma = -(\sigma - \varepsilon l_1(1+\varepsilon l_4))^2 + (\varepsilon l_1(1+\varepsilon l_4))^2 \tag{5.24}$$

we get

$$\int_0^{+\infty} e^{-\frac{s^2}{2} + \varepsilon l_1(1+\varepsilon l_4)s} ds = e^{\varepsilon^2 \kappa(\varepsilon)} \int_0^{+\infty} e^{-\frac{(\sigma - \varepsilon l_1(1+\varepsilon l_4))^2}{2}} d\sigma, \tag{5.25}$$

where

$$\kappa(\varepsilon) := (l_1(1+\varepsilon l_4)^2.$$

Thus, for sufficiently small ε we may assume

$$e^{\varepsilon^2 \kappa(\varepsilon)} \le \sqrt{e}. \tag{5.26}$$

By means of the transformation

$$\tau = \sigma - \varepsilon l_1(1+\varepsilon l_4)$$

we get

$$\int_0^{+\infty} e^{-\frac{(\sigma-\varepsilon l_1(1+\varepsilon l_4))^2}{2}} d\sigma = \int_{-\varepsilon l_1(1+\varepsilon l_4)}^{+\infty} e^{-\frac{\tau^2}{2}} d\tau. \tag{5.27}$$

By (5.17), (5.18) we have

$$\begin{aligned}&\int_0^{+\infty} e^{-\frac{(\sigma-\varepsilon l_1(1+\varepsilon l_4))^2}{2}} \sigma = \int_{-\varepsilon l_1(1+\varepsilon l_4)}^{0} e^{-\frac{\tau^2}{2}} d\tau + \int_0^{+\infty} e^{-\frac{\tau^2}{2}} d\tau = \\ &= \frac{\sqrt{\pi}}{\sqrt{2}}\Big(\operatorname{erf}(\varepsilon l_1(1+\varepsilon l_4)) + 1\Big) \le \sqrt{2\pi},\end{aligned} \tag{5.28}$$

and we obtain from (5.25) and (5.26)

$$\int_0^{\infty} e^{\frac{-s^2}{2}} e^{\varepsilon l_1(1+\varepsilon l_4)s} ds \le \sqrt{2\pi e}. \tag{5.29}$$

Consequently, according to (5.23) we have

$$\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} e^{\varepsilon l_1(1+\varepsilon l_4)|s|} ds \le 2\sqrt{2\pi e}. \tag{5.30}$$

Taking into account this estimate, by (5.20), (5.22) it holds

$$|(Q_h u)(y,\varepsilon) - (Q_h u)(\bar{y},\varepsilon)| \le 8\varepsilon l_2\sqrt{e}|y-\bar{y}|.$$

Therefore, for sufficiently small ε we have by (5.15) and (5.16)

$$|(P_h u)(y,\varepsilon) - (P_h u)(\bar{y},\varepsilon)| \le 2|(Q_h u)(y,\varepsilon) - (Q_h u)(\bar{y},\varepsilon)| \le 16\varepsilon l_2\sqrt{e}|y-\bar{y}|.$$

If we put

$$l_3 := 32 l_2\sqrt{e}, \tag{5.31}$$

then the estimate

$$|(P_h u)(y,\varepsilon) - (P_h u)(\bar{y},\varepsilon)| \le \varepsilon l_3|y-\bar{y}|$$

is valid for sufficiently small ε and we can conclude that P_h maps U into itself.

In the next step we derive conditions assuring P_h to be a contraction operator in U. At first we estimate the difference $Q_h u - Q_h \bar{u}$ for $u, \bar{u} \in U$. According to (3.5), (3.7), (5.11), (5.17) and (5.18) we have

$$|(Q_h u)(y,\varepsilon) - (Q_h \bar{u})(y,\varepsilon)| \le \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} \varepsilon l_2 \varrho(u,\bar{u}) ds = 2\varepsilon l_2 \varrho(u,\bar{u}).$$

Hence, by (5.15) and (5.16) we get

$$|(P_h u)(y,\varepsilon) - (P_h \bar{u})(y,\varepsilon)| \leq 4\varepsilon l_2 \varrho(u,\bar{u}).$$

Thus, for sufficiently small ε, P_h is contraction operator in U, and the equation $u = P_h u$, which is equivalent to (4.5), possesses a unique solution u_h in U. □

Now we study the dependence of the fixed point u_h of P_h on h. Let $u_h(y,\varepsilon)$ and $u_{\bar{h}}(y,\varepsilon)$ be the solutions of (4.5) corresponding to the functions h and $\bar{h}$ respectively. Thus, we have

$$(I + R_h)u_h = Q_h u_h, \quad (I + R_{\bar{h}})u_{\bar{h}} = Q_{\bar{h}} u_{\bar{h}}, \tag{5.32}$$

where in analogy to (5.11), (5.14) it holds

$$(R_{\bar{h}} u_{\bar{h}})(y,\varepsilon) := \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s)[u_{\bar{h}}(\varphi_{\bar{h}}(s;0,y,\varepsilon),\varepsilon) - u_{\bar{h}}(y,\varepsilon)]ds, \tag{5.33}$$

$$(Q_{\bar{h}} u_{\bar{h}})(y,\varepsilon) := -\frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) Z(\cdot) ds, \tag{5.34}$$

with

$$Z(\cdot) = Z(s, \varphi_{\bar{h}}(s;0,y,\varepsilon), \bar{h}(s, \varphi_{\bar{h}}(s;0,y,\varepsilon),\varepsilon), u_{\bar{h}}(\varphi_{\bar{h}}(s;0,y,\varepsilon),\varepsilon),\varepsilon).$$

From (5.32) we obtain

$$u_h - u_{\bar{h}} = (I + R_h)^{-1}[Q_h u - Q_{\bar{h}} u_{\bar{h}} + (R_{\bar{h}} - R_h)u_{\bar{h}}]. \tag{5.35}$$

By (3.7), (3.9), (5.11), (5.21), (5.34) and Lemma 5.1 we have

$$|(Q_h u_h)(y,\varepsilon) - (Q_{\bar{h}} u_{\bar{h}})(y,\varepsilon)| \leq$$

$$\leq \frac{\sqrt{2} l_2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} \Bigg[(\varepsilon + \varepsilon|\tilde{h}| + |\tilde{h}|^2)|\varphi_h(s;0,y,\varepsilon) - \varphi_{\bar{h}}(s;0,y,\varepsilon)| +$$

$$+ (\varepsilon + |\tilde{h}|)|h(s,\varphi_h(s;0,y,\varepsilon),\varepsilon) - \bar{h}(s,\varphi_{\bar{h}}(s;0,y,\varepsilon),\varepsilon)| +$$

$$+ \varepsilon|u_h(\varphi_h(s;0,y,\varepsilon),\varepsilon) - u_{\bar{h}}(\varphi_{\bar{h}}(s;0,y,\varepsilon),\varepsilon)| \Bigg] ds$$

$$\leq \frac{\varepsilon\sqrt{2} l_2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} \Bigg[l_5(\varepsilon)|\varphi_h(s;0,y,\varepsilon) - \varphi_{\bar{h}}(s;0,y,\varepsilon)| +$$

$$+(1+b_4)d(h,\bar h)+\varrho(u_h,u_{\bar h})\Bigg]ds \le \varepsilon l_2\Bigg[\varrho(u_h,u_{\bar h})+(1+b_4)d(h,\bar h)+$$

$$+\frac{\sqrt{2}l_5(\varepsilon)d(h,\bar h)}{\sqrt{\pi}(1+\varepsilon l_4)}\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}}(e^{\varepsilon l_1(1+\varepsilon l_4)|s|}-1)ds\Bigg]. \tag{5.36}$$

Taking into account the estimate (5.30) and the relations (5.17) and (5.18) we have

$$\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}}(e^{\varepsilon l_1(1+\varepsilon l_4)|s|}-1)ds \le \sqrt{2\pi}(2-\sqrt{e}). \tag{5.37}$$

Assuming ε to be sufficiently small such that $1+\varepsilon l_4 \le \frac{3}{2}$ holds, then we get from (5.36), (5.37), (5.22)

$$|(Q_h u_h)(y,\varepsilon)-(Q_{\bar h}u_{\bar h})(y,\varepsilon)| \le \varepsilon l_2\left[\varrho(u_h,u_{\bar h})+\left(1+b_4+6(2-\sqrt{e})\right)d(h,\bar h)\right]. \tag{5.38}$$

Analogously we obtain from (5.14) and (5.33) for sufficiently small ε

$$\begin{aligned}
&|(R_{\bar h}-R_h)u_{\bar h}(y,\varepsilon)| \le \frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}}|u_{\bar h}(\varphi_{\bar h}(s;0,y,\varepsilon),\varepsilon)-u_{\bar h}(\varphi_h(s;0,y,\varepsilon),\varepsilon)|ds \le \\
&\le \frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}}\varepsilon l_3|\varphi_h(s;0,y,\varepsilon)-\varphi_{\bar h}(s;0,y,\varepsilon)|ds \le \\
&\le \frac{\varepsilon\sqrt{2}l_3 d(h,\bar h)}{\sqrt{\pi}(1+\varepsilon l_4)}\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}}(e^{\varepsilon l_1(1+\varepsilon l_4)|s|}-1)ds \le 3\varepsilon l_3(2-\sqrt{e})d(h,\bar h).
\end{aligned} \tag{5.39}$$

Hence, from (5.15), (5.31), (5.35), (5.38), (5.39) we get

$$\varrho(u_h,u_{\bar h}) \le 2\varepsilon l_2\Big[\varrho(u_h,u_{\bar h})+(1+b_4)+102(2-\sqrt{e})d(h,\bar h)\Big].$$

From this inequality we obtain the following result

Lemma 5.3 *Suppose the hypotheses of Theorem 5.2 are satisfied. Then for sufficiently small ε the following estimate is true*

$$\varrho(u_h,u_{\bar h}) \le 2\varepsilon l_2\Big[1+b_4+102(2-\sqrt{e})\Big]d(h,\bar h). \tag{5.40}$$

6 Existence of the integral manifold

As we mentioned in section 4, a fixed point of the operator T defines an integral manifold of system (3.1). In this section we derive conditions guaranteeing that T maps the space H into itself and is strictly contractive in H.

For $h \in H$, $u_h \in U$, and $t \leq 0$ we get from (3.3), (3.7), (3.9), (4.6), (5.18), (5.19)

$$
\begin{aligned}
|(Th)(t,y,\varepsilon)| &\leq \int_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} \Big[|Z(\cdot)| + |u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon)| \Big] ds \leq \\
&\leq \varepsilon \Big(b_2(1+\varepsilon b_4+\varepsilon b_4^2) + b_3 \Big) \int_0^{+\infty} e^{\frac{-s^2}{2}} ds = \varepsilon \frac{\sqrt{\pi}}{\sqrt{2}} b_2(5+\varepsilon b_4+\varepsilon b_4^2).
\end{aligned}
\tag{6.41}
$$

If we set

$$
b_4 := 10 b_2 \frac{\sqrt{\pi}}{\sqrt{2}}, \tag{6.42}
$$

then the boundedness condition in (3.9) is valid for sufficiently small ε and $t \leq 0$. It can be verified that the same result is valid in case $t \geq 0$.

In order to prove that $(Th)(t,y,\varepsilon)$ obeys the Lipschitz condition in (3.9) we estimate for $t \leq 0$ in a similar way

$$
\begin{aligned}
&|(Th)(t,y,\varepsilon) - (Th)(t,\bar{y},\varepsilon)| \leq \\
&\leq \int_{-\infty}^{t} e^{\frac{(t^2-s^2)}{2}} \Big[|Z(s,\varphi_h(s;t,y,\varepsilon), h(s,\varphi_h(s;t,y,\varepsilon),\varepsilon), u(\varphi_h(s;t,y,\varepsilon),\varepsilon),\varepsilon) - \\
&\quad - Z(s,\varphi_h(s;t,\bar{y},\varepsilon), h(s,\varphi_h(s;t,\bar{y},\varepsilon),\varepsilon), u(\varphi_h(s;t,\bar{y},\varepsilon),\varepsilon),\varepsilon)| + \\
&\quad + |u(\varphi_h(s;t,y,\varepsilon),\varepsilon) - u(\varphi_h(s;t,\bar{y},\varepsilon),\varepsilon)| \Big] ds \leq \\
&\leq \int_{-\infty}^{t} e^{\frac{(t^2-s^2)}{2}} [\varepsilon l_2(1+\varepsilon b_4+\varepsilon b_4^2)|\varphi_h(s;t,y,\varepsilon) - \varphi_h(s;t,\bar{y},\varepsilon)| + \\
&\quad + \varepsilon l_2(1+b_4)|h(s,\varphi_h(s;t,y,\varepsilon),\varepsilon) - h(s,\varphi_h(s;t,\bar{y},\varepsilon),\varepsilon)| + \\
&\quad + (\varepsilon l_2 + 1)|u(\varphi_h(s;t,y,\varepsilon),\varepsilon) - u(\varphi_h(s;t,\bar{y},\varepsilon),\varepsilon)|] ds \leq \\
&\leq \varepsilon(l_2 l_5(\varepsilon) + l_3) \int_{-\infty}^{t} e^{\frac{(t^2-s^2)}{2}} |\varphi_h(s;t,y,\varepsilon) - \varphi_h(s;t,\bar{y},\varepsilon)| ds \leq \\
&\leq \varepsilon(l_2 l_5(\varepsilon) + l_3)|y-\bar{y}| \int_0^{+\infty} e^{\frac{-s^2}{2}} e^{\varepsilon l_1(1+\varepsilon l_4)s} ds.
\end{aligned}
$$

Due to (5.22), (5.29) we obtain for $t \leq 0$ and sufficiently small ε

$$
|(Th)(t,y,\varepsilon) - (Th)(t,\bar{y},\varepsilon)| \leq \varepsilon\sqrt{2\pi e}(2l_2 + l_3)|y-\bar{y}|.
$$

Since the same inequality is valid for $t \geq 0$ and if we take into account relation (5.31) it holds for any t

$$
|(Th)(t,y,\varepsilon) - (Th)(t,\bar{y},\varepsilon)| \leq 2\varepsilon l_2\sqrt{2\pi e}(1+16\sqrt{e})|y-\bar{y}|.
$$

Hence, if we set

$$l_4 := 2\sqrt{2\pi e} l_2 (1 + 16\sqrt{e}), \tag{6.43}$$

then T maps H into itself.

Now we prove that T is strictly contractive in H. In the same way as above we obtain from (4.6) for $t \leq 0$ and sufficiently small ε

$$|(Th)(t,y,\varepsilon) - (T\bar{h})(t,y,\varepsilon)| \leq$$

$$\leq \int_{-\infty}^{t} e^{\frac{(t^2-s^2)}{2}} \Bigg[|Z\left(s, \varphi_h(s;t,y,\varepsilon), h(s, \varphi_h(s;t,y,\varepsilon),\varepsilon), u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon),\varepsilon\right) -$$

$$-Z\left(s, \varphi_{\bar{h}}(s;t,y,\varepsilon), \bar{h}(s, \varphi_{\bar{h}}(s;t,y,\varepsilon),\varepsilon), u_{\bar{h}}(\varphi_{\bar{h}}(s;t,y,\varepsilon),\varepsilon),\varepsilon\right)|+$$

$$+|u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon) - u_{\bar{h}}(\varphi_{\bar{h}}(s;t,y,\varepsilon),\varepsilon)| \Bigg] ds \leq$$

$$\leq \int_{-\infty}^{t} e^{\frac{(t^2-s^2)}{2}} \left(\varepsilon l_2 (1 + \varepsilon b_4 + \varepsilon b_4^2)(|\varphi_h(s;t,y,\varepsilon) - \varphi_{\bar{h}}(s;t,y,\varepsilon)| + \right.$$

$$+\varepsilon l_2 (1 + b_4)|h(s, \varphi_h(s;t,y,\varepsilon),\varepsilon) - \bar{h}(s, \varphi_{\bar{h}}(s;t,y,\varepsilon),\varepsilon)|)+$$

$$\left. + (1 + \varepsilon l_2)|u_h(\varphi_h(s;t,y,\varepsilon),\varepsilon) - u_{\bar{h}}(\varphi_{\bar{h}}(s;t,y,\varepsilon),\varepsilon)|\right) ds \leq$$

$$\leq \int_{-\infty}^{0} e^{\frac{-s^2}{2}} \Big(\varepsilon (l_2 l_5 + l_3)|\varphi_h(s;t,y,\varepsilon) - \varphi_{\bar{h}}(s;t,y,\varepsilon)| +$$

$$+\varepsilon l_2 (1 + b_4) d(h, \bar{h}) + (1 + \varepsilon l_2)\varrho(u_h, u_{\bar{h}}) \Big) ds \leq$$

$$\leq \left(\varepsilon l_2 (1 + b_4) d(h, \bar{h}) + (1 + \varepsilon l_2)\varrho(u_h, u_{\bar{h}})\right) \int_{0}^{+\infty} e^{\frac{-s^2}{2}} ds +$$

$$+2\varepsilon l_2 \frac{(1 + 16\sqrt{e})}{1 + \varepsilon l_4} d(h, \bar{h}) \int_{0}^{+\infty} e^{\frac{-s^2}{2}} \left(e^{\varepsilon l_1 (1 + \varepsilon l_4) s} - 1\right) ds.$$

Taking into account (5.18), (5.29), (5.40) we get for sufficiently small ε

$$|(Th)(t,y,\varepsilon) - (T\bar{h})(t,y,\varepsilon)| \leq$$

$$\varepsilon l_2 \frac{\sqrt{\pi}}{\sqrt{2}} \left[\left(1 + b_4 + 2(1 + \varepsilon l_2)\left(1 + b_4 + 102(2 - \sqrt{e})\right)\right) + 3(1 + 16\sqrt{e})(2\sqrt{e} - 1) \right] d(h, \bar{h})$$

Therefore, T is a contraction operator in H for sufficiently small ε.

Thus, we have proved Theorem 3.1

Remark 6.1 Theorem 3.1 can be generalized for the case when the matrix $B(t)$ has the form

$$B(t) = \begin{pmatrix} \alpha(t)\, t & \beta(t) \\ -\beta(t) & \alpha(t)\, t \end{pmatrix},$$

where $\alpha(t), \beta(t)$ are continuous for all $t \in R$ and satisfy

$$0 < \alpha_1 \leq \alpha(t) \leq \alpha_2 < +\infty, \quad 0 < \beta_1 \leq \beta(t) \leq \beta_2 < +\infty.$$

Remark 6.2 If in addition to the conditions of the Theorem 3.1 the functions $Y(t, y, z, \varepsilon)$, $Z(t, y, z, u, \varepsilon)$ on the right hand side of (3.1) have continuous and bounded partial derivatives with respect to y, z, u up to the order $(k+1)$, then the integral manifold $h(t, y, \varepsilon)$ and the control function $u(y, \varepsilon)$ have continuous and bounded partial derivatives with respect to y up to the order k.

Remark 6.3 If the functions $Y(t, y, z, \varepsilon)$ and $Z(t, y, z, u, \varepsilon)$ have bounded partial derivatives with respect to y, z, u, ε of order $(k+1)$, then the integral manifold $z = h(t, y, \varepsilon)$ and the control function $u(y, \varepsilon)$ have the asymptotic representation

$$\begin{aligned} h(t, y, \varepsilon) &= \sum_{i \geq 0}^{k} \varepsilon^i h_i(t, y) + r_h(t, y, \varepsilon), \\ u(y, \varepsilon) &= \sum_{i \geq 0}^{k} \varepsilon^i u_i(y) + r_u(y, \varepsilon), \end{aligned} \tag{6.44}$$

where h_i and u_i are bounded functions which are by Remark 6.2 k-times continuously differentiable with respect y up to the order k, and $r_h = O(\varepsilon^{k+1}), r_u = O(\varepsilon^{k+1})$.

As an example we consider the slow-fast system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= B(t)z + Z(t, y, z, u, \varepsilon) + u(y, \varepsilon), \end{aligned} \tag{6.45}$$

with $y \in R$ and

$$Z(t, y, z, u, \varepsilon) = Z(t, y, \varepsilon) := \begin{pmatrix} \varepsilon \cos t \cos y \\ 0 \end{pmatrix}. \tag{6.46}$$

The function Z satisfies hypotheses (A_0) and (A_1). Then, relation (4.5) takes the form

$$
\begin{aligned}
&\varepsilon \cos y \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \cos^2 s ds + u_1 \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \cos s ds = 0, \\
&-\varepsilon \frac{\cos y}{2} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \sin 2s ds + u_2 \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \sin s ds = 0.
\end{aligned}
\tag{6.47}
$$

Using the relations (2.18), (2.19) we get from (6.47)

$$
u_1(y,\varepsilon) = -\frac{\varepsilon e^{1/2}}{2}(1+e^{-2}) \cos y, \quad u_2(y,\varepsilon) = 0.
$$

Substituting these results into the right hand side of (4.6) we get the following representation of the integral manifold $z = h(t,y,\varepsilon)$ given by

$$
h(t,y,\varepsilon) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} W(t-s)\Big(Z(s,y,\varepsilon) + u(y,\varepsilon)\Big) ds & \text{for} \quad t < 0, \\ \displaystyle -\int_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} W(t-s)\Big(Z(s,y,\varepsilon) + u(y,\varepsilon)\Big) ds & \text{for} \quad t \geq 0. \end{cases}
$$

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